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APPLICATION OF WEAKLY COMPATIBLE MAPPINGS TO COMMON FIXED POINT THEOREMS IN MENGER SPACES

DEEPAK SINGH[1,*], AMIN AHMED[2], M. S. RATHORE[3] AND SURJEET S. TOMAR[4]

[1]Department of Mathematics, Technocrat Institute of Technology, Bhopal, (M.P.), India, 462021.

[2]University Inst. of Tech., Rajiv Gandhi Technical University, Bhopal, (M.P.), India, 462036.

[3]Chandra Shekhar Azad Govt. P.G. College, Sehore (M.P.), India

[4]Gyanganga Institute of Tech. and Manag.,Bhopal, (M.P.), India

Abstract. In this study, coincidence and common fixed point results are presented for two pairs of self maps without continuity under relatively weaker commutativity requirement in Menger spaces. Our results partially extend and improve several known results in Menger spaces. In addition we obtained some related results and furnishing an illustrative example.

Keywords: Menger PM-spaces,coincidence and common fixed point, compatible maps and weakly compatible maps.

2000 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Banach contraction mapping principle is an important result of modern analysis. This principle has been extended and generalized in different directions in metric spaces. The theory of probabilistic metric spaces was introduced in 1942 by Menger [7]. The idea was

*Corresponding author

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to use distribution functions instead of non-negative real numbers as values of the metric. Thus probabilistic metric spaces have notions of uncertainty built within the structure of the space and hence provides a natural framework for the study of quantum mechanical phenomena. It is pertinent to mention at this point that these notions are very important in the context of quantum particle physics, especially in relations with both string and E-infinity theories which have been extensively explored by El Naschie some of which are noted in [9-13]. for instance, in order to analyse the probability involved in the two-slit experiment can be modelled in terms of a probabilistic metric.

Recently the study of fixed point theorems in probabilistic metric spaces is also a topic of recent interest and forms an active direction of research. The first ever effort in this direction appears to be made by Sehgal [20]. since then several authors have already studied fixed point and common fixed point theorems in PM spaces which include [1,3-6,14,15,17,18]. and others have recently initiated work along these lines. Our results partially extend and improve several known results Rashwan and Hedar [15] and Mishra [17] .We cite Cain and Kasreil [5],Sherwood[6],Imdad et. al [8] and Sehgal and Bharucha-Reid [19] and others whose contributions are relevant to the representation of this paper.

2. Preliminaries

Definition 2.1. [2] *A mapping $\mathcal{F} : \Re \to \Re^+$ is called distribution function if it is non-decreasing, left continuous with*

$$\inf\{F(t) : t \in \Re^+\} = 0 \quad \textit{and} \quad \sup\{F(t) : t \in \Re^+\} = 1.$$

Let L be the set of all distribution functions whereas H stands for the specific distribution function (also known as Heaviside function) defined by

$$H(x) = \begin{cases} 0\ ; & x \leq 0 \\ 1\ ; & x > 0 \end{cases}$$

Definition 2.2. [2] *Let X be a non-empty set. An ordered pair $(X, \mathcal{F})$ is called a PM space where $\mathcal{F}$ is a mapping from $X \times X$ into L satisfying the following conditions:*

(i) $F_{x,y}(t) = H(x)$ *if and only if* $x = y$;

(ii) $F_{x,y}(t) = F_{y,x}(t)$;

(iii) $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$, *then* $F_{x,y}(t+s) = 1$, *for all* $x, y, z \in X$ *and* $t, s \geq 0$.

Every metric space(X, d) can always be realized as a PM space by considering $\mathcal{F} : X \times X \to L$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all$x, y \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 2.3. [2] *A mapping* $\Delta : [0, 1] \to [0, 1] \to [0, 1]$ *is called a t-norm if*

(i) $\Delta(a, 1) = a, \Delta(0, 0) = 0$;

(ii) $\Delta(a, b) = \Delta(b, a)$;

(iii) $\Delta(c, d) \geq \Delta(b, a) \quad for \quad c \geq a, d \geq b$;

(iv) $\Delta(\Delta(a, b)c) = \Delta(a, \Delta(b, c)) \quad for \quad all \quad a, b, c \in [0, 1]$.

Remark 2.1. *The following are the four basic t-norms:*

(i) *The minimum t-norm:* $T_M(a, b) = \min\{a, b\}$.

(ii) *The product t-norm:* $T_P(a, b) = a.b$.

(iii) *The Lukasiewicz t-norm:* $T_L(a, b) = \max\{a + b - 1, 0\}$.

(iv) *The weakest t-norm, the drastic product:*

$$H(x) = \begin{cases} \min(a, b) & if \ \max(a, b) = 1 \\ 0 & otherwise \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering :

$T_D < T_L < T_P < T_M$ Throughout this paper, Δ stands for an arbitrary continuous t-norm.

Definition 2.4. [7] *A Menger PM space*$(X, \mathcal{F}, \Delta)$ *is a triplet where* $(X, \mathcal{F})$ *is a PM space and* Δ *is a t-norm satisfying the following condition*

$$F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), (F_{y,z}(s)).$$

Definition 2.5. [2] *A sequence* $\{x_n\}$ *in a Menger PM space* $(X, \mathcal{F}, \Delta)$*is said to converge to a point* x *in* X *if for every* $\in > 0$ *and* $\lambda > 0$*, there is an integer*$M(\in, \lambda)$ *such that* $F_{x_n,x}(\in) > 1 - \lambda$*, for all* $n \geq M(\in, \lambda)$*.*

Definition 2.6. [2] *A sequence* $\{x_n\}$ *in a Menger PM space* $(X, \mathcal{F}, \Delta)$*is said to cauchy if for each* $\epsilon > 0$ *and* $\lambda > 0$*, there is an integer* $M(\epsilon, \lambda)$ *such that* $F_{x_n,x_m}(\epsilon) > 1 - \lambda$*, for all* $n, m \geq M(\epsilon, \lambda)$*.*

Definition 2.7. [2] *A Menger PM space* $(X, \mathcal{F}, \Delta)$*is said to be complete if every Cauchy sequence in it converges to a point of it.*

Lemma 2.1. [4] *Let* $(X, \mathcal{F}, \Delta)$ *be Menger PM space and* $E_{\lambda,\mu} : X \times X \to R^+ \bigcup (0)$ *by* $E_{\lambda,\mu}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\}$ *for each* $\lambda \in (0, 1)$ *and* $x, y \in X$*. then we have*

(*i*) *for any* $\mu \in (0, 1)$ *there exists* $\lambda \in (0, 1)$ *such that*

$$E_{\mu,F}(x_1, x_n) \leq E_{\lambda,F}(x_1, x_2) + \cdots + E_{\lambda,F}(x_{n-1}, x_n) \quad \textit{for all} \;\; x_1, x_2, \cdots, x_n \in X.$$

(*ii*) *The sequence* x_n *is convergent w.r. to Menger PM* f *if and only if* $E_{\mu,F}(x_n, x) \to 0$*. Also the sequence* $\{x_n\}$ *is a Cauchy sequence w.r. to Menger PM spaces* F *if and only if it a Cauchy sequence with* $E_{\lambda,F}$*.*

Lemma 2.2. [14] *A function* $\phi : [0, \infty) \to [0, \infty)$ *is said to satisfy the condition* $(\star)$: *if* ϕ *is nondecreasing and* $\sum_{n=1}^{n} \phi^n(t) < \infty$ *for all* $t > 0$*, where* $\phi^n(t)$ *denotes the* n^th *iterate of* $\phi(t)$*, then* $\phi(t) < t$ *for all* $t > 0$*.*

Lemma 2.3. [4] *Let* $(X, \mathcal{F}, \Delta)$ *be a Menger PM space. suppose that* $x_n \subseteq X$ *is such that* $F_{x_n,x_{n+1}}(\phi^n(t)) \geq F_{x_0,x_1}(t)$ *for all* $t > 0$*, where the function* $\phi : [0, \infty) \to [0, \infty)$ *is onto, strictly increasing and satisfy condition* $(\star)$*. Also assume*

$$E_F(x_0, x_1) = \sup\{E_{\omega,F}(x_0, x_1) : \omega \in (0, 1)\} < \infty.$$

then $\{x_n\}$ *is a Cauchy sequence.*

Lemma 2.4. [4] *If* $(X, \mathcal{F}, \Delta)$ *is a Menger PM space and* $F_{x,y}(t) = C$*, for all* $t > 0$*, then* $C = H(t)$ *and* $x = y$*.*

Lemma 2.5. [4] *Let $(X, \mathcal{F}, \Delta)$ be a Menger PM space. suppose that the function*

$$\phi : [0, \infty) \to [0, \infty)$$

is onto and strictly increasing, then

$$\inf\{\phi^n(s) > 0 : F_{x,y}(s) > 1 - \lambda\} \leq \phi^n(\inf\{s > 0 : F_{x,y}(s) > 1 - \lambda\})$$

for every $x, y \in X$, $\lambda \in (o, 1)$. and $n = 1, 2, 3 \ldots$

Definition 2.8. [17] *A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be weakly commuting if $F_{SAx,ASx}(t) \geq F_{Ax,Sx}(t)$ for all $x \in X$ and $t > 0$.*

Definition 2.9. [18] *A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be compatible if $F_{SAx_n,ASx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to u$ for some u in X as $n \to \infty$.*

Definition 2.10. *A pair (A, S) of self mappings of a nonempty set X is said to be weakly compatible or coincidentally commuting if the mappings commute at their coincidence points, i.e. $Ax = Sx$ for some $x \in X$ implies$ASx = SAx$.*

O'Regan and Saadati [4] proved the following result

Theorem 2.1. *Let A, B, L, M, S and T be self mappings on complete Menger space $(X, \mathcal{F}, \Delta)$ satisfy condition (i) of Theorem (2.1) suppose that*

(i) $LS = SL, MT = TM, AS = SA, BT = TB$,

(ii) either LS or A is continuous,

(iii) the pair (A, LS) is compatible and (B, MT) is weakly compatible,

(iv)

$$F_{Ap,Bq}(\phi(x)) \geq \min\{F_{LSp,Ap}(x), F_{MTq,Bq}(x), F_{MTq,Ap}(\beta x),$$

$$F_{LSp,Bq}((2 - \beta)x), F_{LSp,MTq}(x)\}$$

for all $p, q \in X, \beta \in (0, 2)$ and $x > 0$; where the function $\phi : [0, \infty) \to [0, \infty)$ is onto and strictly increasing, and satisfy condition $(\star)$. In addition there exists $x_0, x_1, x_2 \in X$ with

$Ax_0 = MTx_1$, $Bx_1 = LSX_2$ and

$$E_F(AX_0, Bx_1) = \sup\{E_{\gamma,F}(AX_0, Bx_1) : \gamma \in (0,1)\} < \infty.$$

Then A, B, L, M, S and T have a unique common fixed point in X.

Inspired by the treatment given in [4] we prove our main result for more generalize version by taking both pairs to be weakly compatible maps

3. Main results

Theorem 3.1. *Let A, B, S and T be self mappings of Menger spaces $(X, \mathcal{F}, \Delta)$satisfying the following conditions:*

(i) $A(X) \subset T(X)$, and $B(X) \subset S(X)$,

(ii)

$$F_{Ax,By}(\phi(t)) \geq \min\{F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(\beta t), F_{By,Sx}((2-\beta)t), \frac{F_{Ax,Ty}(2t).F_{Sx,By}(2t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Ty}(t)}{F_{Ax,Sx}(t)+F_{Sx,Ty}(t)}\} \tag{3.1}$$

for all $x, y \in X, \beta \in (0,2)$ and $t > 0$; where the function $\phi : [0,\infty) \to [0,\infty)$ is onto and strictly increasing, and satisfy condition $(\star)$.

In addition there exists $x_0, x_1, x_2 \in X$ with $Ax_0 = Tx_1$, $Bx_1 = Sx_2$ and

$E_F(Ax_0, Bx_1) = \sup\{E_{\gamma,F}(Ax_0, Bx_1) : \gamma \in (0,1)\} < \infty$.

(iii) one of $A(X)$,$B(X)$,$S(X)$ or $T(X)$ is a complete subspace of X. Then

(a)the pair (A, S) (and (B, T)) have a coincidence point,

(b)A, B, S and T have a unique common fixed point provided both the poirs (A, S) and (B, T) are weakly compatible.

Proof. Let x_0 be an arbitrary element in X. By (i) there exists x_1, x_2 in X such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$ and $E_F(Ax_0, Bx_1) < \infty$. Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for

$n = 0, 1, 2, \cdots$. To show that the sequence $\{y_n\}$ is a Cauchy sequence putting $x = x_{2n}$ and $y = x_{2n+1}$ for $x > 0, \beta = 1 - \lambda$ with $\lambda \in (0, 1)$then we get by equation(3.1)

$$\begin{aligned}
F_{y_{2n},y_{2n+1}}(\phi(t)) =& F_{Ax_{2n},Bx_{2n+1}}(\phi(t)),\\
&\geq \min\{F_{Ax_{2n},Sx_{2n}}(t), F_{Bx_{2n+1},Tx_{2n+1}}(t), F_{Ax_{2n},Tx_{2n+1}}(\beta t),\\
&\quad F_{Bx_{2n+1},Sx_{2n}}((2-\beta)t), \frac{F_{Ax_{2n},Tx_{2n+1}}(2t).F_{Sx_{2n},Bx_{2n+1}}(2t)}{F_{Ax_{2n},Sx_{2n}}(t)},\\
&\quad \frac{2F_{Sx_{2n},Tx_{2n+1}}(t)}{F_{Ax_{2n},Sx_{2n}}(t) + F_{Sx_{2n},Tx_{2n+1}}(t)}\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \min\{F_{y_{2n},y_{2n-1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n},y_{2n}}(\beta t), F_{y_{2n+1},y_{2n-1}}((2-\beta)t),\\
&\quad \frac{F_{y_{2n},y_{2n}}(2t)F_{y_{2n-1},y_{2n+1}}(2t)}{F_{y_{2n},y_{2n-1}}(t)}, \frac{2F_{y_{2n-1},y_{2n}}(t),}{F_{y_{2n},y_{2n-1}}(t) + F_{y_{2n-1},y_{2n}}(t)}\Big\}\\
&= \min\{F_{y_{2n},y_{2n-1}}(t), F_{y_{2n+1},y_{2n}}(t), 1, F_{y_{2n+1},y_{2n-1}}((2-1+\lambda)t),\\
&\quad \frac{F_{y_{2n-1},y_{2n+1}}(2t)}{F_{y_{2n},y_{2n-1}}(t)}, 1\Big\}\\
&= \min\{F_{y_{2n},y_{2n-1}}(t), F_{y_{2n+1},y_{2n}}(t), 1, F_{y_{2n+1},y_{2n-1}}((1+\lambda)t),\\
&\quad F_{y_{2n},y_{2n+1}}(t), 1\}\\
&= \min\{F_{y_{2n},y_{2n-1}}(t), F_{y_{2n+1},y_{2n}}(t), 1, (F_{y_{2n+1},y_{2n}}(t) + F_{y_{2n},y_{2n+1}}(\lambda t)),\\
&\quad F_{y_{2n},y_{2n+1}}(t), 1\}\\
&= \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t), F_{y_{2n},y_{2n+1}}(\lambda t)\}
\end{aligned}$$

which on taking $\lambda \to 1$, we get

$$F_{y_{2n},y_{2n+1}}(\phi(t)) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}$$

Similarly we can show that

$$F_{y_{2n+1},y_{2n+2}}(\phi(t)) \geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t)\}$$

for all even or odd n, we have

$$F_{y_n,y_{n+1}}(\phi(t)) \geq \min\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}(t)\}$$

$$F_{y_n,y_{n+1}}(t) \geq \min\{F_{y_{n-1},y_n}(\phi^{-1}t), F_{y_n,y_{n+1}}(\phi^{-1}t)\}$$

continuing this process, we get

$$\begin{aligned} F_{y_n,y_{n+1}}(t) &\geq \min\{F_{y_{n-1},y_n}(\phi^{-1}t), F_{y_{n-1},y_n}(\phi^{-2}t), F_{y_n,y_{n+1}}(\phi^{-2}t)\} \\ &= \min\{F_{y_{n-1},y_n}(\phi^{-1}t), F_{y_n,y_{n+1}}(\phi^{-2}t)\} \\ &\geq \cdots \geq \min\{F_{y_{n-1},y_n}(\phi^{-1}t), F_{y_n,y_{n+1}}(\phi^{-m}t)\}. \end{aligned}$$

For each $\gamma \in (0,1)$ we have

$$\begin{aligned} E_{\gamma,F}(y_n, y_{n+1}) &= \inf\{t > 0 : F_(y_n, y_{n+1})(t) \geq 1 - \gamma\} \\ &= \inf\{t > 0 : \min\{F_{y_{n-1},y_n}(\phi^{-1}t), F_{y_n,y_{n+1}}(\phi^{-m})t\} \geq 1 - \gamma\} \\ &= \max\{\inf\{t > 0 : F_{y_{n-1},y_n}(\phi^{-1}t) \geq 1 - \gamma\}, \\ &\quad \inf\{t > 0 : F_{y_n,y_{n+1}}(\phi^{-m}t) \geq 1 - \gamma\}\} \\ &= \max\{\phi(E_{\gamma,F}(y_{n-1}, y_n)), \phi^m(E_{\gamma,F}(y_n, y_{n+1}))\} \end{aligned}$$

taking $n \to \infty$ we get

$$E_{\gamma,F}(y_n, y_{n+1}) \leq \phi(E_{\gamma,F}(y_{n-1}, y_n)) \leq \phi^n(E_{\gamma,F}(y_0, y_1))$$

Owing to Lemma(2.3), we conclude that $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $S(X)$ is a complete subspace of X then there exists a limit point $u \in S(X)$ such that $Su = z$ as $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of $\{y_n\}$.

To established $Au = z$, set $x = u$ and $y = x_{2n-1}$ with $\beta = 1$ in inequality (3.1)

$$\begin{aligned} F_{Au,B_{2n-1}}(\phi(t)) \geq \min\{&F_{Au,Su}(t), F_{Bx_{2n-1},Tx_{2n-1}}(t), F_{Au,Tx_{2n-1}}(t), F_{Bx_{2n-1},Su}(t), \\ &\frac{F_{Au,Tx_{2n-1}}(2t).F_{Su,Bx_{2n-1}}(2t)}{F_{Au,Su}(t)}, \frac{2F_{Su,Tx_{2n-1}}(t)}{F_{Au,Su}(t) + F_{Su,Tx_{2n-1}}(t)}\} \end{aligned}$$

taking $n \to \infty$

$$\begin{aligned} F_{Au,z}(\phi(t)) &\geq \min\{F_{Au,z}(t), F_{z,z}(t), F_{Au,z}(t), F_{z,z}, \frac{F_{Au,z}(2t).F_{z,z}(2t)}{F_{Au,z}(t)}, \\ &\quad \frac{2F_{z,z}(t)}{F_{Au,z}(t) + F_{z,z}(t)}\} \\ &= \min\{F_{Au,z}(t), 1, F_{Au,z}(t), 1, \frac{F_{Au,z}(2t)}{F_{Au,z}(t)}, \frac{2}{F_{Au,z}(t) + 1}\} \end{aligned}$$

i.e.

$$F_{Au,z}(\phi(t)) \geq F_{Au,z}(t)$$

but

$$F_{Au,z}(\phi(t)) \leq F_{Au,z}(t)$$

and hence $F_{Au,z}(t) = C$ for all $t > 0$ now applying to Lemma(2.4) we have $H(x) = C$ and hence $Au = Su = z$ which shows that the pair (A, S) has a point of coincidence.

Since $A(X) \subset T(X)$ and $Au = z$ hence $z \in T(x)$. Let $Tv = z$. If $Bv \neq z$ then by inequality (3.1) with $x = x_{2n}$, $y = v$ with $\beta = 1$, we have

$$\begin{aligned} F_{Ax_{2n},Bv}(\phi(t)) \geq & \min\{F_{Ax_{2n},Sx_{2n}}(t), F_{Bv,Tv}(t), F_{Ax_{2n},Tv}(t), F_{Bv,Sx_{2n}}(t), \\ & \frac{F_{Ax_{2n},Tv}(2t).F_{Sx_{2n},Bv}(2t)}{F_{Ax_{2n},Sx_{2n}}(t)}, \frac{2F_{Sx_{2n},Tv}(t)}{F_{Ax_{2n},Sx_{2n}}(t) + F_{Sx_{2n},Tv}(t)}\right\} \end{aligned}$$

which on making $n \to \infty$

$$\begin{aligned} F_{z,Bv}(\phi(t)) \geq & \min\{F_{z,z}(t), F_{Bv,z}(t), F_{z,z}(t), F_{Bv,z}(t), \frac{F_{z,z}(2t).F_{z,Bv}(2t)}{F_{z,z}(t)}, \\ & \frac{2F_{z,z}(t)}{F_{z,z}(t) + F_{z,z}(t)}\right\} \\ = & \min\{F_{z,z}(t), F_{Bv,z}(t), F_{z,z}(t), F_{Bv,z}(t), F_{z,Bv}(2t), 1\} \end{aligned}$$

i.e.

$$F_{z,Bv}(\phi(t)) \geq F_{z,Bv}(t)$$

but

$$F_{z,Bv}(\phi(t)) \leq F_{z,Bv}(t)$$

therefore $F_{z,Bv}(t) = C$. Now again appealing to Lemma(2.4) we have $H(x) = C$ for all $t > 0$ and hence $Bv = z$. Thus we get $Bv = Tv = z$. Which shows that the pair (B, T) has a point of coincidence.

If we take $T(X)$ is complete subspace of X, then analogous arguments establish $(iii)(a)$. The remaining two cases pertain essentially to the previous cases. Indeed, if $B(X)$ is complete subspace of X, then $z \in B(X) \subset S(X)$ and if $A(X)$ is complete then $z \in A(X) \subset T(X)$. Hence $(iii)(a)$ completely established.

Now since the pairs (A,S) and (B,T) are weakly compatible at u and v respectively, i.e. $Au = Su = Bv = Tv = z$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$, which shows that z is a common coincidence point of pairs (A,S) and (B,T). Now we have to show that $Az = Bz = Sz = Tz = z$. we put $x = x_{2n}$, $y = z$ with $\beta = 1$ in inequality (3.1)

$$F_{Ax_{2n},Bz}(\phi(t)) \geq \min\{F_{Ax_{2n},Sx_{2n}}(t), F_{Bz,Tz}(t), F_{Ax_{2n},Tz}(t), F_{Bz,Sx_{2n}}(t), \frac{F_{Ax_{2n},Tz}(2t).F_{Sx_{2n},Bz}(2t)}{F_{Ax_{2n},Sx_{2n}}(t)}, \frac{2F_{Sx_{2n},Tz}(t)}{F_{Ax_{2n},Sx_{2n}}(t) + F_{Sx_{2n},Tz}(t)}\}$$

as $n \to \infty$ we have

$$\begin{aligned} F_{z,Bz}(\phi(t)) &\geq \min\{F_{z,z}(t), F_{Bz,Bz}(t), F_{z,Tz}(t), F_{Bz,z}(t), \frac{F_{z,Tz}(2t).F_{Bz,z}(2t)}{F_{z,z}(t)}, \frac{2F_{z,Tz}(t)}{F_{z,z}(t) + F_{z,Tz}(t)}\} \\ &= \min\{1, 1, F_{z,Bz}(t), F_{Bz,z}(t), \frac{(F_{z,Bz}(2t))^2}{1}, \frac{2.F_{z,Bz}(t)}{F_{z,Bz}(t)+1}\} \end{aligned}$$

i.e.

$$F_{z,Bz}(\phi(t)) \geq F_{z,Bz}(t)$$

but

$$F_{z,Bz}(\phi(t)) \leq F_{z,Bz}(t)$$

and hence $F_{z,Bz}(t) = C$ now by Lemma(2.4) we have $H(t) = C$ for all $t > 0$ and $Bz = z$. Hence $Bz = Tz = z$. Similarly we can show that $Az = Sz = z$. combine all the result then we get $Az = Bz = Sz = Tz = z$ Thus z is common fixed point of A, B, S and T. Following the lines of proved theorem (3.1), one can easily prove the existance of unique common fixed point of mappings A, B, S and T.. Thus concludes the proof.

We can deduce corollaries for two and three mappings which run as follows:

Corollary 3.1. *Let A, S and T be self mappings on complete Menger space $(X, \mathcal{F}, \Delta)$ satisfying the conditions*

(i) $A(X) \subset T(X) \cap S(X)$,

(ii)

$$F_{Ax,By}(\phi(t)) \geq \min\{F_{Ax,Sx}(t), F_{Ay,Ty}(t), F_{Ax,Ty}(\beta t), F_{Ay,Sx}((2-\beta)t), \frac{F_{Ax,Ty}(2t).F_{Sx,By}(2t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Ty}(t)}{F_{Ax,Sx}(t)+F_{Sx,Ty}(t)}\}$$

for all $x, y \in X$, $\beta \in (0,2)$ *and* $t > 0$; *where the function* $\phi : [0,\infty) \to [0,\infty)$ *is onto and strictly increasing, and satisfy condition* $(\star)$.

In addition there exists $x_0, x_1, x_2 \in X$ *with* $Ax_0 = Tx_1$, $Ax_1 = Sx_2$ *and* $E_F(Ax_0, Ax_1) = \sup\{E_{\gamma,F}(Ax_0, Ax_1) : \gamma \in (0,1)\} < \infty$.

(iii) one of $A(X)$, $S(X)$ *or* $T(X)$ *is a complete subspace of* X. *Then*

(a)the pair (A,S) *(and* (A,T)*) have a coincidence point,*

(b)A, S *and* T *have a unique common fixed point provided both the poirs* (A,S) *and* (A,T) *are weakly compatible.*

Corollary 3.2. *Let* A, B *and* T *be self mappings on complete Menger space* $(X, \mathcal{F}, \Delta)$ *satisfying the conditions*

(i) $A(X) \cup B(X) \subset T(X)$,

(ii)

$$F_{Ax,By}(\phi(t)) \geq \min\{F_{Ax,Tx}(t), F_{By,Ty}(t), F_{Ax,Ty}(\beta t), F_{By,Tx}((2-\beta)t), \frac{F_{Ax,Ty}(2t).F_{Tx,By}(2t)}{F_{Ax,Tx}(t)}, \frac{2.F_{Tx,Ty}(t)}{F_{Ax,Tx}(t)+F_{Tx,Ty}(t)}\}$$

for all $x, y \in X$, $\beta \in (0,2)$ *and* $t > 0$; *where the function* $\phi : [0,\infty) \to [0,\infty)$ *is onto and strictly increasing, and satisfy condition* $(\star)$.

In addition there exists $x_0, x_1, x_2 \in X$ *with* $Ax_0 = Tx_1$, $Bx_1 = Tx_2$ *and* $E_F(Ax_0, Ax_1) = \sup\{E_{\gamma,F}(Ax_0, Ax_1) : \gamma \in (0,1)\} < \infty$.

(iii) one of $A(X)$, $B(X)$ *or* $T(X)$ *is a complete subspace of* X. *Then*

(a)the pair (A,T) *(and* (B,T)*) have a coincidence point,*

(b)A, B *and* T *have a unique common fixed point provided both the poirs* (A,T) *and* (B,T) *are weakly compatible.*

Corollary 3.3. *Let A and S be two self mappings on complete Menger space $(X, \mathcal{F}, \Delta)$ satisfying the conditions*

(i) $A(X) \subset S(X)$,

(ii)

$$F_{Ax,Ay}(\phi(t)) \geq \min\{F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ax,Sy}(\beta t), F_{Ay,Sx}((2-\beta)t), \frac{F_{Ax,Sy}(2t).F_{Sx,Ay}(2t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Sy}(t)}{F_{Ax,Sx}(t)+F_{Sx,Sy}(t)}\}$$

for all $x, y \in X$, $\beta \in (0,2)$ and $t > 0$; where the function $\phi : [0,\infty) \to [0,\infty)$ is onto and strictly increasing, and satisfy condition $(\star)$.

In addition there exists $x_0, x_1 \in X$ with $Ax_0 = Sx_1$, and

$E_F(Ax_0, Ax_1) = \sup\{E_{\gamma,F}(Ax_0, Ax_1) : \gamma \in (0,1)\} < \infty$.

(iii) one of $A(X)$,or $S(X)$ is a complete subspace of X. Then

(a)the pair (A, S) has a coincidence point,

(b)A and S have a unique common fixed point provided the poir (A, S) and (A, T) are weakly compatible.

Remark 3.1. *Corollary (3.3) improves the result of O'Regan and Saadati[4] proved for a pair of mappings in respect of continuity, compliteness and commutativity considerations.*

Example 3.1. *consider $X = [0, 6]$ with*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|};\ t > 0 \\ 0\ ;\ t = 0 \end{cases}$$

for all $x, y \in X$. then (X, F, Δ) is a Menger PM space. Define self mappings A, B, S and T on X as follows:

$$A0 = 0, Ax = 1, 0 < x \leq 6;$$

$$B0 = 0, Bx = 3, 0 < x < 6, B6 = 0;$$

$$S0 = 0, Sx = 5, 0 < x < 6, S6 = 3;$$

and

$$T0 = 0, Tx = 6, 0 < x < 6, T6 = 1.$$

Notice that all the four mappings A, B, S and T are discontinuous at 0 which is also their common fixed point. Also the pairs (A, S) and (B, T) are weakly compatible with $A(X) = \{0, 1\} \subset \{0, 1, 6\} = T(X)$ $B(X) = \{0, 3\} \subset \{0, 3, 5\} = S(X)$. define $\phi(t) = kt$ with $k = 1/2$ and choose $\beta = 1$. Now, in order to verify the contraction condition (3.1), with $t > 0$ we get

Case I *if $x = 0$ and $y = 6$, then $F_{Ax,By}(t/2) = 1 = F_{Sx,By}(t) \geq m(t, s)$,*

$$m(t,s) = \min\{F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(\beta t), F_{By,Sx}((2-\beta)t), \frac{F_{Ax,Ty}(2t).F_{Sx,By}(2t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Ty}(t)}{F_{Ax,Sx}(t) + F_{Sx,Ty}(t)}\}$$

Case II *if $x = 6$ and $y = 0$, then $F_{Ax,By}(t/2) = \frac{t}{t+2} \geq F_{Sx,By}(t) = m(t, s)$.*

Case III *if $x = 0$ and $y \in (0, 6)$, then $F_{Ax,By}(t/2) = \frac{t}{t+6} = F_{Sx,Ty}(t) = m(t, s)$.*

Case IV *if $x \in (0, 6)$, and $y = 0$ then $F_{Ax,By}(t/2) \geq \frac{t}{t+2} \geq \frac{t}{t+5} = F_{Sx,Ty}(t) = m(t, s)$.*

Case V *if $x = 6$ and $y \in (0, 6)$, then $F_{Ax,By}(t/2) \geq \frac{t}{t+4} \geq \frac{t}{t+5} = F_{Ty,Ax}(t) = m(t, s)$.*

Case VI *if $x \in (0, 6)$, and $y = 6$ then $F_{Ax,By}(t/2) \geq \frac{t}{t+2} \geq \frac{t}{t+5} = F_{Sx,By}(t) = m(t, s)$.*

Thus all the conditions of Theorem (3.1) are satisfied and 0 is the unique common fixed point of the mappings A, B, S and T.

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