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ON β-ADMISSIBLE CONTRACTION AND COMMON FIXED POINT THEOREMS FOR L-FUZZY MAPPINGS

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Abstract. In this paper, we introduce the concept of generalized β-admissible contraction for a pair of L-fuzzy mappings. By using the new idea, some common fixed point theorems are established. A few examples to illustrate the validity of the main result are also provided.

Keywords: fuzzy sets; L-fuzzy sets; general β-admissible contraction; g-fixed point.

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1. Introduction and Preliminary

A number of practical problems in economics, management sciences, engineering, environment sciences, medical sciences, robotics, computer science, meteorology and a large number of other fields involve vagueness and the difficulty of modeling uncertain data. Classical mathematical techniques are not usually successful because the imprecisions in these domains may be of various kinds. Dating back to about five decades, researchers have been proposing a number of theories for handling imprecise environments. One of these is the theory of fuzzy sets introduced by Zadeh[23]. Fuzzy set theory does not only have applications in physical and

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applied sciences but also in mathematical analysis, decision making, clustering, data mining and in almost all the soft sciences. As a result, more than a handful of generalizations of fuzzy sets have so far appeared in the literature, see, for example, [2, 13, 15]. The theory of fuzzy sets provides a firm mathematical framework in which vague conceptual phenomena can be rigorously studied. For some applications of fuzzy sets, the interested reader may see, for example, [7, 21]) and the reference therein. More than a few authors have extended the concepts of fuzzy sets in different directions. Heilpern [10] introduced the idea of fuzzy contraction mapping and consequently proved the existence of fuzzy fixed point theorem which is a fuzzy generalization of Banach contraction theorem ([5]) and Nadler's [14] fixed point theorem for multivalued mappings. As an extension of the notion of fuzzy sets, Goguen introduced the concept of L-Fuzzy sets in [9], which is a real generalization of fuzzy sets by replacing the range set $[0,1]$ of the membership function by a lattice L. The ideas behind L-fuzzy sets are basically two. First is when L is taken as a complete lattice endowed with a multiplication operator satisfying certain postulates and the second point is when L is viewed as a complete distributive lattice (see, for example, [22, 24]). Along the lane, the concepts of β-admissible mappings was introduced by Samet et al. [19]; the idea of which is used to establish fixed point theorems in partially ordered spaces and coupled fixed point theorems.

Thereafter, Asl et al. [1] refined the notion of β-admissible for single-valued mappings to multi-valued mappings. Along the lane, Azam et al [4], obtained common fixed theorems for Chatterjea type fuzzy mappings on closed ball in a complete metric space. The result hinges on the fact that fuzzy fixed point can be obtained via fixed point theory of mappings defined on closed balls. As a further extension of the work of [3, 19], Maliha et al. [18] presented the notion of β_{FL}-admissible for a pair of L-fuzzy mappings and established the existence of common L-fuzzy fixed point.

In this paper, encouraged by the work of Maliha et al [18], we introduce the concept of generalized β-admissible contraction for L-fuzzy mappings. Our result is an extension of [18, Theorem 14] into integral version. An example is provided to support the main result .

2. NOTATIONS AND PRELIMINARIES

In this section, we recall some basic concepts/definitions relevant to the next sections as follows:

Let $(X,d) = X$ be a metric space and

$$CB(X) = \{A : A \text{ is a nonempty closed and bounded subset of } \}X.$$

For $A, B \in CB(X)$, define

$$d(x,B) = \inf_{y\in B} d(x,y) \quad \text{and} d(A,B) = \inf_{x\in A, y\in B} d(x,y).$$

For any closed and bounded subsets A and B of a metric space X, their Hausdorff distance is defined as :

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \quad \sup_{y\in B} d(A,y)\}, & \text{if it exists} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 2.1. *A relation R on a set L is called a partial order if it is*

(i) *Reflexive*

(ii) *Antisymmetric*

(iii) *Transitive.*

A set L together with a partial ordering R is called a partially ordered set (poset, for short) and is denoted by (L,R) *or* $(L, \preceq_L)$ *. Recall that partial orderings are used to give an order to sets that may not have a natural one.*

Definition 2.2. *Let* X *be a nonempty set and* $(X, \preceq)$ *be a partially ordered set. Then any two elements* $x, y \in X$ *are said to be comparable if either* $x \preceq y$ *or* $y \preceq x$.

Definition 2.3. [9] *A partially ordered set* $(L, \preceq_L)$ *is called*

(i) *a lattice , if* $x \vee y \in L$, $x \wedge y \in L$ *for any* $x, y \in L$;

(ii) *a complete lattice, if* $\bigvee A \in L$, $\bigwedge A \in L$ *for any* $A \subseteq L$;

(iii) *distributive lattice if* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *for any* $x, y, z \in L$.

Definition 2.4. [9] *Let L be a lattice with top element 1_L and bottom element 0_L and let $x,y \in L$. Then y is called a complement of x , if $x \vee y = 1_L$ and $x \wedge y = 0_L$.*

Definition 2.5. [9] *An L-fuzzy set M on a nonempty set X is a function with domain X and whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L respectively.*

Denote the class of all L-fuzzy sets on a nonempty set X by $F_L(X)$. The characteristic function χ_{L_M} of an L-fuzzy set M is defined by :

$$\chi_{L_{M}(x)} = \begin{cases} 0_L, & \text{if } x \notin M \\ 1_L & \text{otherwise} \quad x \in M. \end{cases}$$

Remark 2.6. *Setting $L = [0,1]$, reduces an L-fuzzy set to a fuzzy sets.*

Example 2.7. *Let $L = \{p,q,r,s\}$ be such that for all $\{x,y\} \subseteq L$, with $x \leq y$, we have $x \bigwedge y = 0$ and $x \bigvee y = 1$; then L is a complete distributive lattice with bottom and to elements 0_L and 1_L respectively. Let $X = \{a_1,a_2,a_3,a_4,a_5,a_6\}$ and define $M : X \longrightarrow L$ by*

$$M(a_1) = p, M(a_2) = r, M(a_3) = 0, M(a_4) = p, M(a_5) = s, M(a_6) = 0.$$

Then M is an L-fuzzy set in X and may be denoted by

$$M = \{(a_1,p),(a_2,r),(a_3,1),(a_4,p),(a_5,s),(a_6,0)\}.$$

Definition 2.8. *The α_L-level set of an L-fuzzy set M is denoted by $M_{\alpha L}$ and is defined as follows:*

$$M_{\alpha L} = \{x : \alpha_L \preceq_L M(x), if \quad \alpha_L \in L \setminus \{0_L\}\},$$

and

$M_{0L} = cl(\{x : 0_L \preceq_L M(x)\})$, where $cl(Y)$ is the closure of a crisp set Y.

Definition 2.9. [17] *Let X be an arbitrary set and Y a metric space. A mapping $S : X \longrightarrow F_L(Y)$ is called an L-fuzzy mapping. An L-fuzzy mapping S from X into $F_L(Y)$, is an L-fuzzy subset of $X \times Y$ with membership function $S(x)(y)$. The function $S(x)(y)$ is the degree of belongingness of y in $S(x)$ with respect to the lattice L.*

Definition 2.10. [17] *Let (X,d) be a metric space and $S,T : X \longrightarrow F_L(X)$ be L-fuzzy mappings. A point $p \in X$ is called an L-fuzzy fixed point of S if $p \in [Sp]_{\alpha L}$, where $\alpha_L \in L \setminus \{0_L\}$. The point p is a common L-fuzzy fixed point of S and T if $p \in [Sp]_{\alpha L} \cap [Tp]_{\alpha L}$.*

Lemma 2.11. [14] *Let (X,d) be a metric space and $A,B \in X$. Then for each $x \in A$,*

$$d(x,B) \leq H(A,B).$$

Lemma 2.12. [14] *Let (X,d) be a metric space and $A,B \in X$. Then for each $x \in A$ and any $\varepsilon > 0$, there exists an element $y \in B$ such that*

$$d(x,y) \leq H(A,B) + \varepsilon.$$

Let $\psi = \{\varphi : \varphi : [0,\infty) \longrightarrow \mathbb{R}\}$ be such that φ is nonnegative , Lebesgue integrable and integrably subadditive.

3. Main Results

In this section, we present the new concepts of generalized β-admissible pair and generalized β-admissible contraction for L-fuzzy mappings.

Definition 3.1. *Let (X,d) be a metric space, $g : X \longrightarrow X$, $\beta : X \times X \longrightarrow [0,\infty)$ and $S,T : X \longrightarrow F_L(X)$ be L-fuzzy mappings. The ordered pair (S,T) is called a generalized β-admissible pair if the following axioms hold:*

(i) *for each $g(x) \in X$ and $g(y) \in [Sg(x)]_{\alpha_L(x)}$, where $\alpha_L(x) \in L\setminus\{0_L\}$, with $\beta(g(x),g(y)) = k \geq 1$,, we have $\beta(g(y),g(z)) \geq 1$ for all $g(z) \in [Ty]_{\alpha_L(y)}$, where $\alpha_L(y) \in L\setminus\{0_L\}$;*

(ii) *for each $g(x) \in X$ and $g(y) \in [Tg(x)]_{\alpha_L(x)}$, where $\alpha_L(x) \in L\setminus\{0_L\}$, with $\beta(g(x),g(y)) = k \geq 1$, we have $\beta(g(y),g(z)) \geq 1$, for all $g(z) \in [Sg(y)]_{\alpha_L(y)}$, where $\alpha_L(y) \in L\setminus\{0_L\}$.*

Remark 3.2. *Notice that if (S,T) is β_{F_L}-admissible in the sense of Maliha et al* [18]*, then (S,T) is a generalized β-admissible pair , where $g \equiv I$, the identity mapping.*

Remark 3.3. *If $S = T$, then S is called generalized β-admissible. Also, it is clear that if (S,T) is a generalized β-admissible pair, then (T,S) is also a generalized β-admissible pair.*

Definition 3.4. *Let (X,d) be a complete metric space, $\beta : X \times X \longrightarrow [0,\infty)$, $\varphi \in \psi$ and S,T be L-Fuzzy mappings from X into $F_L(X)$. The ordered pair (S,T) is said to be a generalized β-admissible contraction if there exists an arbitrary function $g : X \longrightarrow X$ such that for all $x,y \in X$ and any $\zeta \geq 0$, the following axioms hold:*

(a) *For each $g(x) \in X$, there exists $\alpha_L(x) \setminus \{0_L\}$ such that $[Sg(x)]_{\alpha_L(x)}$, $[Tg(x)]_{\alpha_L(x)}$ are nonempty closed and bounded subsets of X, and for any $g(x_0) \in X$, there exists $g(x_1) \in [Sg(x_0)]_{\alpha_L(x_0)}$ such that $\beta(g(x_0),g(x_1)) \geq 1$,*

(b) *For all $g(x),g(y) \in X$, we have*

$$\begin{aligned}\int_0^{\delta}\varphi(t)dt \quad &\leq \quad p_1\int_0^{d(g(x),[Sg(x)]_{\alpha_L(x)})}\varphi(t)dt + p_2\int_0^{d(g(y),[Tg(y)]_{\alpha_L(y)})}\varphi(t)dt \\ &\quad + p_3\int_0^{d(g(x),[Tg(y)]_{\alpha_L(y)})}\varphi(t)dt + p_4\int_0^{d(g(y),[Sg(x)]_{\alpha_L(x)})}\varphi(t)dt \\ &\quad p_5\int_0^{d(g(x),g(y))}\varphi(t)dt + \zeta, \qquad (3.1)\end{aligned}$$

where

$\delta = \max\{\beta(g(x),g(y)),\beta(g(y),g(x))\}H\left([Sg(x)]_{\alpha_L(x)},[Tg(y)]_{\alpha_L(y)}\right) + \zeta$; $p_j(1 \leq j \leq 5)$ *are nonnegative reals satisfying $\sum_{j=1}^{5} p_j < 1$ and either $p_1 = p_2$ or $p_3 = p_4$,*

(c) *(S,T) is a generalized β-admissible pair,*

(d) *if $\{g(x_n)\}_{n\in\mathbb{N}}$ is a sequence in X such that $\beta(g(x_n),g(x_{n+1})) \geq 1$, and $g(x_n) \longrightarrow g(x)(n \longrightarrow \infty)$, then $\beta(g(x_n),g(x)) \geq 1$.*

Definition 3.5. *Let (X,d) be a metric space, $\varphi \in \psi$ and S,T be L-Fuzzy mapping from X into $F_L(X)$. A point $u \in X$ is called a g-fixed point of T if there exists an arbitrary function $g : X \longrightarrow X$ such that for any $\lambda \in (0,1)$ there exists $\zeta \geq 0$, such that*

$$\int_0^{d(g(u),[Tg(u)]_{\alpha_L(u)})}\varphi(t)dt \leq \lambda\int_0^{d(g(u),[Tg(u)]_{\alpha_L(u)})}\varphi(t)dt + \zeta. \qquad (3.2)$$

We say $u \in [Tg(u)]_{\alpha_L(u)}$, where $\alpha_L(u) \in L \setminus \{0_L\}$, and hence call u a g-fixed point of the L-fuzzy mapping T if (3.2) holds. u is said to be a common g-fixed point of S and T if $u \in [Sg(u)]_{\alpha_L(u)} \cap [Tg(u)]_{\alpha_L(u)}$. Clearly, if u is a common g-fixed point of S and T, then $g(u)$ is a *common coincidence point of S and T. In particular, if $g \equiv I$, the identity mapping and $\zeta = 0$, then every g- fixed point of T reduces to a fixed point of an L-fuzzy mapping T.*

Next, we give an example in line with the above definitions .

Example 3.6. *Let $X = \{1,2,3\}$ and $g : X \longrightarrow X$ be defined by*

$$g(x) = \begin{cases} 1, & \text{if } x = 2 \\ 2, & \text{if } x = 3 \\ 3, & \text{if } \quad x = 1. \end{cases}$$

Also, for any $x, y \in X$, define $d : g(X) \times g(X) \longrightarrow \mathbb{R}$ by

$$d(g(x), g(y)) = \begin{cases} 0, & \text{if } g(x) = g(y) \\ 1, & \text{if } g(x) \neq g(y) \quad \text{and} \quad g(x), g(y) \in \{1,3\} \\ 2, & \text{if } g(x) \neq g(y) \quad \text{and} \quad g(x), g(y) \in \{1,2\} \\ 3, & \text{if } g(x) \neq g(y) \quad \text{and} \quad g(x), g(y) \in \{2,3\}. \end{cases}$$

Obviously, (X, d) is a complete metric space.

Further, let $L = \{p, q, r, u\}$ with $p \preceq_L q \preceq_L u$, $p \preceq_L r \preceq_L u$, q and r are not comparable. Moreover, define L-fuzzy mappings $S, T : X \longrightarrow F_L(X)$ by

$$(Tg(1))(t) = (T3)(t) = \begin{cases} q, & \text{if } t = 1,3 \\ u, & \text{if } \quad t = 2. \end{cases}$$

$$(Tg(2))(t) = (T1)(t) = \begin{cases} u, & \text{if } t = 1 \\ p, & \text{if } \quad t = 2,3. \end{cases}$$

$$(Tg(3))(t) = (T2)(t) = \begin{cases} r, & \text{if } t = 1,2 \\ u, & \text{if } \quad t = 3. \end{cases}$$

and

$$(Sg(1))(t) = (Sg(2))(t) = (Sg(3))(t) = \begin{cases} u, & \text{if } t = 3 \\ q, & \text{if } \quad t = 1,2. \end{cases}$$

Let $\alpha = u$. Then, we get

$$[Tg(x)]_\alpha = \begin{cases} \{2\}, & \text{if } g(x) = g(1) \\ \{1\}, & \text{if } g(x) = g(2) \\ \{3\}, & \text{if } \quad g(x) = g(3). \end{cases}$$

and $[Sg(x)]_\alpha = \{3\}$, *for all* $g(x) \in X$.

Again, define $\beta : g(X) \times g(X) \longrightarrow [0,\infty)$ *by*

$$\beta(g(x),g(y)) = \begin{cases} k \geq 1, & \text{if } g(x),g(y) \in \{2,3\} \\ 0, & \text{otherwise}. \end{cases}$$

Now, using the above constructions, we show that (S,T) *is a generalized* β*-admissible pair.*

Now, for $g(x) \in X$, $g(y) \in [Sg(x)]_\alpha = [3]$ *with* $\beta(g(x),g(y)) = k \geq 1$, *we have either* $g(x) = 3$, $g(y) = 3$ *or* $g(x) = 2$, $g(y) = 3$. *If* $g(x) = 3, g(y) = 3$, *then* $\beta(g(x),g(z)) \geq 1$ *for all* $g(z) \in [Tg(y)]_\alpha = \{2\}$.
If $g(x) = 3, g(y) = 2$, *then* $\beta(g(y),g(z)) \geq 1$, *for all* $g(z) \in [Tg(y)]_\alpha = \{2\}$.

Next, for $g(x) \in X$ *and* $g(y) \in [Tg(x)]_\alpha$ *with* $\beta(g(x),g(y)) = k \geq 1$, *we get* $g(x) = g(y) = 3$ *so that* $\beta(g(y),g(z)) \geq 1$ *for all* $g(z) \in [Sg(x)]_\alpha$.
Therefore, (S,T) *is a general* β*-admissible pair.*

With direct calculation, we obtain

$$H([Sg(x)]_\alpha,[Tg(y)]_\alpha) = \begin{cases} 1, & \text{if } g(y) = 1 \quad \text{and for all} \quad g(x) \in X \\ 0, & \text{if } g(y) = 2 \quad \text{and for all} \quad g(x) \in X \\ 3, & \text{if } \quad g(y) = 3 \quad \text{and for all } g(x) \in X. \end{cases}$$

Now, letting $p_1 = p_2 = \frac{1}{10}$, $p_3 = p_5 = 0$, $p_4 = \frac{1}{8}$ *and*

$$\varphi(t) = \begin{cases} \frac{\sin(nt)}{t}, & \text{if} \quad t > 0, \quad \text{and } n \in \mathbb{N} \\ n, & \text{if} \quad t = 0, \text{and } n \in \mathbb{N} \end{cases}$$

it is immediate that the contractive condition in (b) *of definition* (3.4) *holds. Also, there exists* $g(x_0) = 2 \in X$ *and* $g(x_1) = 3 \in [Sg(x_0)]_\alpha = \{3\}$ *such that* $\beta(g(x_0),g(x_1)) = k \geq 1$.

In what follows, we provide a common g-fixed point theorem for generalized β-admissible contraction.

Theorem 3.7. *Let (X,d) be a complete metric space, $\varphi \in \psi$ and S,T be L-fuzzy mappings from X into $F_L(X)$ such that (S,T) is a generalized β-admissible contraction. Then S and T have a common g-fixed point in X.*

Proof. We consider the following three possible cases:

(i) $p_1+p_3+p_5=0$;

(ii) $p_2+p_4+p_5=0$;

(iii) $p_1+p_3+p_5\neq 0, \quad p_2+p_4+p_5\neq 0$

Case(i): $p_1+p_3+p_5=0$.

For $g(x_0)\in X$ in condition (b), there exists $\alpha_L(x)\in L\setminus\{0_L\}$ and $g(x_1)\in [Sg(x_0)]_{\alpha L(x_0)}$ such that $\beta(g(x_0),g(x_1))\geq 1$. Also, there exists $\alpha_{L(x_1)}\in L\setminus\{0_L\}$ such that $[Sg(x_0)]_{\alpha_L(x_0)}$ and $[Tg(x_1)]_{\alpha_L(x_1)}$ are nonempty closed and bounded subsets of X. Therefore, from Lemma 2.11, we have

$$\begin{aligned}
d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}}) &\leq H\left([Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right)\\
&\leq \beta(g(x_0),g(x_1))\left[H\left([Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right)\right]\\
&\leq \max\{\beta(g(x_0),g(x_1)),\beta(g(x_1),g(x_0))\}\\
&\quad\times\left[H\left([Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right)\right]+\zeta
\end{aligned}$$

By ineq. (3.18), we have

$$\begin{aligned}
\int_0^{d(g(x_1),[T(g(x_1))]_{\alpha_{L(x_1)}})}\varphi(t)dt &\leq p_1\int_0^{d(g(x_0),[Sg(x_0)]_{\alpha_{L(x_0)}})}\varphi(t)dt\\
&+p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\
&+p_3\int_0^{d(g(x_0),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\
&+p_4\int_0^{d(g(x_1),[Sg(x_0)_{\alpha_{L(x_0)}}])}\varphi(t)dt\\
&+p_5\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\zeta.
\end{aligned}$$

Using $p_1+p_3+p_5=0$, couple with the fact $d(g(x_1),[Sg(x_0)]_{\alpha_{L(x_0)}})=0$, we obtain

$$\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt \leq p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt+\zeta. \tag{3.3}$$

Ineq. (3.3) implies x_1 is a g-fixed point of T and so, $g(x_1)\in[Tg(x_1)]_{\alpha_{L(x_1)}}$. Again, by Lemma 2.11, it follows that

$$d(g(x_1),[Sg(x_1)]_{\alpha_{L(x_1)}})\leq H\left([Tg(x_1)]_{\alpha_{L(x_1)}},[Sg(x_1)]_{\alpha_{L(x_1)}}.\right)$$

By condition (c), for $g(x_0)\in X$ and $g(x_1)\in[Sg(x_0)]_{\alpha_{L(x_0)}}$ such that $\beta(g(x_0),g(x_1))\geq 1$, we have $\beta(g(x_1),g(z))\geq 1$ for all $g(z)\in[Tg(x_1)]_{\alpha_{L(x_1)}}$. Since $g(x_1)\in[Tg(x_1)]_{\alpha_{L(x_1)}}$, it follows $\beta(g(x_1),g(x_1))\geq 1$. Therefore,

$$d(g(x_1),[Sg(x_1)]_{\alpha_{L(x_1)}})\leq\beta(g(x_1),g(x_1))\left[H\left([Sg(x_1)]_{\alpha_{L(x_1)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right)\right]+\zeta.$$

Again, from ineq. 3.18 and above expression, we get

$$\begin{aligned}\int_0^{d(g(x_1),[Sg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt \quad\leq\quad & p_1\int_0^{d(g(x_1),[Sg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\ &+p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\ &+p_3\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\ &+p_4\int_0^{d(g(x_1),[Sg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt\\ &+p_5\int_0^{d(g(x_1),g(x_1))}\varphi(t)dt+\zeta.\end{aligned}$$

Since $p_1+p_3+p_5=0$, and $d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})=0$, the above ineq. becomes

$$\int_0^{d(g(x_1)),[Sg(x_1)]_{\alpha_{L(x_1)}}}\varphi(t)dt\leq\int_0^{d(g(x_1)),[Sg(x_1)]_{\alpha_{L(x_1)}}}\varphi(t)dt+\zeta. \tag{3.4}$$

Ineq. (3.4) implies x_1 is a g-fixed point of S, and so $g(x_1)\in[Sg(x_1)]_{\alpha_{L(x_1)}}$. Hence, $g(x_1)\in[Sg(x_1)]_{\alpha_{L(x_1)}}\cap[Tg(x_1)]_{\alpha_{L(x_1)}}$.

Case (ii): $p_2+p_4+p_5=0$.

For $g(x_0) \in X$ in condition (a), there exists $\alpha_{L(x_0)} \in L \setminus \{0_L\}$ and $g(x_1) \in [Sg(x_0)]_{\alpha_{L(x_0)}}$ such that $\beta(g(x_0), g(x_1)) \geq 1$. Also, there exists $\alpha_{L(x_1)} \in L \setminus \{0_L\}$ such that $[Sg(x_0)]_{\alpha_{L(x_0)}}$ and $[Tg(x_0)]_{\alpha_{L(x_0)}}$ are nonempty closed and bounded subsets of X. By condition (c), we have $\beta(g(x_1), g(x_2)) \geq 1$ for all $x_2 \in X$ and $g(x_2) \in [Tg(x_1)]_{\alpha_{L(x_1)}}$. From Lemma 2.11, it follows that

$$\begin{aligned} d(g(x_2), [Sg(x_2)]_{\alpha_{L(x_2)}}) &\leq H\left([Tg(x_1)]_{\alpha_{L(x_1)}}, [Sg(x_2)]_{\alpha_{L(x_2)}}\right) \\ &\leq \beta\left(g(x_1), g(x_2)\right) H\left([Tg(x_1)]_{\alpha_{L(x_1)}}, [Sg(x_2)]_{\alpha_{L(x_2)}}\right) \\ &\leq \max\left\{\beta\left(g(x_1), g(x_2)\right)\right\} \\ &\quad \times H\left([Tg(x_1)]_{\alpha_{L(x_1)}}, [Sg(x_2)]_{\alpha_{L(x_2)}}\right) + \zeta. \end{aligned} \tag{3.5}$$

From condition (b) and ineq. (3.5),

$$\begin{aligned} \int_0^{d(g(x_1),[Sg(x_2)]_{\alpha_{L(x_1)}})} \varphi(t)dt &\leq p_1 \int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})} \varphi(t)dt \\ &\quad + p_2 \int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})} \varphi(t)dt \\ &\quad + p_3 \int_0^{d(g(x_2),[Tg(x_1)]_{\alpha_{L(x_1)}})} \varphi(t)dt \\ &\quad + p_4 \int_0^{d(g(x_1),[Sg(x_2)]_{\alpha_{L(x_2)}})} \varphi(t)dt \\ &\quad + p_5 \int_0^{d(g(x_2),g(x_1))} \varphi(t)dt + \zeta. \end{aligned}$$

Using $p_2 + p_4 + p_5 = 0$ together with the fact that $d(g(x_2), [Tg(x_1)]_{\alpha_{L(x_1)}}) = 0$, we obtain

$$\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})} \varphi(t)dt \leq p_1 \int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})} \varphi(t)dt + \zeta. \tag{3.6}$$

Ineq. (3.6) implies x_2 is a g-fixed point of S. It follows that $g(x_2) \in [Sg(x_2)]_{\alpha_{L(x_2)}}$. Thus, by Lemma 2.11 ,

$$d(g(x_2), [Tg(x_2)]_{\alpha_{L(x_2)}}) \leq H\left([Sg(x_2)]_{\alpha_{L(x_2)}}, [Tg(x_2)]_{\alpha_{L(x_2)}}\right).$$

By condition (c), $\beta(g(x_2), g(x_2)) \geq 1$. Hence,

$$d(g(x_2), [Tg(x_2)]_{\alpha_{L(x_2)}}) \leq \beta(g(x_2), g(x_2)) H\left([Sg(x_2)]_{\alpha_{L(x_2)}}, [Tg(x_2)]_{\alpha_{L(x_2)}}\right).$$

Again, using ineq.(3.18), we have

$$\begin{aligned}\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt \quad &\leq \quad p_1\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt\\ &+p_2\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt\\ &+p_3\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt\\ &+p_4\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt\\ &+p_5\int_0^{d(g(x_2),g(x_2))}\varphi(t)dt+\zeta.\end{aligned}$$

Since $p_2+p_4+p_5=0$ and $d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})=0$, we have

$$\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt \leq p_3\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt+\zeta. \tag{3.7}$$

Ineq.(3.7) implies x_2 is a g-fixed point of T and hence, $g(x_2)\in[Tg(x_2)]_{\alpha_{L(x_2)}}$. Consequently, $g(x_2)\in[Sg(x_2)]_{\alpha_{L(x_2)}}\cap[Tg(x_2)]_{\alpha_{L(x_2)}}$.

Case (iii): $p_1+p_3+p_5\neq 0, \quad p_2+p_4+p_5\neq 0$.

Let $\tau=\left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right), \quad \theta=\left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right)$, and $\sigma=\max\{\tau,\theta\}$.

Notice that if $\sigma=0$, then $p_1=p_2=p_3=p_4=p_5=0$, and so the proof holds trivially. So assume $\sigma\neq 0$.

Next, we show that if $p_1=p_2$ or $p_3=p_4$, then $0<\tau\theta<1$.

If $p_3=p_4$, then

$$\tau=\left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)=\left(\frac{p_1+p_3+p_5}{1-p_2-p_4}\right)<1,$$

and

$$\theta=\left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right)=\left(\frac{p_2+p_4+p_5}{1-p_1-p_3}<1\right);$$

therefore, $0<\tau\theta<1$.

If $p_1=p_2$, then

$$\begin{aligned} 0<\tau\theta &= \left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right) \\ &= \left(\frac{p_1+p_3+p_5}{1-p_1-p_3}\right)\left(\frac{p_1+p_4+p_5}{1-p_1-p_4}\right) \\ &= \left(\frac{p_1+p_3+p_5}{1-p_1-p_4}\right)\left(\frac{p_1+p_4+p_5}{1-p_1-p_3}\right)<1. \end{aligned}$$

Now, by condition (a), for $g(x_1)\in X$, there exists $\alpha_{L(x_1)}\in L\setminus\{0_L\}$ such that $[Tg(x_1)]_{\alpha_{L(x_1)}}$ is a nonempty closed and bounded subset of X. Since $p_1+p_3+p_5>0$, therefore, by Lemma 2.12, there exists $g(x_2)\in[Tg(x_1)]_{\alpha_{L(x_1)}}$ such that

$$\begin{aligned} d(g(x_1),g(x_2)) &\leq H\left([Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right)+\sigma \\ &\leq \beta(g(x_0),g(x_1))H\left[[Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right]+\sigma(p_1+p_3+p_5) \\ &\leq \max\{\beta(g(x_0),g(x_1),\beta(g(x_1),g(x_0))\} \\ &\quad \times H\left[[Sg(x_0)]_{\alpha_{L(x_0)}},[Tg(x_1)]_{\alpha_{L(x_1)}}\right]+\sigma(p_1+p_3+p_5). \end{aligned}$$

Therefore, by condition (b), it follows that

$$\begin{aligned} \int_0^{d(g(x_1),g(x_2))}\varphi(t)dt &\leq p_1\int_0^{d(g(x_0),[Sg(x_0)]_{\alpha_{L(x_0)}})}\varphi(t)dt+p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt \\ &\quad +p_3\int_0^{d(g(x_0),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt+p_4\int_0^{d(g(x_1),[Sg(x_0)]_{\alpha_{L(x_0)}})}\varphi(t)dt \\ &\quad +p_5\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sigma(p_1+p_3+p_5) \\ &\quad \leq p_1\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+p_2\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt \\ &\quad +p_3\int_0^{d(g(x_0),g(x_2))}\varphi(t)dt+p_4\int_0^{d(g(x_1),g(x_1))}\varphi(t)dt \\ &\leq (p_1+p_5)\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+p_2\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt \\ &\quad +p_5\int_0^{d(g(x_0),g(x_2))}\varphi(t)dt+\sigma(p_1+p_3+p_5). \end{aligned}$$

Using the subadditivity of the integral, we get

$$\begin{aligned}\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt &\leq (p_1+p_5)\int_0^{d(g(x_0,g(x_1)))}\varphi(t)dt+p_2\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt\\ &\quad +p_3\left[\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt\right]+\sigma(p_1+p_3+p_5)\\ &\leq (p_1+p_3+p_5)\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt\\ &\quad +(p_2+p_3)\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma(p_1+p_3+p_5).\end{aligned}$$

From which we have

$$\begin{aligned}\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt &\leq \left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sigma\left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\\ &\leq \sigma\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sigma^2. \end{aligned}\tag{3.8}$$

On similar arguments, for $g(x_2)\in X$, there exists $\alpha_{L(x_2)}\in L\setminus\{0_L\}$ such that $[Sg(x_2)]_{\alpha_{L(x_2)}}$ is a nonempty closed and bounded subset of X. Since $p_2+p_4+p_5>0$, therefore, by Lemma 2.12, we can find $g(x_3)\in[Sg(x_2)]_{\alpha_{L(x_2)}}$ such that

$$d(g(x_2),g(x_3))\leq H\left([Tg(x_1)]_{\alpha_{L(x_1)}},[Sg(x_2)]_{\alpha_{L(x_2)}}\right)+\sigma^2(p_2+p_4+p_5).\tag{3.9}$$

By condition (c), for $g(x_0)\in X$ and $g(x_1)\in[Sg(x_0)]_{\alpha_{L(x_1)}}$ such that $\beta(g(x_0),g(x_1))\geq 1$, we have $\beta(g(x_1),g(x_2))\geq 1$ for $g(x_2)\in[Tg(x_1)]_{\alpha_{L(x_1)}}$. Therefore, (3.9) becomes

$$\begin{aligned}d(g(x_2),g(x_3)) &\leq H\left([Tg(x_1)]_{\alpha_{L(x_1)}},[Sg(x_2)]_{\alpha_{L(x_2)}}\right)+\sigma^2(p_2+p_4+p_5).\\ &\leq \beta(g(x_1),g(x_2))H\left([Tg(x_1)]_{\alpha_{L(x_1)}},[Sg(x_2)]_{\alpha_{L(x_2)}}\right)+\sigma^2(p_2+p_4+p_5)\\ &\leq \max\{\beta(g(x_1),g(x_2)),\beta(g(x_2),g(x_1))\}\\ &\quad\times H\left([Tg(x_1)]_{\alpha_{L(x_1)}},[Sg(x_2)]_{\alpha_{L(x_2)}}\right)+\sigma^2(p_2+p_4+p_5).\end{aligned}$$

Therefore, by condition (b), we have

$$\begin{aligned}
\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt &\le p_1\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt+p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\\
&\quad +p_3\int_0^{d(g(x_2),[Tg(x_1)]_{\alpha_{L(x_1)}})}\varphi(t)dt+p_4\int_0^{d(g(x_1),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_2),g(x_1))}\varphi(t)dt+\sigma^2(p_2+p_4+p_5).\\
&\le p_1\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+p_2\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(x_2),g(x_2))}\varphi(t)dt+p_4\int_0^{d(g(x_1),g(x_3))}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^2(p_2+p_4+p_5).\\
&= p_1\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt+p_2\int_0^{d(g(x_1),[Tg(x_1)]_{\alpha_{L(x_1)}})}\\
&\quad +p_4\int_0^{d(g(x_1),g(x_3))}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^2(p_2+p_4+p_5).
\end{aligned}$$

Using the subadditivity of the integral, we have

$$\begin{aligned}
\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt &\le p_1\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+p_2\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt\\
&\quad +p_4\left[\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt\right]\\
&\quad +p_5\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^2(p_2+p_4+p_5)\\
&\le (p_2+p_4+p_5)\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt\\
&\quad +(p_1+p_4)\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\sigma^2(p_2+p_4+p_5).
\end{aligned}$$

Simplifying, we get

$$\begin{aligned}
\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt &\le \left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right)\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^2\left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right)\\
&\le \sigma\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^3. \qquad (3.10)
\end{aligned}$$

By repeating the above steps, for $g(x_3) \in X$, there exists $\alpha_{L(x_3)} \in L\{0_L\}$ such that $[Tg(x_3)]_{\alpha_{L(x_3)}}$ is a nonempty closed and bounded subset of X. Hence, by lemma 2.12, we can find $g(x_4) \in [Tg(x_3)]_{\alpha_{L(x_3)}}$ such that

$$d(g(x_3),g(x_4)) \leq H\left([Sg(x_2)]_{\alpha_{L(x_2)}},[Tg(x_3)]_{\alpha_{L(x_3)}}\right)+\sigma^3(p_1+p_3+p_5).$$

By condition (c), for $g(x_1) \in X$ and $g(x_2) \in [Tg(x_1)]_{\alpha_{L(x_1)}}$ such that $\beta(g(x_1),g(x_2)) \geq 1$, we have $\beta(g(x_2),g(x_3)) \geq 1$ for $g(x_3) \in [Sg(x_2)]_{\alpha_{L(x_2)}}$. Thus,

$$\begin{aligned}
d(g(x_3),g(x_4)) &\leq H\left([Sg(x_2)]_{\alpha_{L(x_2)}},[Tg(x_3)]_{\alpha_{L(x_3)}}\right)+\sigma^3(p_1+p_3+p_5)\\
&\leq \beta(g(x_2),g(x_3))H\left([Sg(x_2)]_{\alpha_{L(x_2)}},[Tg(x_3)]_{\alpha_{L(x_3)}}\right)+\sigma^3(p_1+p_3+p_5)\\
&\leq \max\{\beta(g(x_2),g(x_3)),\beta(g(x_3),g(x_2))\}\\
&\quad \times H\left([Sg(x_2)]_{\alpha_{L(x_2)}},[Tg(x_3)]_{\alpha_{L(x_3)}}\right)+\sigma^3(p_1+p_3+p_5).
\end{aligned}$$

Therefore, conditon (c) yields

$$\begin{aligned}
\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt &\leq p_1\int_0^{d(g(x_2),[Sg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt+p_2\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(x_2),[Tg(x_2)]_{\alpha_{L(x_2)}})}\varphi(t)dt+p_4\int_0^{d(g(x_3),[Sg(x_2)])}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\sigma^3(p_1+p_3+p_5).\\
&\leq p_1\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+p_2\int_0^{d(g(x_3),g(x_4))}\varphi\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(x_2),g(x_4))}\varphi(t)dt+p_4\int_0^{d(g(x_3),g(x_3))}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\sigma^3(p_1+p_3+p_5).\\
&= p_1\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+p_2\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(x_2),g(x_4))}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\sigma^3(p_1+p_3+p_5).
\end{aligned}$$

Using the subadditivity of the integral, we have

$$\begin{aligned}\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt &\leq p_1\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+p_2\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt\\ &\quad +p_3\left[\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt\right]\\ &\quad +p_5\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt+\sigma^3(p_1+p_3+p_5).\\ &= (p_1+p_3+p_5)\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt\\ &\quad +(p_2+p_3)\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt+\sigma^3(p_1+p_3+p_5).\end{aligned}$$

The above expression gives

$$\begin{aligned}\int_0^{d(g(x_3),g(x_4))}\varphi(t)dt &\leq \left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\int_0^{d(g(x_2),g(x_3)}\varphi(t)dt+\sigma^3\left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\\ &\leq \sigma\int_0^{d(g(x_2),g(x_3)}\varphi(t)dt+\sigma^4. \qquad (3.11)\end{aligned}$$

Continuing this process inductively, we can find a sequence $\{x_n\}$ in X such that

$$x_{2k+1}\in[Sg(x_{2k})]_{\alpha_{L(x_{2k})}},\quad x_{2k+2}\in[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}},\quad k\in\mathbb{N},$$

and $\beta(g(x_{n-1}),g(x_n))\geq 1$, for all $n\in\mathbb{N}$. Therefore, we have

$$\begin{aligned}d(g(x_{2k+1}),g(x_{2k+2})) &\leq H\left([Sg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right)+\sigma^{2k+1}(p_1+p_3+p_5)\\ &\leq \beta(g(x_{2k}),g(x_{2k+1}))\\ &\quad\times H\left([Sg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right)+\sigma^{2k+1}(p_1+p_3+p_5)\\ &\leq \max\{\beta(g(x_{2k}),g(x_{2k+1})),\beta(g(x_{2k+1}),g(x_{2k}))\}\\ &\quad\times H\left([Sg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right)+\sigma^{2k+1}(p_1+p_3+p_5)\end{aligned}$$

Hence, by condition (c), we get

$$\begin{aligned}
\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt \quad &\leq \quad p_1\int_0^{d(g(x_{2k}),[Sg(x_{2k})]_{\alpha_{L(x_{2k})}})}\varphi(t)dt \\
&\quad +p_2\int_0^{d(g(x_{2k+1}),[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}})}\varphi(t)dt \\
&\quad +p_3\int_0^{d(g(x_{2k}),[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}})}\varphi(t)dt \\
&\quad +p_4\int_0^{d(g(x_{2k+1}),[Sg(x_{2k})]_{\alpha_{L(x_{2k})}})}\varphi(t)dt \\
&\quad +p_5\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+\sigma^{2k+1}(p_1+p_3+p_5) \\
&\leq \quad p_1\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+p_2\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt \\
&\quad +p_3\int_0^{d(g(x_{2k}),g(x_{2k+2}))}\varphi(t)dt+p_4\int_0^{d(g(x_{2k+1}),g(x_{2k+1}))}\varphi(t)dt \\
&\quad +p_5\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+\sigma^{2k+1}(p_1+p_3+p_5) \\
&= \quad p_1\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+p_2\int_0^{d(g(x_{2k+1}),g(g(x_{2k+2})))}\varphi(t)dt \\
&\quad +p_3\int_0^{d(g(x_{2k}),g(x_{2k+2}))}\varphi(t)dt \\
&\quad +p_5\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+\sigma^{2k+1}(p_1+p_3+p_5)
\end{aligned}$$

The subadditivity of the integral yields

$$\begin{aligned}
\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt \quad &\leq \quad p_1\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+p_2\int_0^{d(g(x_{2k+1}),g(x_{2k+2})}\varphi(t)dt \\
&\quad +p_3\left[\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt\right] \\
&\quad +p_5\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt+\sigma^{2k+1}(p_1+p_3+p_5) \\
&\leq \quad (p_1+p_3+p_5)\int_0^{d(g(x_{2k}),g(x_{2k+1}))}\varphi(t)dt \\
&\quad +(p_2+p_3)\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt+\sigma^{2k+1}(p_1+p_3+p_5).
\end{aligned}$$

Factorizing the above inequality, gives

$$\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt \leq \left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right)\int_0^{d(g(x_{2k}),g(x_2k+1))}\varphi(t)dt + \sigma^{2k+1}\left(\frac{p_1+p_3+p_5}{1-p_2-p_3}\right) \tag{3.12}$$

$$\leq \sigma\int_0^{d(g(x_{2k}),g(x_2k+1))}\varphi(t)dt+\sigma^{2k+2}. \tag{3.13}$$

Similarly, we get

$$\begin{aligned} d(g(x_{2k+2}),g(x_{2k+3})) &\leq H\left([Sg(x_{2k+2})]_{\alpha_{L(x_{2k+2})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right) \\ &\quad +\sigma^{2k+2}(p_2+p_4+p_5) \\ &\leq \beta(g(x_{2k+1}),g(x_{2k+2})) \\ &\quad \times H\left([Sg(x_{2k+2})]_{\alpha_{L(x_{2k+2})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right) \\ &\quad +\sigma^{2k+2}(p_2+p_4+p_5) \\ &\leq \max\{\beta(g(x_{2k+1}),g(x_{2k+2})),\beta(g(x_{2k+2}),g(x_{2k+1}))\} \\ &\quad \times H\left([Sg(x_{2k+2})]_{\alpha_{L(x_{2k+2})}},[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}}\right) \\ &\quad +\sigma^{2k+2}(p_2+p_4+p_5). \end{aligned}$$

By condition (b), we have

$$\begin{aligned} \int_0^{d(g(x_{2k+2}),g(x_{2k+3}))}\varphi(t)dt &\leq p_1\int_0^{d(g(x_{2k+2}),[Sg(x_{2k+2})]_{\alpha_{L(x_{2k+2})}})}\varphi(t)dt \\ &\quad +p_2\int_0^{d(g(x_{2k+1}),[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}})}\varphi(t)dt \\ &\quad +p_3\int_0^{d(g(x_{2k+2}),[Tg(x_{2k+1})]_{\alpha_{L(x_{2k+1})}})}\varphi(t)dt \\ &\quad +p_4\int_0^{d(g(x_{2k+1}),[Sg(x_{2k+2})]_{\alpha_{L(x_{2k+2})}})}\varphi(t)dt \\ &\quad +p_5\int_0^{d(g(x_{2k+2}),g(x_{2k+1}))}\varphi(t)dt+\sigma^{2k+2}(p_2+p_4+p_5). \\ &\leq p_1\int_0^{d(g(x_{2k+2}),g(x_{2k+3}))}\varphi(t)dt+p_2\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))}\varphi(t)dt \end{aligned}$$

$$
\begin{aligned}
& +p_3 \int_0^{d(g(x_{2k+2}),g(x_{2k+2}))} \varphi(t)dt + p_4 \int_0^{d(g(x_{2k+1}),g(x_{2k+3}))} \varphi(t)dt \\
& +p_5 \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+2}(p_2+p_4+p_5). \\
= \; & p_1 \int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt + p_2 \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt \\
& +p_4 \int_0^{d(g(x_{2k+1}),g(x_{2k+3}))} \varphi(t)dt \\
& +p_5 \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+2}(p_2+p_4+p_5).
\end{aligned}
$$

Since the integral is subadditive, we have

$$
\begin{aligned}
\int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt \;\leq\; & p_1 \int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt + p_2 \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt \\
& +p_4 \left[\int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt\right] \\
& +p_5 \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+2}(p_2+p_4+p_5). \\
\leq\; & (p_1+p_4) \int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt \\
& +(p_2+p_4+p_5) \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+2}(p_2+p_4+p_5).
\end{aligned}
$$

Simplifying the above expression, results in

$$
\begin{aligned}
\int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt \;\leq\; & \left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right) \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt \\
& +\sigma^{2k+2}\left(\frac{p_2+p_4+p_5}{1-p_1-p_4}\right) \\
\leq\; & \sigma \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+3}. \qquad (3.14)
\end{aligned}
$$

Now, from (3.14) and (3.12), we obtain

$$
\begin{aligned}
\int_0^{d(g(x_{2k+2}),g(x_{2k+3}))} \varphi(t)dt \;\leq\; & \sigma \int_0^{d(g(x_{2k+1}),g(x_{2k+2}))} \varphi(t)dt + \sigma^{2k+3} \\
\leq\; & \sigma\left[\sigma \int_0^{d(g(x_{2k}),g(x_{2k+1}))} \varphi(t)dt + \sigma^{2k+2}\right] + \sigma^{2k+3} \\
\leq\; & \sigma^2 \int_0^{d(g(x_{2k}),g(x_{2k+1}))} \varphi(t)dt + 2\sigma^{2k+3} \\
\leq\; & \sigma^2 \left[\sigma \int_0^{d(g(x_{2k-1}),g(x_{2k}))} \varphi(t)dt + \sigma^{2k+1}\right] + 2\sigma^{2k+3} \\
\leq\; & \sigma^3 \int_0^{d(g(x_{2k-1}),g(x_{2k}))} \varphi(t) + 3\sigma^{2k+3}. \qquad (3.15)
\end{aligned}
$$

Similarly, from ineqs. (3.10) and and (3.8), we have

$$\begin{aligned}\int_0^{d(g(x_2),g(x_3))}\varphi(t)dt &\leq \sigma\int_0^{d(g(x_1),g(x_2))}\varphi(t)dt+\sigma^3\\ &\leq \sigma\left[\sigma\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sigma^2\right]+\sigma^3\\ &\leq \sigma^2\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+2\sigma^3. \qquad (3.16)\end{aligned}$$

Therefore, from (3.15) and (3.16), we see that for $n1,2,3,\cdots$,

$$\int_0^{d(g(x_n),g(x_{n+1}))}\varphi(t)dt \leq \sigma^n\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+n\sigma^{n+1}. \qquad (3.17)$$

Thus, for each positive integer $m,n(n>m)$, it follows that

$$\begin{aligned}\int_0^{d(g(x_m),g(x_n))}\varphi(t)dt &\leq \int_0^{d(g(x_m),g(x_{m+1}))}\varphi(t)dt+\int_0^{d(g(x_{m+1}),g(x_{m+2}))}\varphi(t)dt\\ &\quad+\cdots+\int_0^{d(x_{n-1},g(x_n))}\varphi(t)dt\\ &\leq \sigma^m\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+m\sigma^{m+1}\\ &\quad+\sigma^{m+1}\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+(m+1)\sigma^{m+2}\\ &\quad+\cdots+\sigma^{n-1}\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+(n-1)\sigma^n\\ &\leq (\sigma^m+\sigma^{m+1}+\cdots+\sigma^{n-1})\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt\\ &\quad+(m\sigma^{m+1}+(m+1)\sigma^{m+2}+\cdots+(n-1)\sigma^n)\\ &\leq \sum_{i=m}^{n-1}\sigma^i\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sum_{i=m}^{n-1}i\sigma^{i+1}\\ &\leq \frac{\sigma^m}{1-\sigma}\int_0^{d(g(x_0),g(x_1))}\varphi(t)dt+\sum_{i=m}^{n-1}i\sigma^{i+1}.\end{aligned}$$

Observe that $(u_n)^{\frac{1}{n}}=(n\sigma^{n+1})^{\frac{1}{n}}=\sigma<1$ as $n\longrightarrow\infty$. Hence, by Cauchy's root test, $\sum_{i=m}^{n-1}i\sigma^{i+1}$ is convergent. Therefore, $\{g(x_n)\}$ is a Cauchy sequence of elements of X. Since X is complete, there exists $g(z)\in X$ for some $z\in X$ such that $g(x_n)\longrightarrow g(z)$ as $n\longrightarrow$. By condition (d), $\beta(g(x_{n-1}),g(z))\geq 1$, $\forall n\in\mathbb{N}$.

Now, consider

$$\begin{aligned}
d(g(x_n),[Sg(z)]_{\alpha_{L(z)}}) &\leq H\left([Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}},[Sg(z)]_{\alpha_{L(z)}}\right)\\
&\leq \beta(g(x_{n-1}),g(z))\\
&\quad \times H\left([Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}},[Sg(z)]_{\alpha_{L(z)}}\right)+\sigma^n\\
&\leq \max\{\beta(g(x_{n-1}),g(z)),\beta(g(z),g(x_{n-1})\}\\
&\quad \times H\left([Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}},[Sg(z)]_{\alpha_{L(z)}}\right)+\sigma^n.
\end{aligned}$$

Therefore, condition (a) gives

$$\begin{aligned}
\int_0^{d(g(x_n),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt &\leq p_1\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt+p_2\int_0^{d(g(x_{n-1}),[Tg(x_{n-1})])}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(z),[Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}})}\varphi(t)dt+p_4\int_0^{d(g(x_{n-1}),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt+\sigma^n.
\end{aligned}$$

Since

$$d(g(z),[Sg(z)]_{\alpha_{L(z)}})\leq d(g(z),g(x_n))+d(g(x_n),[Sg(z)]_{\alpha_{L(z)}}),$$

we have

$$\begin{aligned}
\int_0^{g(z),[Sg(z)]_{\alpha_{L(z)}}}\varphi(t)dt &\leq \int_0^{d(g(z),g(x_n))}\varphi(t)dt+\int_0^{d(g(x_n),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\leq \int_0^{d(g(z),g(x_n))}\varphi(t)d+p_1\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_2\int_0^{d(g(x_{n-1}),[Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}})}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(z),[Tg(x_{n-1})]_{\alpha_{L(x_{n-1})}})}\varphi(t)dt+p_4\int_0^{g(x_{n-1},[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt+\sigma^n
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{d(g(z),g(x_n))}\varphi(t)dt + p_1\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_3\int_0^{d(g(z),g(x_n))}\varphi(t)dt + p_4\int_0^{d(g(x_{n-1}),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_5\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt+\sigma^n\\
&\leq (1+p_3)\int_0^{d(g(z),g(x_n))}\varphi(t)dt + p_1\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\\
&\quad +p_2\int_0^{d(g(x_{n-1}),g(x_n)}\varphi(t)dt\\
&\quad +p_4\left[\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt+\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt\right]\\
&\quad +p_5\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt+\sigma^n\\
&\leq (1+p_3)\int_0^{d(g(z),g(x_n))}\varphi(t)dt+(p_4+p_5)\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt\\
&\quad +p_2\int_0^{d(g(x_{n-1}),g(x_n))}\varphi(t)dt+(p_1+p_4)\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt+\sigma^n.
\end{aligned}
$$

Factorizing the above expression, produces

$$
\begin{aligned}
\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt \;\leq\; &\left(\frac{1+p_3}{1-p_1-p_4}\right)\int_0^{d(g(z),g(x_n))}\varphi(t)dt\\
&+\left(\frac{p_4+p_5}{1-p_1-p_4}\right)\int_0^{d(g(z),g(x_{n-1}))}\varphi(t)dt\\
&+\left(\frac{p_2}{1-p_1-p_4}\right)\int_0^{d(g(x_{n-1}),g(x_n))}\varphi(t)dt+\frac{\sigma^n}{1-p_1-p_4}.
\end{aligned}
$$

As $n\longrightarrow\infty$, we have $\int_0^{d(g(z),[Sg(z)]_{\alpha_{L(z)}})}\varphi(t)dt\leq 0$. Hence, z is a g-fixed point of S. This implies $g(z)\in[Sg(z)]_{\alpha_{L(z)}}$.

On similar steps, one can show that $g(z)\in[Tg(z)]_{\alpha_{L(z)}}$, by using

$$
d(g(z),[Tg(z)]_{\alpha_{L(z)}})\leq d(g(z),g(x_{n_1}))+d(g(x_n),[Tg(z)]_{\alpha_{L(z)}}).
$$

Consequently, $z\in X$ is a common g-fixed point of S and T,

which means $g(z)\in[Sg(z)]_{\alpha_{L(z)}}\cap[Tg(z)]_{\alpha_{L(z)}}$. □

Remark 3.8. *If we set $g \equiv I \equiv \varphi$, the identity mapping, then using Theorem 3.7, all the results of* [18] *are obtained as corollaries.*

Example 3.9. *In continuation of Example 3.6, if $g(x_n)$ is a sequence in X and $g(x) \in X$ such that $\beta(g(x_n), g(x)) \geq 1$, for all $n \in \mathbb{N}$, then $g(x_n) \in \{2,3\}$ for all $n \in \mathbb{N}$, which implies $g(x) \in \{2,3\}$. Therefore, $\beta(g(x_n), g(x)) \geq 1$ for all $n \in \mathbb{N}$. Thus, all the hypothesis of Theorem 3.7 are satisfied to have $g(3) \in X$ such that*

$$g(3) \in [Sg(3)]_{\alpha_{L(3)}} \cap [Tg(3)]_{\alpha_{L(3)}} = \{3\}.$$

This means $3 \in X$ is a common g-fixed point of S and T; which also implies that $g(3) \in X$ is a common coincidence point of S and T.

Since every fuzzy mapping is an L-fuzzy mapping, we deduce the following corollary.

Corollary 3.10. *Let (X,d) be a complete metric space, $\beta : X \times X \longrightarrow [0,\infty)$, $\varphi \in \psi$ and $S,T : X \longrightarrow I^X$ be a pair of fuzzy mappings. If the pair (S,T) is a generalized β-admissible contraction, then S and T have a common g-fixed point in X.*

As a direct consequence of Theorem 3.7, we have the next result.

Theorem 3.11. *Let (X,d) be a complete metric space, $\beta : X \times X \longrightarrow [0,\infty)$, $\varphi \in \psi$ and $g : X \longrightarrow X$ be an arbitrary function. Also, let $S,T : X \longrightarrow CB(Y)$ be a pair of multi-valued mappings such that*

(a) *For each $g(x_0) \in X$, there exists $g(x_1) \in Sg(x_0)$ such that $\beta(g(x_0), g(x_1)) \geq 1$,*

(b) *For all $g(x), g(y) \in X$, we have*

$$\begin{aligned}\int_0^{\delta} \varphi(t)dt \quad \leq \quad & p_1 \int_0^{d(g(x),Sg(x))} \varphi(t)dt + p_2 \int_0^{d(g(y),Tg(y))} \varphi(t)dt \\ & + p_3 \int_0^{d(g(x),Tg(y))} \varphi(t)dt + p_4 \int_0^{d(g(y),Sg(x))} \varphi(t)dt \\ & p_5 \int_0^{d(g(x),g(y))} \varphi(t)dt + \zeta, \end{aligned} \tag{3.18}$$

where

$\delta = \max\{\beta(g(x),g(y)), \beta(g(y),g(x))\} H(Sg(x),Tg(y)) + \zeta$; $p_j (1 \leq j \leq 5)$ are nonnegative reals satisfying $\sum_{j=1}^{5} p_j < 1$ and either $p_1 = p_2$ or $p_3 = p_4$,

(c) (S,T) *is a generalized* β*-admissible pair,*

(d) *if* $\{g(x_n)\}_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $\beta(g(x_n),g(x_{n+1}))\geq 1$*, and* $g(x_n)\longrightarrow g(x)(n\longrightarrow\infty)$*, then* $\beta(g(x_n),g(x))\geq 1$.

Then S *and* T *have a common* g*-fixed point in* X.

Proof. Define $A,B : X\longrightarrow I^X$ by

$$A(g(x))=\chi_{Sg(x)},\quad B(g(x))=\chi_{Tg(x)}.$$

Then for $\alpha\in(0,1]$,

$$\begin{aligned}
[Ag(x)]_\alpha &= \{t : A(g(x))(t)\geq\alpha\}\\
&= \{t : \chi_{Sg(x)}\geq\alpha\}\\
&= \{t : \chi_{Sg(x)}=1\}\\
&= \{t : t\in Sg(x)\}=Sg(x).
\end{aligned}$$

On similar steps, $[Bg(x)]_\alpha=Tg(x)$. Therefore,

$$H\left([Ag(x)]_\alpha,[Bg(x)]_\alpha\right)=H\left(Sg(x),Tg(y)\right).$$

It follows consequently from Cor. 3.10 that S and T have a common g-fixed point in X. □

CONCLUSION

In this article, the concepts of generalized β-admissible contraction and g-fixed point theorems are established. These ideas are used to obtain a common g-fixed point of a pair of L-fuzzy mappings. we provided a few examples to support the validity of our concepts. Consequently, our derivation improves some known results existing in the field of fuzzy fixed point theory.

Conflict of Interests

The authors declare that there is no conflict of interests.

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