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BIFURCATION AND CHAOS OF AN SIS MODEL WITH LOGISTIC TERM

NING CUI[1,*], XIAOCHUN ZHONG[2], JUNHONG LI[1,3]

[1]Department of Mathematics and Sciences, Hebei Institute of Architecture and Civil Engineering, Zhangjiakou, Hebei, 075000, China

[2]Office of Academic Affairs, Hebei Institute of Architecture and Civil Engineering, Zhangjiakou, Hebei, 075000, China

[3]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, China

Abstract. This paper presents a discrete-time SIS epidemic model with Logistic term. The transcritical bifurcation, flip bifurcation and Hopf bifurcation are investigated. The results show that the epidemic system with Logistic growth of susceptible population has rich dynamic behaviors, including period-6, 7, 8, 10, 11-orbits and chaotic phenomena.

Keywords: SIS model; logistic term; bifurcation; chaotic attractor.

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1. Introduction

Studies of epidemic models that incorporate the disease causing death and variation in total population have become one of the important areas in the mathematical theory of epidemiology. In the theory of epidemics, there are two kinds of mathematical models: the continuous-time models described by differential equations and the discrete-time models described by difference

*Corresponding author
E-mail address: biomathematics@163.com

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equations. The continuous-time epidemic models have been widely studied in many literatures [1-6]. Recently, discrete-time epidemic models have been used to study [7-14]. The advantages of a discrete-time approach are multiple. Firstly, difference models are more realistic than differential ones since the epidemic statistics are compiled from given time intervals and are discontinuous. Secondly, the discrete-time models can provide natural simulators for the continuous cases. One can thus not only study the behaviors of the continuous-time model with good accuracy, but also assess the effect of lager time steps. Finally, the use of discrete-time models makes it possible to use the entire arsenal of methods recently developed for the study of mappings and lattice equations, either from the integrability and/or chaos points of view.

On the other hand, simple models, by their own nature, cannot incorporate many of the complex biological factors. However, they often provide useful insights to help our understanding of complex process. For such reasons, assume only susceptible population are capable of reproducing with logistic equation [15-18], we firstly focus on the SIS model with Logistic term

$$\text{(1)} \qquad \begin{cases} \frac{dS}{dt} = rS(1-\frac{N}{K}) - \beta SI + \gamma I, \\ \frac{dI}{dt} = \beta SI - dI - \gamma I, \end{cases}$$

Where S, I are denoted the susceptible and infected, respectively. $N = S + I$ is the total population. All these of course are functions of time. r is the intrinsic birth rate. K, β are the carrying capacity and the infection rate respectively. $\gamma \geq 0$ is the recovery rate. d is the death rate of I.

Let

$S = \frac{d+\gamma}{\beta}, I = \frac{(d+\gamma)K}{r+\beta K} I_1, \tau = (d+\gamma)t.$

We obtain the following system analogous to (1)

$$\text{(2)} \qquad \begin{cases} \frac{dS_1}{d\tau} = m_1 S_1 - m_2 S_1^2 - S_1 I_1 + m_3 I_1, \\ \frac{dI}{d\tau} = S_1 I_1 - I_1, \end{cases}$$

where

$m_1 = \frac{r}{d+\gamma}, m_2 = \frac{r}{K\beta}, m_3 = \frac{\beta K \gamma}{(d+\gamma)(r+\beta K)}.$

In this paper, we apply the Euler scheme to discrete the SIS epidemic model and investigate the dynamical behaviors in detail by using bifurcation theory and center manifold theory [19-21]. It is verified that there are phenomena of the transcritical bifurcation, flip bifurcation, Hopf bifurcation types and chaos.

This paper is organized as follows. In Section 2, we give sufficient conditions of existence for transcritical bifurcation, flip bifurcation and Hopf bifurcation. In Section 3, a series of numerical simulations show that there are bifurcation and chaos in the discrete-time epidemic model. Finally, we give remarks to conclude this paper in Section 4.

2. Dynamic analysis

2.1. Existence of fixed points. Applying the Euler scheme to (2), we obtain the discrete system

$$\begin{cases} S_{1n+1} = (m_1+1)S_{1n} - m_2 S_{1n}^2 - S_{1n}I_{1n} + m_3 I_{1n}, \\ I_{1n+1} = S_{1n}I_{1n}. \end{cases} \tag{3}$$

The fixed points of (3) satisfy the following equations

$$\begin{cases} m_1 S_1 - m_2 S_1^2 - S_1 I_1 + m_3 I_1 = 0, \\ I_1 - S_1 I_1 = 0. \end{cases} \tag{4}$$

By the analysis of roots for Eq. (4), we obtain the following theorem

Theorem 1. (i). (3) has three fixed points $P_0(0,0)$, $P_1(\frac{m_1}{m_2},0)$, and $P_2(1,\frac{m_1-m_2}{1-m_3})$, when $m_1 > m_2$. (ii) If $m_1 \le m_2$, (3) has two fixed points $P_0(0,0)$, $P_1(\frac{m_1}{m_2},0)$.

The Jacobian matrix of the system (3) at fixed point $P(S_{1n},I_{1n})$ takes the form

$$J(P) = \begin{pmatrix} 1+m_1-2m_2S_{1n}-I_{1n} & m_3-S_{1n} \\ I_{1n} & S_{1n} \end{pmatrix} \tag{5}$$

2.2. Bifurcations. It is easy to see that the fixed point $P_0(0,0)$ is a saddle. In the following, we focus on investigating the bifurcations of P_1, P_2.

Theorem 2. If $m_2 = m_1$ and $m_1 \neq 2$, (3) undergoes a transcritical bifurcation at P_1.

Proof. The Jacobian matrix $J(P_1)$ has eigenvalues $\lambda_1 = 1-m_1$, $\lambda_2 = 1$ when $m_2 = m_1$, and $m_1 \neq 2$ implies $|\lambda_1| \neq 1$.

Let

$x_n = S_{1n} - \frac{m_1}{m_2}$, $y_n = I_{1n}$, $c_n = m_1 - m_2$,

(3) becomes

$$
\begin{cases}
x_{n+1} = (1-m_2)x_n + (m_3-1)y_n - x_n(m_2x_n + c_n + y_n) - \frac{c_n y_n}{m_2}, \\
y_{n+1} = y_n + \frac{c_n y_n}{m_2} + x_n y_n, \\
c_{n+1} = c_n.
\end{cases}
\tag{6}
$$

By the following transformation

$$
\begin{pmatrix} x_n \\ y_n \\ c_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \frac{m_2}{m_3-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \\ \sigma_n \end{pmatrix}
$$

(6) becomes

$$
\begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \\ \sigma_{n+1} \end{pmatrix} = \begin{pmatrix} 1-m_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \\ \sigma_n \end{pmatrix} + \begin{pmatrix} f_1(\phi_n, \psi_n, \sigma_n) \\ f_2(\phi_n, \psi_n, \sigma_n) \\ 0 \end{pmatrix}
$$

where

$$f_1(\phi_n, \psi_n, \sigma_n) = -m_2\phi_n^2 - \left(\frac{(2m_2m_3+m_3-m_2-1)\psi_n}{m_3-1} + \sigma_n\right)\phi_n - \frac{m_2m_3+m_3-1}{m_3-1}\psi_n^2 - \frac{m_2m_3+m_3-1}{m_2(m_3-1)}\psi_n,$$

$$f_2(\phi_n, \psi_n, \sigma_n) = \psi_n^2 + \frac{\psi_n\sigma_n}{m_2} + \phi_n\psi_n.$$

Then, we can consider

$$\phi_n = g(\psi_n, \sigma_n) = \varepsilon_1\psi_n^2 + \varepsilon_2\psi_n\sigma_n + \varepsilon_3\sigma_n^2 + o((|\psi_n| + |\sigma_n|)^3),$$

which must satisfy

$$g(\psi_n + f_2(g(\psi_n, \sigma_n), \psi_n, \sigma_n), \sigma_{n+1}) = (1-m_2)g(\psi_n, \sigma_n) + f_1(g(\psi_n, \sigma_n), \psi_n, \sigma_n).$$

Thus, we have

$$\varepsilon_1 = \frac{m_2m_3+m_3-1}{m_2-m_2m_3},\ \varepsilon_2 = \frac{m_2m_3+m_3-1}{m_2^2-m_2^2m_3},\ \varepsilon_3 = 0.$$

And (6) is restricted to the center manifold, which is given by

$$f : \psi_{n+1} = \psi_n + \psi_n^2 + \frac{\psi_n\sigma_n}{m_2} + \frac{m_2m_3+m_3-1}{m_2-m_2m_3}\psi_n^3 + \frac{m_2m_3+m_3-1}{m_2^2-m_2^2m_3}\psi_n^2\sigma_n + o((|\psi_n| + |\sigma_n|)^4).$$

Since

$$f(0, \sigma_n) = 0,\ \frac{\partial f}{\partial \psi}\Big|_{(0,0)} = 1,\ \frac{\partial^2 f}{\partial \psi^2}\Big|_{(0,0)} = 2,\ \frac{\partial^2 f}{\partial \psi \partial \sigma}\Big|_{(0,0)} = \frac{1}{m_2} \neq 0.$$

the system (3) undergoes a transcritical bifurcation at P_1. This proves the theorem.

Theorem 3. (3) undergoes a flip bifurcation at P_1 when $m_1 = 2$, $m_1 \neq m_2$. Furthermore, the stable periodic-2 point bifurcates from this fixed point.

Proof. Let

$x_n = S_{1n} - \frac{m_1}{m_2}$, $y_n = I_{1n}$, $c_n = m_1 - 2$,

the system (3) becomes

$$
(7) \qquad \begin{cases} x_{n+1} = -x_n + (m_3 - \frac{2}{m_2})y_n - (m_2 x_n + c_n + y_n)x_n - \frac{c_n y_n}{m_2}, \\ c_{n+1} = -c_n, \\ y_{n+1} = \frac{2}{m_2} y_n + y_n x_n + \frac{c_n y_n}{m_2}. \end{cases}
$$

By the following transformation

$$
\begin{pmatrix} x_n \\ c_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & (m_2+2)/(m_2 m_3 - 2) \end{pmatrix} \begin{pmatrix} p_n \\ \sigma_n \\ q_n \end{pmatrix}
$$

(7) becomes

$$
\begin{pmatrix} p_{n+1} \\ \sigma_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2/m_2 \end{pmatrix} \begin{pmatrix} p_n \\ \sigma_n \\ q_n \end{pmatrix} + \begin{pmatrix} f_1(p_n, q_n, \sigma_n) \\ 0 \\ f_2(p_n, q_n, \sigma_n) \end{pmatrix}
$$

where

$f_1(p_n, q_n, \sigma_n) = \frac{1}{m_2 m_3 - 2}(2p_n + q_n - m_2 m_3 p_n - m_2 m_3 q_n + m_3 q_n)(m_2 p_n + m_2 q_n + \sigma_n)$,
$f_2(p_n, q_n, \sigma_n) = q_n^2 + p_n q_n + \frac{q_n \sigma_n}{m_2}$.

Then, we can consider

$$q_n = g(p_n, \sigma_n) = \theta_1 p_n^2 + \theta_2 p_n \sigma_n + \theta_3 \sigma_n^2 + o((|p_n| + |\sigma_n|)^3),$$

which must satisfy

$$g(-p_n + f_1(p_n, g(p_n, \sigma_n), \sigma_n), \sigma_{n+1}) = \frac{2}{m_2} g(p_n, \sigma_n) + g^2(p_n, \sigma_n) + p_n g(p_n, \sigma_n) + \frac{g(p_n, \sigma_n)\sigma_n}{m_2}$$

Thus, we have

$$\theta_k = 0,\ k = 1, 2, 3.$$

We obtain the center manifold as followed

$$q_n = g(p_n, \sigma_n) = \theta_4 p_n^3 + \theta_5 p_n^2 \sigma_n + \theta_6 p_n \sigma_n^2 + \theta_7 \sigma_n^3 + o((|p_n| + |\sigma_n|)^4).$$

And (7) is restricted to the center manifold, which is given by

$f_1 : p_{n+1} = -p_n - \frac{1}{m_2m_3-m_2}(m_2m_3p_n - 2p_n - q_n + m_2m_3q_n + m_3q_n)(m_2p_n + m_2q_n + \sigma_n).$

Direct calculations show that

$(\frac{\partial f}{\partial \sigma}\frac{\partial^2 f}{\partial p^2} + 2\frac{\partial^2 f}{\partial p \partial \sigma})\big|_{(0,0)} = -2,\ (\frac{1}{2}(\frac{\partial^2 f}{\partial p^2})^2 + \frac{1}{3}(\frac{\partial^3 f}{\partial p^3}))\big|_{(0,0)} = 2m_2^2.$

Hence, the system (3) undergoes a flip bifurcation at P_1. This completes the proof.

The positive fixed point is so important to the biological system that people usually are very interested in it. We will next pay attention to the unique positive fixed point P_2.

Theorem 4. If $m_1 \neq m_2$ and $m_2 = 2$, (3) undergoes a flip bifurcation at P_2.

Since the analysis is similar to the case at P_1, the above proofs are omitted.

We next give the condition of existence of Hopf bifurcation using the Hopf bifurcation theorem in [21]. The characteristic equation of the Jacobian matrix $J(P_2)$ can be written as

$\lambda^2 - t_2\lambda + d_2 = 0.$

where

$t_2 = 2 + m_1 - 2m_2 - \frac{m_1-m_2}{1-m_3},\ d_2 = 1 + 2m_1 - 3m_2 - \frac{m_1-m_2}{1-m_3}.$

The eigenvalues of the Jacobian matrix of (3) at P_2 are

$\lambda_{1,2} = \frac{t_2 \pm \sqrt{t_2^2 - 4d_2}}{2}.$

The eigenvalues $\lambda_{1,2}$ are complex conjugates for $t_2^2 < 4d_2$.

We transform the fixed point $P_2(1, \frac{m_1-m_2}{1-m_3})$ to the origin by

$x_n = S_{1n} - 1,\ y_n = I_{1n} - \frac{m_1-m_2}{1-m_3}.$

the system (3) becomes

$$(8) \qquad \begin{cases} x_{n+1} = (1 + m_1 - 2m_2 - \frac{m_1-m_2}{1-m_3})x_n + (m_3-1)y_n - m_2x_n^2 - x_ny_n, \\ y_{n+1} = \frac{m_1-m_2}{1-m_3}x_n + y_n + x_ny_n. \end{cases}$$

The eigenvalues of the matrix associated with the linearized map (8) at fixed point (0.0) are complex conjugates which are written as $\lambda, \bar{\lambda} = \frac{t_2 \pm i\sqrt{4d_2 - t_2^2}}{2}$, where

$|\lambda| = \sqrt{1 + 2m_1 - 3m_2 - \frac{m_1-m_2}{1-m_3}}.$

Now assume $m_3 < \frac{1}{2}$ and let $m_{10} = \frac{m_2(2-3m_3)}{1-2m_3}$. Then we have

$\frac{d(|\lambda|)}{dm_1}\big|_{m_1=m_{10}} = \frac{1-2m_3}{2(1-m_3)},\ |\lambda(m_{10})| = 1,\ \lambda(m_{10}) = 1 + \frac{m_2-m_2m_3}{4m_3-2} \pm i\frac{\sqrt{m_2(1-m_3)(m_2m_3+4-m_2-8m_3)}}{2-4m_3},$

and $\lambda^j \neq 1$, $j = 1,2,3,4$. Let

$$T=\begin{pmatrix} m_3-1 & 0\\ \frac{2-t_2}{2} & -\frac{\sqrt{4d_2-t_2^2}}{2}\end{pmatrix},\quad \begin{pmatrix} x_n\\ y_n\end{pmatrix}=T\begin{pmatrix}\phi_n\\ \xi_n\end{pmatrix},$$

(8) becomes

$$\begin{pmatrix}\phi_{n+1}\\ \xi_{n+1}\end{pmatrix}=\begin{pmatrix}\frac{t_2}{2} & -\frac{\sqrt{4d_2-t_2^2}}{2}\\ \frac{\sqrt{4d_2-t_2^2}}{2} & \frac{t_2}{2}\end{pmatrix}\begin{pmatrix}\phi_n\\ \xi_n\end{pmatrix}+\begin{pmatrix} f_1(\phi_n,\xi_n)\\ f_2(\phi_n,\xi_n)\end{pmatrix}, \tag{9}$$

where

$$f_1(\phi_n,\xi_n)=\frac{1}{2m_3-2}(2m_2m_3+m_1m_3-m_2-2m_2m_3^2)\phi_n^2-\frac{\sqrt{4d_2-t_2^2}}{2}\phi_n\xi_n,$$
$$f_2(\phi_n,\xi_n)=\frac{1}{(4d_2-t_2^2)(m_3-1)^2}(2m_2m_3-m_2-2m_2m_3^2+m_1m_3-2+4m_3-2m_3^2)(m_1m_3+m_2$$
$$-2m_2m_3)\phi_n^2+(m_2-\frac{m_1}{2}+\frac{m_1-m_2}{2-2m_3}-1+m_3)\phi_n\xi_n.$$

Notice that (9) is exactly on the center manifold in the form, in which the coefficient l [20] is given by

$$l=-\mathrm{Re}[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}l_{11}l_{20}]-\frac{1}{2}|l_{11}|^2-|l_{02}|^2+\mathrm{Re}(\bar{\lambda}l_{21}),$$

where

$$l_{11}=\frac{1}{4}[(f_{1\phi\phi}+f_{1\xi\xi})+(f_{2\phi\phi}+f_{2\xi\xi})i]=\frac{1}{4(1-m_3)}(m_2-2m_2m_3+2m_2m_3^2-m_1m_3)$$
$$-\frac{1}{4(1-m_3)^2\sqrt{4d_2-t_2^2}}(m_1m_3+m_2-2m_2m_3)(2m_2m_3-m_2-2m_2m_3^2+m_1m_3-2+4m_3-2m_3^2)i,$$
$$l_{20}=\frac{1}{8}[(f_{1\phi\phi}-f_{1\xi\xi}+2f_{2\phi\xi})+(f_{2\phi\phi}-f_{2\xi\xi}-2f_{1\phi\xi})i]=\frac{1-m_3}{4(m_2-1)}-[\frac{1}{4(1-m_3)^2\sqrt{4d_2-t_2^2}}(m_1m_3$$
$$+m_2-2m_2m_3)(2m_2m_3-m_2-2m_2m_3^2+m_1m_3-2+4m_3-2m_3^2)+\frac{\sqrt{4d_2-t_2^2}}{8}]i,$$
$$l_{02}=\frac{1}{8}[(f_{1\phi\phi}-f_{1\sigma\sigma}-2f_{2\phi\sigma})+(f_{2\phi\phi}-f_{2\sigma\sigma}+2f_{1\phi\sigma})i]=\frac{m_1m_3-1+2m_3-m_3^2-m_2m_3^2}{4(m_3-1)}$$
$$-[\frac{1}{8(1-m_3)^2\sqrt{4d_2-t_2^2}}(m_1m_3+m_2-2m_2m_3)(2m_2m_3-m_2-2m_2m_3^2+m_1m_3-2+4m_3-2m_3^2)$$
$$+\frac{\sqrt{4d_2-t_2^2}}{8}]i,$$
$$l_{21}=0.$$

By direct calculating, we obtain that $l\neq 0$. From the above analysis, we get the theorem 5.

Theorem 5. If $t_2^2<4d_2$, $m_3<\frac{1}{2}$, $m_{10}=\frac{m_2(2-3m_3)}{1-2m_3}$, and $m_1\neq\frac{1}{m_3}(2-2m_3-m_2+2m_2m_3)$, (3) undergoes a Hopf bifurcation at fixed point P_2.

3. Numerical simulations

It has been long supposed that the existence of chaotic behaviour in the microscopic motions is responsible for their equilibrium and non-equilibrium properties[22]. From above results, it shows that (3) has rich dynamic behaviors. In this section, we use the bifurcation diagrams, Lyapunov exponents and phase portraits to illustrate the above analytic results and find new dynamic behaviors of the model (3) as the parameters varies. The bifurcation parameters are considered in the following three cases:

(I) Varying m_1 in range $1 \leq m_1 \leq 2.5$, and fixing $m_2 = 0.7$, $m_3 = 0.12$.

(II) Varying m_2 in range $0.5 \leq m_2 \leq 1.2$, and fixing $m_1 = 2$, $m_3 = 0.12$.

(III)Varying m_3 in range $0 \leq m_3 \leq 0.8$, and fixing $m_1 = 2$, $m_2 = 0.7$.

Since that the bifurcation diagrams of $m_k - S_{1n}$ are similar with the bifurcation diagrams of $m_k - I_{1n}$ $(k = 1,2,3)$,we will only show the former. For case (I), the bifurcation diagram of system (3) in $m_1 - S_{1n}$ space is given in Fig. 1(i) to show the dynamical changes of susceptible and infective respectively as m_1 varying. The spectrum of Lyapunov exponents of (3) with respect to parameter m_1 is given in Fig. 1(ii). Thus, we can see that the orbit with initial values approaches to the stable fixed point P_2 for $m_1 < 1.4$ approximately, and Hopf bifurcation occurs at $m_1 \approx 1.4$. When increases to $m_1 \approx 2.3$, (3) becomes stable. In Fig. 1(i), we observe the period-10, period-11 windows within the chaotic regions and boundary crisis at $m_1 \approx 2.3$. For $m_1 \in (1.9,2.3)$ the maximum Lyapunov exponents are mostly positive which corresponding to chaotic region. To well see the dynamics, the attractor in the system and time series of (3) with $m_1 = 2.2$ and time series of S_{1n} and I_{1n} are given in Fig. 1(iii) and (iv) respectively. For case (II), Fig. 2(i) is shows that there are period-6 windows, period-7 windows, period-8 windows and invariant in the chaotic regions and boundary crisis at $m_2 \approx 0.92$. Fig. 2(ii) shows that there is a part of Lyapunov exponents are positive which corresponding to chaotic region when $m_2 \in (0.65,0.75)$.

For case (III), Fig . 3(i) depicts that there are period-6, period-8 windows and invariant in the chaotic regions, and two onsets of chaos at $m_3 \approx 0.32$ and $m_3 \approx 0.68$, , respectively. Fig 3(ii) shows a part of Lyapunov exponents are positive which corresponding to chaotic region when $m_3 \in (0.1,0.2)$.

4. CONCLUSION

In this paper, we investigate the behaviors of the SIS epidemic with logistic term as a discrete-time dynamical system, and find many complex and new interesting dynamical phenomena. Without the recovery rate of infectious, (1) becomes the SI model in [15]. Our theoretical

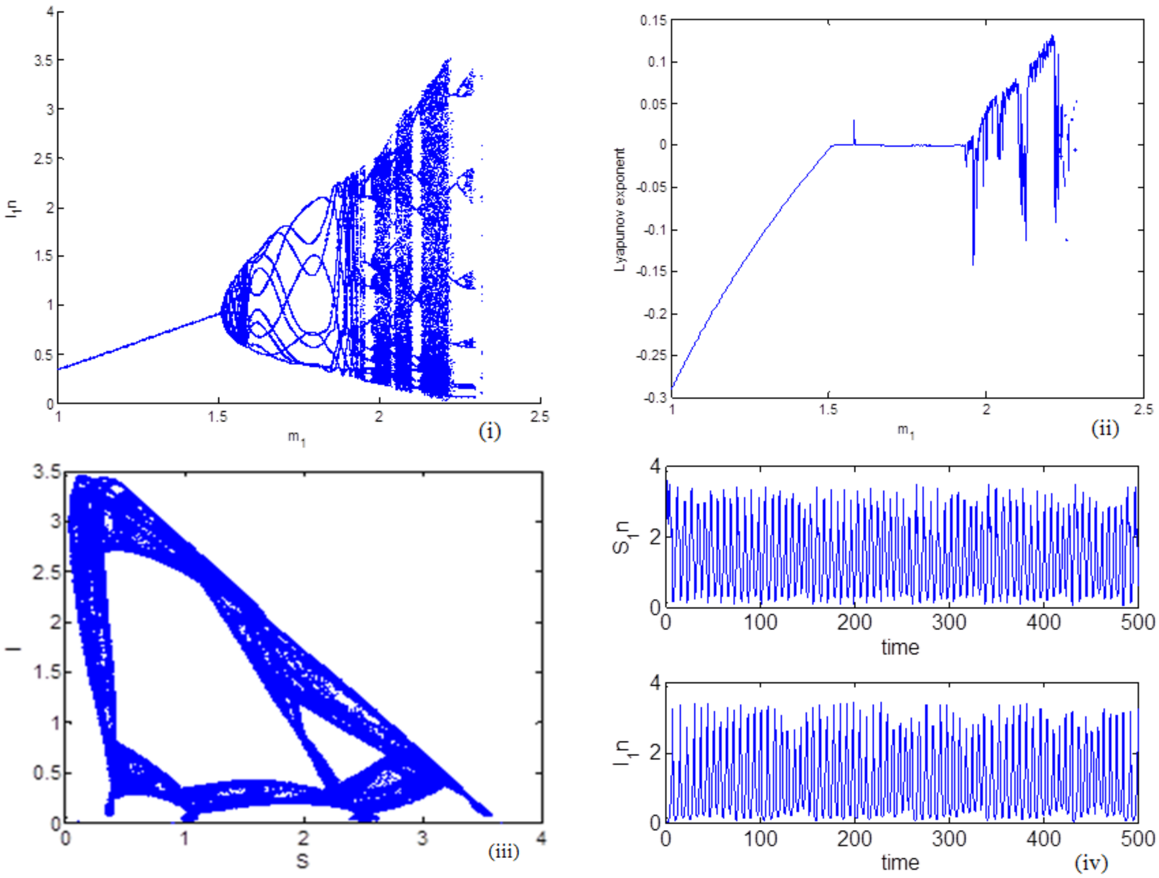

Fig .1. (i) Bifurcation diagram in $m_1 - S_{1n}$ plane. (ii) Spectrum of Lyapunov exponents corresponding to (i). (iii) The attractor of the system (3) with $m_1 = 2.2$. (iv) Time series of S_{1n} and I_{1n} with $m_1 = 2.2$.

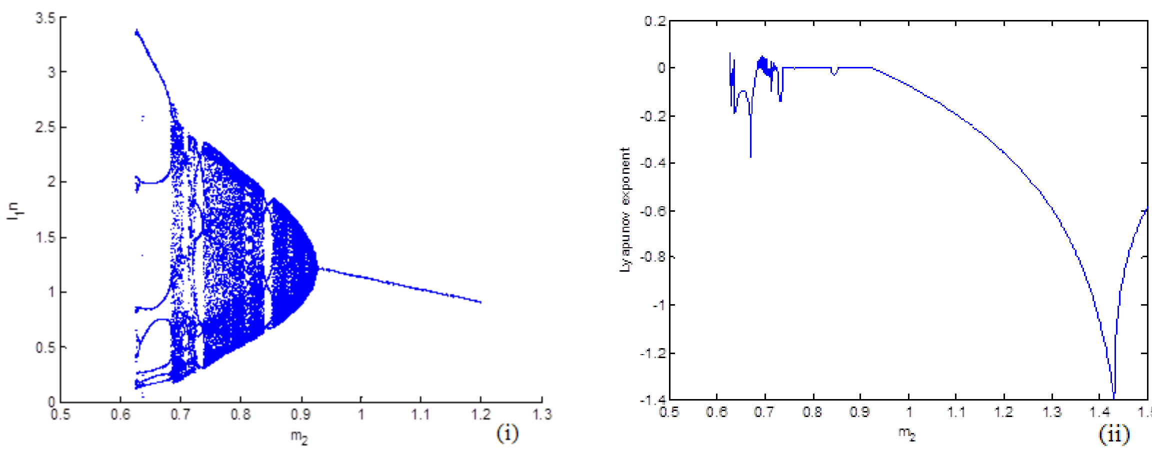

Fig. 2. (i) Bifurcation diagram in $m_2 - S_{1n}$ plane. (ii) Spectrum of Lyapunov exponents corresponding to (i).

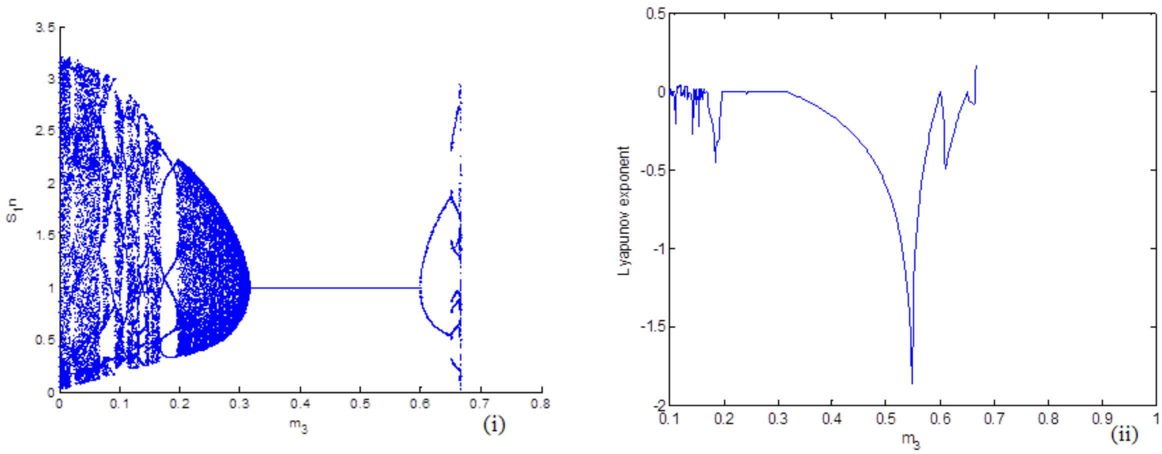

Fig. 3. (i) Bifurcation diagram in $m_3 - S_{1n}$ plane. (ii) The spectrum of Lyapunov exponents of (3) with respect to parameter m_3 .

analysis and numerical simulations have demonstrated that the model exhibits the variety of dynamical behaviors, which include the discrete epidemic model undergoes transcritical bifurcation, flip bifurcation, Hopf bifurcation and chaos. The results show that there are different dynamical behaviors between discrete system and its corresponding continuous system and the results are different from [23]. Furthermore, chaos can cause the population to run a higher risk of extinction due to the unpredictably [24-25]. Thus, how to control chaos in the epidemic model is very important, which needs further consideration.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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