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SOME CONDITIONS IMPLYING THE EXISTENCE OF COINCIDENCE POINTS OF A PAIR OF INTUITIONISTIC FUZZY MAPPINGS

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Abstract. In this article, two novel conditions implying the existence of a pair of intuitionistic fuzzy mappings with an integral type contractive conditions are discussed. An example validating the main result is given. As an application of our result, an existence theorem of solution for a class of nonlinear Volterra equation of the first type is established. This way, the work extends a few existing results in the literature .

Keywords: coincidence points; fuzzy sets; intuitionistic fuzzy mappings.

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1. Introduction

Let $(X,d) = X$ be a metric space and $CB(X)$ be the set of all nonempty closed and bounded subsets of X. Let $A,B \in CB(X)$. Then,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\},$$

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where

$$d(x,A) = \inf_{y\in A} d(x,y), \qquad d(A,B) = \inf_{x\in A, y\in B} d(x,y).$$

A fuzzy set A in X is an object of the form

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A : X \longrightarrow [0,1]$ is called the membership function of A and $\mu_A(x)$ is the degree of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and is defined as:

$$[A]_\alpha = \{x : A(x) \geq \alpha, \quad \text{if} \quad \alpha \in (0,1]\},$$

$$[A]_0 = \{x : \overline{A(x) > 0}, \text{if} \quad \alpha = 0\};$$

where $\overline{B}$ denotes the closure of a crisp set B.

A fuzzy mapping T is a fuzzy set-valued mapping of X into I^X. A fuzzy mapping T is a fuzzy subset of $X \times Y$ with membership function $T(x)(y)$. The function value $T(x)(y)$ is the grade of membership of y in $T(x)$. An $x^* \in X$ is called a fuzzy fixed point of T if there exists an $\alpha \in (0,1]$ such that $x^* \in [Tx^*]_\alpha$. An element x^* of X is called a fixed point of T if $T(x^*)(x^*) \geq T(x^*)(x), \forall x \in X$. Similarly, x^* is known as Heilpern fixed point of T if $\{x^*\} \subset T(x^*)$. For some of these modification of classical fixed point, see [6, 8] and the reference therein.

The earliest most known result on fixed points for contractive-type mappings is the Banach's theorem introduced in 1922 (see, [8, 13]) . The Banach's contraction theorem has been extended in different directions (see, [1, 2, 14, 15, 18], and the reference therein). The idea of intuitionistic fuzzy set (IFS) was first studied in 1986 by Atanassov [7], using the concepts of t-norm and t-conorm as an extension of fuzzy set initiated by Zadeh [22] in 1965. Intuitionistic fuzzy sets do not only characterize the degree of membership of an element, but also tell the degree of nonmembership. As a result, (IFS) gained applications in the areas of modeling real life problems such as psychological investigation, career determination, to mention but a few. For some development in the field of (IFS), see [10, 11, 12].Thereafter, the idea of fuzzy mappings was introduced by Heilpern [16] which is also fuzzy extension of Banach contraction mapping and a direct expansion of the concept of fuzzy sets.

Along this development , Akbar [2] used generalized contractive conditions involving a rational inequality to study common fixed point theorems for fuzzy set-valued mappings. The result was further developed in [5] with particular applications in function space. As a result, some new common intuitionistic fixed point theorems for a pair of intuitionistic fuzzy mappings in a complete metric space with (α,β)-cut set of intuitionistic fuzzy sets were derived.

Along the lane, fixed point theorem for mappings satisfying contractive condition of integral type was brought up by Branciari [9]. Given a metric space (X,d) with $x,y\in X$, and for some $\lambda\in(0,1)$, Branciari studied the self mapping T on X satisfying the contractive conditions of the type,

$$\int_0^{d(Tx,Ty)}\varphi(t)dt\leq\lambda\int_0^{d(x,y)}\varphi(t)dt,$$

for any Lebesgue integrable function $\varphi:[0,+\infty)\longrightarrow[0,+\infty)$ which is summable on each compact subset of $[0,+\infty)$ and satisfies $\int_0^{\varepsilon}\varphi(t)dt>0$, for all $\varepsilon>0$.

In this paper, we study two conditions guaranteing the existence of coincidence point of a given pair of intuitionistic fuzzy mappings. The main contractive condition used here is an extension of the work of [3, 5] into integral form. In this direction, it is hoped this work will attract in a way, interests in the field of fuzzy fixed point theory.

2. Preliminaries

For the use of our main results, the following salient preliminary definitions and results are recalled.

Let Ψ represents the class of functions $\varphi:[0,\infty)\longrightarrow[0,\infty)$ satisfying the following conditions:

(i) φ is Lebesgue integrable, summable on each compact subset of $[0,\infty)$,

(ii) $\int_0^{\tau}\varphi(t)dt>0$, for each $\tau>0$.

Definition 2.1.[7] Let X be a universal set. An intuitionistic fuzzy set (IFS) A in X is an object of the form :

$$A=\{\langle x,\mu_A(x),\nu_A(x)|x\in X\rangle\};$$

where the function $\mu_A : X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$ define the degree of membership and nonmembership of the element $x \in X$ to the set A, respectively, such that for every $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.2.[7] Let A be an intuitionistic fuzzy set and $x \in X$. The α-level set of A is denoted by $[A]_\alpha$ and is defined as:

$$[A]_\alpha = \{x \in X : \mu_A(x) \geq \alpha \quad \text{and} \quad \nu_A(x) \leq 1-\alpha\}, \text{if} \quad \alpha \in [0,1].$$

Definition 2.3.[20]

Let $L = \{(\alpha,\beta) : \alpha+\beta \leq 1, \alpha,\beta \in (0,1]\times[0,1)\}$ and A is an IFS on X. The (α,β)-cut set of A is defined as:

$$A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \quad \text{and} \quad \nu_A(x) \leq \beta\}.$$

Definition 2.4.[21] Let X be an arbitrary set and Y a metric space. A mapping $T : X \longrightarrow (IFX)^X$ is called an intuitionistic fuzzy mapping. A point $x \in X$ is called an intuitionistic fuzzy fixed point of T if there exists $(\alpha,\beta) \in (0,1]\times[0,1)$ such that $x^* \in [Tx^*]_{(\alpha,\beta)}$.

Lemma 2.5.[19] Let A and B be nonempty closed and bounded subsets of a metric space X. If $x \in A$, then

$$d(x,B) \leq H(A,B).$$

Lemma 2.6.[19] Let A and B be nonempty closed and bounded subsets of a metric space X. Then, for any $\varepsilon > 0$, and $x \in A$, there exists $y \in B$ such that

$$d(x,y) \leq H(A,B) + \varepsilon.$$

3. Main Results

Now, we discuss the two conditions for the existence of point of coincidence of a pair of intuitionistic fuzzy mappings with an integral type contractive conditions.

Theorem 3.1. Let X be a nonempty set, Y a metric space , $\varphi \in \Psi$ and $S,T : X \longrightarrow (IFS)^X$. Suppose that for each $x \in X$, there exists $(\alpha,\beta)_{Sx}, (\alpha,\beta)_{Tx} \in (0,1]\times[0,1)$ such that $[Sx]_{(\alpha,\beta)_{Sx}}, [Tx]_{(\alpha,\beta)_{Tx}} \in CB(Y)$, $\bigcup[Sx]_{(\alpha,\beta)_{Sx}} \subseteq \bigcup[Tx]_{(\alpha,\beta)_{Tx}}$ and either $\bigcup[Sx]_{(\alpha,\beta)_{Sx}}$ or $\bigcup[Tx]_{(\alpha,\beta)_{Tx}}$

is complete. If there exists $\gamma \in (0,1)$ and any $\rho > 0$ such that for all $x, y \in X$,

$$\int_0^{H\left([Sx]_{(\alpha,\beta)_{Sx}},[Sy]_{(\alpha,\beta)_{Sy}}\right)+\rho} \varphi(t)dt \leq \gamma \int_0^{d\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)+\frac{\rho}{\gamma}} \varphi(t)dt, \tag{1}$$

then, there exists $u \in X$ such that $[Su]_{(\alpha,\beta)_{Su}} \cap [Tu]_{(\alpha,\beta)_{Tu}} \neq \emptyset$.

Proof. Let x_0 be an arbitrary element of X. Suppose that $y \in [Sx_0]_{(\alpha,\beta)_{Sx_0}}$. Since $\bigcup[Sx]_{(\alpha,\beta)_{Sx}} \subseteq \bigcup[Tx]_{(\alpha,\beta)_{Tx}}$, then there exists $x_1 \in X$ such that $y_1 \in [Tx_1]_{(\alpha,\beta)_{Tx_1}}$. This implies

$$[Sx_0]_{(\alpha,\beta)Sx_0} \cap [Tx_1]_{(\alpha,\beta)Tx_1} \neq \emptyset.$$

If $d([Tx_0]_{(\alpha,\beta)Tx_0}, [Tx_1]_{(\alpha,\beta)_{Tx_1}}) = 0$,then it follows from ineq (1) that $[Sx_0]_{(\alpha,\beta)_{Sx_0}} = [Sx_1]_{(\alpha,\beta)_{Sx_1}}$. Hence, $y_1 \in [Sx_1]_{(\alpha,\beta)_{Sx_1}} \cap [Tx_1]_{(\alpha,\beta)_{Tx_1}}$. This means y is the required coincidence point of S and T.

Now, if $d([Tx_0]_{(\alpha,\beta)_{Tx_0}}, [Tx_1]_{(\alpha,\beta)_{Tx_1}}) \neq 0$, then by Lemma 2.6 , we can choose $y_2 \in [Sx_1]_{(\alpha,\beta)_{Sx_1}}$ such that

$$d(y_1, y_2) \leq H\left([Sx_0]_{(\alpha,\beta)_{Sx_0}}, [Sx_1]_{(\alpha,\beta)_{Sx_1}}\right) + \gamma^2. \tag{2}$$

For this $y_2 \in [Sx_1]_{(\alpha,\beta)_{Sx_1}}$, we may use the fact that $\bigcup[Sx]_{(\alpha,\beta)_{Sx}} \subseteq \bigcup[Tx]_{(\alpha,\beta)_{Tx}}$ to obtain $x_2 \in X$ such that $y_2 \in [Tx_2]_{(\alpha,\beta)_{Tx_2}}$. Again, by Lemma 2.6 , we can choose $y_3 \in [Sx_2]_{(\alpha,\beta)Sx_2}$ such that

$$d(y_2, y_3) \leq H([Sx_1]_{(\alpha,\beta)_{Sx_1}}, [Sx_2]_{(\alpha,\beta)Sx_2}) + \gamma^4. \tag{3}$$

Continuing this process repeatedly produces $x_n \in X$ such that
$y_n \in [Tx_n]_{(\alpha,\beta)_{Tx_n}}, \quad n \in \mathbb{N}$.
and

$$d(y_n, y_{n+1}) \leq H\left([Sx_{n-1}]_{(\alpha,\beta)_{Sx_{n-1}}}, [Sx_n]_{(\alpha,\beta)_{Sx_n}}\right) + \gamma^n$$

Now, from ineqs.(1) and (2), we get

$$\begin{aligned} \int_0^{d(y_1,y_2)} \varphi(t)dt &\leq \int_0^{H([Sx_0]_{(\alpha,\beta)_{Sx_0}},[Sx_1]_{(\alpha,\beta)_{Sx_1}})+\gamma^2} \varphi(t)dt \\ &\leq \gamma \int_0^{d([Tx_0]_{(\alpha,\beta)_{Tx_0}},[Tx_1]_{(\alpha,\beta)_{Tx_1}})+\gamma} \varphi(t)dt = \gamma \int_0^{d(y_0,y_1)+\gamma} \varphi(t)dt. \end{aligned} \tag{4}$$

If $d([Tx_1]_{(\alpha,\beta)_{Tx_1}},[Tx_2]_{(\alpha,\beta)_{Tx_2}})=0$, then by similar arguments, the conclusion is obtained. That is, $y_2 \in [Sx_2]_{(\alpha,\beta)Sx_2} \cap [Tx_2]_{(\alpha,\beta)Tx_2}$.

Again, from ineqs. (1) and (3), we have

$$\begin{aligned}
\int_0^{d(y_2,y_3)} \varphi(t)dt &\leq \int_0^{H([Sx_1]_{(\alpha,\beta)_{Sx_1}},[Sx_2]_{(\alpha,\beta)_{Sx_2}})+\gamma^4} \varphi(t)dt \\
&\leq \gamma \int_0^{d([Tx_1]_{(\alpha,\beta)_{Tx_1}},[Tx_2]_{(\alpha,\beta)_{Tx_2}})+\gamma^3} \varphi(t)dt \\
&\leq \gamma \int_0^{d(y_1,y_2)+\gamma^3} \varphi(t)dt \leq \gamma \int_0^{H([Sx_0]_{(\alpha,\beta)_{Sx_0}},[Sx_1]_{(\alpha,\beta)_{Sx_1}})+\gamma^2+\gamma^3} \varphi(t)dt \\
&\leq \gamma^2 \int_0^{d([Tx_0]_{(\alpha,\beta)_{Tx_0}},[Tx_1]_{(\alpha,\beta)_{Tx_1}})+\gamma^2+\gamma^3} \varphi(t)dt \\
&\leq \gamma^2 \int_0^{d(y_0,y_1)+\gamma^2+\gamma^3} \varphi(t).
\end{aligned}$$

Continuing this process, having picked $x_n \in X$, $y_n \in [Sx_{n-1}]_{(\alpha.\beta)_{Sx_{n-1}}} \cap [Tx_n]_{(\alpha,\beta)_{Tx_n}}$, we obtain $x_{n+1} \in X$ and $y_{n+1}[Sx_n]_{(\alpha,\beta)_{Sx_n}} \cap [Tx_{n+1}]_{(\alpha,\beta)_{Tx_{n+1}}}$ such that

$$\begin{aligned}
\int_0^{d(y_n,y_{n+1})} \varphi(t) &\leq \gamma^n \int_0^{d(y_0,y_1)+\gamma+\gamma^2+\gamma^3+\cdots+\gamma^n} \varphi(t)dt \\
&\leq \gamma^n \int_0^{d(y_0,y_1)+\frac{\gamma}{1-\gamma}} \varphi(t)dt.
\end{aligned}$$

Now, for $m > n \geq 1$, and $\varepsilon = \frac{\gamma}{1-\gamma}$, consider

$$\begin{aligned}
\int_0^{d(y_m,y_n)} \varphi(t)dt &\leq \int_0^{d(y_m,y_{m-1})} \varphi(t)dt + \int_0^{d(y_{m-1},y_{m-2})} \varphi(t)dt \cdots + \int_0^{d(y_{n+1},y_n)} \varphi(t)dt \\
&\leq \gamma^{m-1} \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt + \gamma^{m-2} \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt + \cdots + \gamma^n \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt \\
&= \left(\gamma^{m-1}+\gamma^{m-2}+\cdots+\gamma^n\right) \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt \\
&= \gamma^n \left(1+\gamma+\gamma^2+\cdots+\gamma^{m-n-1}\right) \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt \\
&= \left[\frac{\gamma^n}{1-\gamma}\right] \int_0^{d(y_0,y_1)+\varepsilon} \varphi(t)dt.
\end{aligned}$$

As $n \longrightarrow \infty$, $\frac{\gamma^n}{1-\gamma} \longrightarrow 0$.

Hence, $\int_0^{d(y_m,y_n)} \varphi(t)dt \longrightarrow 0 (m,n \longrightarrow \infty)$. Therefore, $\{y_n\}$ is a Cauchy sequence in $\bigcup [Tx]_{(\alpha,\beta)_{Tx}}$. By the completeness of $\bigcup [Tx]_{(\alpha,\beta)_{Tx}}$, there exists $z \in \bigcup [Tx]_{(\alpha,\beta)_{Tx}}$ such that $y_n \longrightarrow z, (n \longrightarrow \infty)$.

(This also hold if $\bigcup[Sx]_{(\alpha,\beta)_{Sx}}$ is complete with $z \in \bigcup[Sx]_{(\alpha,\beta)_{Sx}} \subseteq \bigcup[Tx]_{(\alpha,\beta)_{Tx}}$).
Consequently, $z \in [Tu]_{(\alpha,\beta)_{Tu}}$ for some $u \in X$.

On the other hand, suppose on the contrary that $d(z,[Su]_{(\alpha,\beta)_{Su}}) = \tau > 0$. Then, by triangle inequality, we have

$$d(z,[Su]_{(\alpha,\beta)_{Su}}) \leq d(z,y_n) + d(y_n,[Su]_{(\alpha,\beta)_{Su}}).$$

Now, consider:

$$\begin{aligned}\int_0^{d(y_n,[Su]_{(\alpha,\beta)_{Su}})} \varphi(t)dt &\leq \int_0^{H\left([Sx_{n-1}]_{(\alpha,\beta)_{Sx_{n-1}}},[Su]_{(\alpha,\beta)_{Su}}\right)+\gamma\tau} \varphi(t)dt \\ &\leq \gamma \int_0^{d([Tx_{n-1}]_{(\alpha,\beta)_{Tx_{n-1}}},[Tu]_{(\alpha,\beta)_{Tu}})+\tau} \varphi(t)dt \\ &< \gamma \int_0^{d(y_{n-1},z)+\tau} \varphi(t)dt.\end{aligned}$$

Letting $n \longrightarrow \infty$, gives $\int_0^\tau < \gamma \int_0^\tau$, which is a contradiction. It follows that $d(z,[Su]_{(\alpha,\beta)Su}) = 0$.
Hence, $z \in [Su]_{(\alpha,\beta)_{Su}} \cap [Tu]_{(\alpha,\beta)_{Tu}}$.

Application

In the following result, we apply Theorem 3.1 with the completeness property of the function space $(C[a,b],\mathbb{R})$ to establish an existence theorem of solution for a class of nonlinear Volterra equation of the first type. This technique is borrowed from [2].

Theorem 3.2. Let $A : \mathbb{R} \times [a,b] \longrightarrow \mathbb{R}$, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous mappings, $p_0 \in \mathbb{R}$, $\varphi \in \Psi$. Let $X = (C[a,b],\mathbb{R})$. Assume that for each $x \in X$, there exists $y \in X$ such that

$$(h \circ y)(t) = p_0 + \int_a^t [A(u,x(u))]du$$

and $\{h \circ x : x \in X\}$ is closed. If there exists $\lambda < \frac{1}{b-a}$ such that for all $x,y \in \mathbb{R}$, $t \in [a,b]$,

$$| A(t,x) - A(t,y) | \leq \lambda |h(x) - h(y)|$$

then the integral equation

$$h(x(t)) = p_0 + \int_a^t [A(u,x(u))]du, \quad a \leq t \leq b, \tag{5}$$

has a solution in $(C[a,b],\mathbb{R})$.

Proof. Let $X = Y = (C[a,b],\mathbb{R})$ and $d : X \times X \longrightarrow \mathbb{R}$ be defined by

$$d(x,y) = \max_{a\leq t\leq b} |x(t) - y(t)|.$$

Suppose $\eta, \theta, \varphi, \psi : X \longrightarrow (0,1]$ are any four mappings. Assume that for $x \in X$, we have

$$\pi_x(t) = p_0 + \int_a^t [A(u,x(u))]du, \quad a \leq t \leq b. \tag{6}$$

Define a pair of intuitionistic fuzzy mappings $S,T : X \longrightarrow (IFS)^X$ as follows:

$$\mu_{Sx}(r) = \begin{cases} \eta(x), & \text{if } r(t) = \pi_x(t), \quad a \leq t \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{Sx}(r) = \begin{cases} 0, & \text{if } r(t) = \pi_x(t), \quad a \leq t \leq b, \\ \theta(x), & \text{otherwise.} \end{cases}$$

and

$$\mu_{Tx}(r) = \begin{cases} \varphi(x), & \text{if } r(t) = h(x(t)), \quad a \leq t \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_{Tx}(r) = \begin{cases} 0, & \text{if } r(t) = h(x(t)), \quad a \leq t \leq b, \\ \psi(x), & \text{otherwise.} \end{cases}$$

If $\alpha_{Sx} = \eta(x), \beta_{Sx} = 0, \quad$ and $\quad \alpha_{Tx} = \varphi(x), \beta_{Tx} = 0$, then

$$\begin{aligned} \bigcup_{x\in X} [Sx]_{(\alpha,\beta)_{Sx}} &= \bigcup_{x\in X} \{r \in X : \mu_{Sx}(r) = \eta(x) \quad \text{and} \quad \nu_{Sx}(r) = 0\} \\ &= \bigcup_{x\in X} \{\pi_x(t)\}. \end{aligned}$$

and

$$\begin{aligned} \bigcup_{x\in X} [Tx]_{(\alpha,\beta)_{Tx}} &= \bigcup_{x\in X} \{r \in X : \mu_{Tx}(r) = \varphi(x) \quad and \quad \nu_{Tx}(r) = 0\} \\ &= \bigcup_{x\in X} \{h \circ x : x \in X\}. \end{aligned}$$

Since X is complete and $\{h\circ x : x\in X\}$ is closed in X , therefore $\bigcup_{x\in X}[Tx]_{(\alpha,\beta)_{Tx}}$ is also complete.

Further, for $x\in\bigcup_{x\in X}[Sx]_{(\alpha,\beta)_{Sx}}$, we obtain $x\in X$ such that $\mu_{Sx}(r)=\alpha_{Sx}$ and hence $r(t)=\pi_x(t), a\le t\le b$.

Then, by assumption, there exists $y\in X$ such that $r=h\circ y$. Consequently,

$$\bigcup_{x\in X}[Sx]_{(\alpha,\beta)_{Sx}}=\bigcup_{x\in X}\{\pi_x\}\subseteq\bigcup_{x\in X}[Tx]_{(\alpha,\beta)_{Tx}}=\bigcup_{x\in X}\{h\circ x : x\in X\}\cdot$$

Hence,

$$\int_0^{H\left([Sx]_{(\alpha,\beta)_{Sx}},[Sy]_{(\alpha,\beta)_{Sy}}\right)}\varphi(t)dt\le\int_0^{\max|\pi_x(t)-\pi_y(t)|}\varphi(t)dt,\quad a\le t\le b.$$

and

$$\int_0^{d\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)}\varphi(t)dt\le\int_0^{\max|h(x(t))-h(y(t))|}\varphi(t)dt,\quad a\le t\le b.$$

Now, from eqs. (5) and (6), we obtain

$$\begin{aligned}|\pi_x(t)-\pi_y(t)|&=\left|\int_a^t[A(u,x(u))]du-\int_a^t[A(u,y(u))]du\right|\\&\le\int_a^t|A(u,x(u))-A(u,y(u))|\,du\\&\le\int_a^t\lambda\,|h(x(u))-h(y(u))|\,du\\&\le\lambda\,(\sup|(Tx)(t)-(Ty)(t)|)\left|\int_a^t du\right|\\&\le\lambda(b-a)d\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right).\end{aligned}$$

Notice that for $\gamma=\lambda(b-a)$ and recalling that ρ was arbitrary, all the hypotheses of Theorem 3.1 are satisfied to find a continuous function

$z:[a,b]\longrightarrow\mathbb{R}$ such that $[Sz]_{(\alpha,\beta)_{Sz}}\cap[Tz]_{(\alpha,\beta)_{Tz}}\neq\emptyset$.

This shows that z is a solution of the integral equation (5).

Below, as an additional application, we obtain a solution of an initial value problem(IVP) using Theorem 3.1.

Example 3.3. Consider an IVP given as

$$\frac{d}{dt}x(t) = \left(x(t) + \frac{t}{x^8(t)}\right)\frac{t}{9}, \quad x(0) = \rho, \quad 0 \le t \le b, b < 1. \tag{7}$$

From eq. (7), we have the integral equation:

$$x^9(t) = \rho^9 + \int_0^t \left(x^9(u) + u\right) u du, \quad 0 \le t \le b, b < 1. \tag{8}$$

Observe that for all $t \in [0,b]$,

$$\left|[x^9 + t]t - [y^9 + t]t\right| = |t| \left|x^9 - y^9\right| \le b|x^9 - y^9|.$$

Taking $X = (C[0,b], \mathbb{R})$, we see that all conditions of Theorem 3.2 are satisfied for $\lambda = 1, a = 0$. Now, we approximate the solution z by constructing the iterative sequences as follows:

$$x_n \in (C[0,b], \mathbb{R}), \quad y_n \in [Sx_{n-1}]_{(\alpha,\beta)_{Sx_{n-1}}} \cap [Tx_n]_{(\alpha,\beta)_{Tx_n}},$$

in connection with the intuitionistic fuzzy mappings $S, T : X \longrightarrow (IFS)^X$ defined by:

$$\mu_{Sx}(r) = \begin{cases} \alpha_S, & \text{if } r(t) = \pi_x(t), \quad 0 \le t \le b, \\ 0, & \text{otherwise.} \end{cases}$$

$$v_{Sx}(r) = \begin{cases} \beta_S, & \text{if } r(t) = \pi_x(t), \quad 0 \le t \le b, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mu_{Tx}(r) = \begin{cases} \alpha_T, & \text{if } r(t) = x^9, \quad 0 \le t \le b, \\ 0, & \text{otherwise.} \end{cases}$$

$$v_{Tx}(r) = \begin{cases} \beta_T, & \text{if } r(t) = x^9, \quad 0 \le t \le b, \\ 1, & \text{otherwise.} \end{cases}$$

Where $(\alpha,\beta)_{Sx}, (\alpha,\beta)_{Tx} \in (0,1] \times [0,1)$ and

$$\pi_x(t) = \rho^9 + \int_0^t (x^9(u) + u) u du, \quad 0 \le t \le b.$$

We see that

$$\begin{aligned} [Sx]_{(\alpha,\beta)_{Sx}} &= \{r \in X : \mu_{Sx}(r) = \alpha_S \quad \text{and} \quad v_{Sx}(r) = \beta_S\} \\ &= \{\pi_x\}. \end{aligned}$$

$$[Tx]_{(\alpha,\beta)_{Tx}} = \{r \in X : \mu_{Tx}(r) = \alpha_T \quad \text{and} \quad \nu_{Tx}(r) = \beta_T\}$$
$$= \{x^9\}.$$

Take $x_0 = 0$, for all $t \in [0,b]$. Then $[Sx_0]_{(\alpha,\beta)_{Sx_0}} \cap [Tx_1]_{(\alpha,\beta)_{Tx_1}} = \{\pi_{x_0}\}$.

Therefore,

$$y_1 = \pi_{x_0} = \rho^9 + \int_0^t \left[x^9{}_0(u) + u\right] udu$$
$$\rho^9 + \frac{t^3}{3}.$$

It follows that

$$x_1(t) = \left(\rho^9 + \frac{t^3}{3}\right)^{\frac{1}{9}}.$$

Now,

$$y_2 = \pi_{x_1}(t) \in [Sx_1]_{(\alpha,\beta)_{Sx_1}} \cap [Tx_2]_{(\alpha,\beta)_{Tx_2}},$$

where

$$\pi_{x_1}(t) = \rho^9 + \int_0^t [x_1^9(u) + u]udu$$
$$= \rho^9 + \int_0^t [\rho^9 + \frac{u^3}{3} + u]udu$$
$$= \rho^9 + \rho^9\frac{t^2}{2} + \frac{t^5}{3\cdot 5} + \frac{t^3}{3}.$$

Therefore,

$$x_2(t) = \left(\rho^9 + \rho^9\frac{t^2}{2} + \frac{t^5}{3\cdot 5} + \frac{t^3}{3}\right)^{\frac{1}{9}}.$$

Similarly,

$$y_3 = \pi_{x_2}(t) = \int_0^t [x_2^9(t) + u]udu$$
$$= \rho^9 + \rho^9\frac{t^2}{2} + \rho^9\frac{t^4}{2\cdot 4} + \frac{t^7}{3\cdot 5\cdot 7} + \frac{t^5}{3\cdot 5} + \frac{t^2}{2}.$$

and

$$x_3 = \left(\rho^9 + \rho^9\frac{t^2}{2} + \rho^9\frac{t^4}{2\cdot 4} + \frac{t^7}{3\cdot 5\cdot 7} + \frac{t^5}{3\cdot 5} + \frac{t^2}{2}\right)^{\frac{1}{9}}.$$

Hence,

$$\lim_{n\longrightarrow\infty} y_n = \left[\rho^9+\rho^9\sum_{k=1}^{\infty}\frac{t^{2k}}{(k)(2k)\cdots 2}+\sum_{k=1}^{\infty}\frac{t^{2k+3}}{(2k+1)(2k+3)\cdots 3}\right] \in [Sz]_{(\alpha,\beta)_{Sz}} \cap [Tz]_{(\alpha,\beta)_{Tz}}.$$

Consequently,

$$z(t) = \left[\rho^9+\rho^9\sum_{k=1}^{\infty}\frac{t^{2k}}{(k)(2k)\cdots 2}+\sum_{k=1}^{\infty}\frac{t^{2k+3}}{(2k+1)(2k+3)\cdots 3}\right]^{\frac{1}{9}}$$

is a solution of the initial value problem (7) and the integral equation (8).

Knowing that intuitionistic sets generalizes fuzzy sets, the following corollary is immediate from Theorem 3.1.

Corollary 3.1. *Let X be a nonempty set, Y a metric space , $\varphi \in \Psi$ and $S,T : X \longrightarrow I^X$ be fuzzy mappings. Suppose that for each $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in CB(Y)$, $\bigcup[Sx]_{\alpha_S(x)} \subseteq \bigcup[Tx]_{\alpha_T(x)}$ and one of $\bigcup[Sx]_{\alpha_S(x)}$ or $\bigcup[Tx]_{\alpha_T(x)}$ is complete. If there exists $\gamma \in (0,1)$ and any $\rho > 0$ such that for all $x,y \in X$,*

$$\int_0^{H\left([Sx]_{(\alpha,\beta)_{Sx}},[Sy]_{(\alpha,\beta)_{Sy}}\right)+\rho}\varphi(t)dt \leq \gamma\int_0^{\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)+\frac{\rho}{\gamma}}\varphi(t)dt,$$

then there exists $z \in X$ such that $z \in [Su]_{(\alpha,\beta)_{Su}} \cap [Tu]_{(\alpha,\beta)_{Tu}}$. .

3. Conclusion

This paper establishes coincidence and common fixed points of a pair of intuitionistic fuzzy mappings on a metric space. In Theorem 3.1 , an integral type contractive condition is used to obtain a coincidence point of two intuitionistic fuzzy mappings. Thereafter, an example illustrating the applicability of the main result is provided. Consequently, the work generalizes some results of [2, 3, 5, 9] and is therefore hoped to contribute positively in some ways to metric fixed point and fuzzy optimization theory.

Conflict of Interests

The authors declare that there is no conflict of interests.

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