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BEST PROXIMITY POINT THEOREMS FOR F_ρ-PROXIMAL CONTRACTION IN MODULAR FUNCTION SPACES

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Abstract. In this paper, we introduce Wardowski's F-contractions in the context of modular function spaces. We also introduce the concept of F_ρ-proximal contraction and prove some best proximity point theorems by unifying and generalizing some recent results in modular function spaces. Moreover, we discuss some illustrative examples to highlight the realized improvements.

Keywords: best proximity points; modular function space; proximal contraction.

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1. INTRODUCTION

The study of modular function spaces has recently attracted a good attention of researchers since its initiation by Nakano [1] in connection with the theory of order spaces. In particular, the fixed point theory in such spaces has got a remarkable treatment by researchers like Benavides [2], Khamsi et al. [3], Kozlowski [4] and Khan and Abbas [5].

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Best proximity point results are the ones that provide sufficient conditions for the existence of a best proximity point and algorithms for finding best proximity points (see Definition 2). It is interesting to note that best proximity point results extend fixed point results in a natural fashion. Indeed, if the mapping under consideration is a self-mapping, a best proximity point problem becomes a fixed point problem. Banach contraction principle is of paramount importance in metrical fixed pint theory and it has been generalized by many mathematicians in different directions. One of these generalizations is its extension to the case of nonself mappings. In fact, given nonempty closed subsets A and B of a complete metric space (X,d), a contractive nonself mapping $T: A \to B$ does not necessarily have a fixed point. Eventually, it is quite natural to find an element x such that $d(x,Tx)$ is minimum over a set A which implies that x and Tx are in close proximity to each other. A best proximity point theorem for contractive mappings has been detailed in Basha [6]. Eldred et al. [7] have elicited a best proximity point theorem for relatively nonexpansive mappings, an alternative treatment to which has been focused in [8].

Recently, Omidvari et at. [9] proved the existence of a best proximity point for F-contraction nonself mappings in complete metric spaces. Also they defined F-proximal contractions of first and second kind and extended some comparable best proximity theorems and improved the recent results. Latif et al. [10] proved coincidence best proximity point results for F_g-weak contractive mappings in ordered metric spaces. Jleli et al. [11] introduced the class of proximal quasi contractions for nonself mappings in modular spaces, and provided sufficient conditions assuring the existence and uniqueness of the best proximity point in modular spaces with Fatou property.

Motivated by the above results, we first present some best proximity point results for F_ρ-proximal contractions in modular function spaces. Then we give some sufficient conditions guaranteeing the existence and uniqueness of best proximity points for nonself Ciric type generalized F_ρ-proximal contractions in modular function spaces.

2. Preliminaries

Some basic facts and notation about modular spaces are recalled here from Kozlowski [4]. We refer the reader to Kilmer et al. [12] and the references therin for an exposition of the theory.

Definition 2.1. Let X be an arbitrary vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A functional $\rho : X \to [0,\infty]$ is called modular if, for any x, y in X, the following hold:

(m_1) $\rho(x) = 0$ if and only if $x = 0$.

(m_2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$.

(m_3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$, provided that $\alpha + \beta = 1$, and $\alpha, \beta \ge 0$.

If (m_3) is replaced by $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$, if $\alpha + \beta = 1$, and $\alpha, \beta \ge 0$, then ρ is called a convex modular.

The vector space X_ρ given by

$$X_\rho = \{x \in X : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a modular space.

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. Consistent with Kilmer et al. [13], the ρ-distance of $x \in X_\rho$ from a set $D \subset X_\rho$ is given as follows:

$$d_\rho(x, D) = \inf\{\rho(x - h) : h \in D\}.$$

Definition 2.2. Let X_ρ be a modular space. The sequence $\{x_n\} \subset X_\rho$ is called:

(i) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) ρ-Cauchy, if $\rho(x_n - x_m) \to 0$ as n, $m \to \infty$.

Note that, ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality.

Definition 2.3. A subset $D \subset X_\rho$ is called

(i) ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D.

(ii) ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of D always belongs to D.

(iii) ρ-compact if every sequence in D has a ρ-convergent subsequence in D.

(iv) ρ-a.e. compact if every sequence in D has a ρ-a.e. convergent subsequence in D.

(v) ρ-bounded if

$$diam_\rho(D) = \sup\{\rho(x - y) : x, y \in D\} < \infty.$$

Definition 2.4. Let $\digamma$ be the collection of all mappings $F:\mathbb{R}^{+}\to\mathbb{R}$ that satisfy the following conditions:

(C_1) F is strictly increasing, i.e. for all $\alpha,\beta\in\mathbb{R}$ such that $\alpha<\beta$ implies that $F(\alpha)<F(\beta)$;

(C_2) For any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$;

(C_3) There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^{+}}\alpha^{k}F(\alpha)=0$.

Recently, Wardowski [14] introduced a new type of contraction called F-contraction. We now reformulate it in the context of modular function spaces as follows.

Definition 2.5. Let ρ be a modular and X_ρ be a modular function space. A self mapping T on X_ρ is called F_ρ-contraction if there exist $F\in\digamma$ and $\tau>0$ such that

$$\tau+F(\rho(c(Tx-Ty)))\leq F(\rho(x-y)),$$

for all $x,y\in L_\rho$ with $\rho(Tx-Ty)>0$, where $c,l\in\mathbb{R}^{+}$ *with* $c>l$.

We also reshape Mınak et al. [15] definition of Ciric type generalized F-contraction in the context of modular function spaces as below:

Definition 2.6. Let $\digamma$ and X_ρ be same as in the above definition. A self mapping T on X_ρ is called Ciric type generalized F_ρ-contraction if there exist $F\in\digamma$ and $\tau>0$ such that

$$\tau+F(\rho(c(Tx-Ty)))\leq F(M(x,y))$$

where, $M(x,y)=\max\left\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{1}{2}[\rho(x-Ty)+\rho(y-Tx)]\right\}$, for all $x,y\in X_\rho$ with $\rho(c(Tx-Ty))>0$, and $c,l\in\mathbb{R}^{+}$ *with* $c>l$.

Khamsi [16] introduced the concept of self-mappings in modular spaces. We extend Khamsi's definition to F_ρ-quasi contraction as below:

Definition 2.7. Considering the same family $\digamma$ of mappings as in Definition 2. Let X_ρ be a modular function space with modular ρ . A self mapping T on X_ρ is called F_ρ-quasicontraction if there exist $F\in\digamma$ and $\tau>0$ such that

$$\tau+F(\rho(c(Tx-Ty)))\leq F(M(x,y))$$

where, $M(x,y)=\max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\rho(x-Ty),\rho(y-Tx)\}$, for all $x,y\in X_\rho$ with $\rho(c(Tx-Ty))>0$ and $c,l\in\mathbb{R}^{+}$ *with* $c>l$.

It is remarked that, by considering $F(\alpha) = \ln \alpha$ in above definitions, a Ciric type generalized F_ρ-contraction reduces to Ciric type generalized ρ-contraction and an F_ρ-contraction becomes a ρ-contraction. Note also that every ρ-quasicontraction is an F_ρ-quasicontraction.

Let (A,B) be a pair of nonempty ρ-closed subsets of a modular function space X_ρ. Then we have the following notation and notions which are to be used in the subsequent sections.

$$\begin{aligned} d_\rho(A,B) &: = \inf\{\rho(x-y) : x \in A \text{ and } y \in B\} \\ A_0 &: = \{x \in A : \rho(x-y) = d_\rho(A,B) \text{ for some } y \in B\}, \\ B_0 &: = \{y \in B : \rho(x-y) = d_\rho(A,B) \text{ for some } x \in A\}. \end{aligned}$$

Definition 2.8. Let (A,B) be a pair of nonempty subsets of a modular function space X_ρ. A point $x \in A$ is said to be a best proximity point of the mapping $T : A \to B$ if $\rho(x - Tx) = d_\rho(A,B)$. It can be observed that a best proximity reduces to a fixed point for self mappings.

Definition 2.9. Let (A,B) be a pair of nonempty subsets of a modular function space X_ρ. Then (A,B) is said to have the $P-$property if and only if

$$\left.\begin{array}{l} \rho(x_1 - y_1) = d_\rho(A,B) \\ \rho(x_2 - y_2) = d_\rho(A,B) \end{array}\right\} \Rightarrow \rho(x_1 - x_2) = \rho(y_1 - y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 2.10. A is said to be approximately ρ-compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition $\rho(y - x_n) \to \rho(y - A)$ for some y in B has a ρ-convergent subsequence.

We now first introduce the concept of F_ρ-proximal contractions of the first and the second kind in the setting of modular function spaces. By doing so, we actually generalize the concepts of $F-$proximal contraction of the first and the second kind due to Omidvari et at. [9].

Definition 2.11. Let A and B be two ρ-closed subset of X_ρ. A mapping $T : A \to B$ is said to be a F_ρ-proximal contraction of first kind if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\left.\begin{array}{c} \rho(u_1 - Tx_1) = d(A,B) \\ \rho(u_2 - Tx_2) = d(A,B) \\ \rho(u_1 - u_2) > 0, \rho(x_1 - x_2) > 0 \end{array}\right\} \Rightarrow \tau + F(\rho(c(u_1 - u_2))) \leq F(\rho(l(x_1 - x_2)))$$

where $u_1, u_2, x_1, x_2 \in A$ and $c, l \in \mathbb{R}^+$ with $c > l$.

Definition 2.12. Let A and B be two ρ-closed subset of X_ρ. A mapping $T : A \to B$ is said to be a F_ρ-proximal contraction of the second kind if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\left.\begin{array}{r} \rho\left(u_1 - Tx_1\right) = d_\rho\left(A,B\right) \\ \rho\left(u_2 - Tx_2\right) = d_\rho\left(A,B\right) \\ \rho\left(Tu_1 - Tu_2\right) > 0, \rho\left(Tx_1 - Tx_2\right) > 0 \end{array}\right\} \Rightarrow \tau + F\left(\rho\left(c\left(Tu_1 - Tu_2\right)\right)\right) \leq F\left(\rho\left(l\left(Tx_1 - Tx_2\right)\right)\right)$$

where $u_1, u_2, x_1, x_2 \in A$ and $c, l \in \mathbb{R}^+$ with $c > l$.

Next, we present the concept of Ciric type generalized F_ρ-proximal contraction which a modular extension of an F_ρ-proximal contraction.

Definition 2.13. Let X_ρ be modular function space and (A, B) a pair of nonempty ρ-closed subset of X_ρ. A nonself mapping $T : A \to B$ is said to be Ciric type generalized F_ρ-proximal contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\left.\begin{array}{r} \rho\left(u_1 - Tx_1\right) = dist_\rho\left(A,B\right) \\ \rho\left(u_2 - Tx_2\right) = dist_\rho\left(A,B\right) \end{array}\right\},$$

implies

$$\begin{aligned} & \tau + F\left(\rho\left(c\left(u_1 - u_2\right)\right)\right) \\ \leq\ & F\left(\max\left\{\begin{array}{c} \rho\left(l\left(x_1 - x_2\right)\right), \rho\left(l\left(x_1 - u_1\right)\right), \rho\left(l\left(x_2 - u_2\right)\right), \\ \frac{1}{2}\left\{\rho\left(\frac{l}{2}\left(x_1 - u_2\right)\right) + \rho\left(\frac{l}{2}\left(x_2 - u_1\right)\right)\right\} \end{array}\right\}\right) \end{aligned} \tag{1}$$

for all $x_1, x_2, u_1, u_2 \in A$, where $c, l \in \mathbb{R}^+$ with $c > l$.

We also need the following.

Definition 2.14. Given that $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to be modular ρ isometric with respect to g if the following holds:

$$\rho\left(Tgx_1 - Tgx_2\right) = \rho\left(Tx_1 - Tx_2\right), \text{ for all } x_1, x_2 \in A.$$

3. SOME BEST PROXIMITY POINT RESULTS

We divide this section into several subsections dealing with best proximity point results of different kinds of proximal contractions.

3.1. F_ρ-proximal contractions.

Here we deal with F_ρ-proximal contractions. First we present some results on existence and uniqueness of best proximity points and then examples to testify our results. We start with the following result.

Theorem 3.1. Suppose that the pair (A,B) of nonempty ρ-closed subsets of a modular function space X_ρ has the $P-$property. Also, it is supposed that A_0 is nonempty. If $T : A \to B$ is F_ρ-proximal contraction such that $T(A_0) \subseteq B_0$. Then there exists a unique x^* in A such that $\rho\left(x^* - Tx^*\right) = d_\rho\left(A,B\right)$, that is, x^* is a best proximity point of T.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T\left(A_0\right) \subseteq B_0$, there exists x_1 in A_0 such that $\rho\left(x_1 - Tx_0\right) = d_\rho\left(A,B\right)$. Moreover, $Tx_1 \in T\left(A_0\right) \subseteq B_0$ implies the existence of an $x_2 \in A_0$ such that $\rho\left(x_2 - Tx_1\right) = d_\rho\left(A,B\right)$. Continuing in this way, we obtain a sequence $\{x_n\}$ in A_0 such that

$$\rho\left(x_{n+1} - Tx_n\right) = d_\rho\left(A,B\right), \text{ for all } n \in \mathbb{N}. \tag{2}$$

Since pair (A,B) has the $P-$property, from (2) we have,

$$\rho\left(x_n - x_{n+1}\right) = \rho\left(Tx_{n-1} - Tx_n\right), \text{ for all } n \in \mathbb{N}. \tag{3}$$

We now prove that the sequence $\{x_n\}$ is ρ-convergent in A_0.If there exists $n_0 \in \mathbb{N}$ such that $\rho\left(Tx_{n_0-1} - Tx_{n_0}\right) = 0$, then $\rho\left(x_{n_0} - x_{n_0+1}\right) = 0 \Leftrightarrow x_{n_0} - x_{n_0+1} = 0 \Leftrightarrow x_{n_0} = x_{n_0+1}$ by (3). Thus

$$Tx_{n_0} = Tx_{n_0+1} \Leftrightarrow Tx_{n_0} - Tx_{n_0+1} = 0 \Leftrightarrow \rho\left(Tx_{n_0} - Tx_{n_0+1}\right) = 0. \tag{4}$$

From (3) and (4), we obtain

$$\rho\left(x_{n_0+2} - x_{n_0+1}\right) = \rho\left(Tx_{n_0+1} - Tx_{n_0}\right) = 0 \Rightarrow x_{n_0+2} = x_{n_0+1}.$$

Thus $x_n = x_{n_0}$ for all $n \geq n_0$ and hence $\{x_n\}$ is ρ-convergent in A_0.

Next let $\rho\left(Tx_{n-1} - Tx_n\right) \neq 0$ for all $n \in \mathbb{N}$. Then, for any positive integer n, using (3), we have

$$\tau + F\left(\rho\left(c\left(Tx_n - Tx_{n-1}\right)\right)\right) \leq F\left(\rho\left(l\left(x_n - x_{n-1}\right)\right)\right).$$

because T is an F_ρ-contraction and this implies that

$$\begin{aligned} F(\rho(c\ (x_{n+1}-\ x_n))) &\leq F(\rho(l(x_n-x_{n-1})))-\tau \\ F(\rho(c\ (x_{n+1}-\ x_n))) &\leq F(\rho(c(x_n-x_{n-1})))-\tau \\ F(\rho(c(x_{n+1}-x_n))) &\leq F(\rho(l(x_{n-1}-x_{n-2})))-2\tau \\ &\leq F(\rho(c(x_{n-2}-x_{n-3})))-3\tau\leq\ldots\leq F(\rho(c(x_1-x_0)))-n\tau. \end{aligned} \tag{5}$$

Denote $\beta_n := (\rho(c(x_{n+1}-x_n)))$. Then by (5), $\lim_{n\to\infty} F(\beta_n) = -\infty$. Appealing to (C_2), we get

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \rho(x_{n+1}-x_n) = 0. \tag{6}$$

A use of (C_3) guarantees the existence of a $k\in(0,1)$ such that

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) = 0, \tag{7}$$

and so by (5), for all $n\in\mathbb{N}$, we have

$$\beta_n^k(F(\beta_n)-F(\beta_0)) \leq -\beta_n^k n\tau \leq 0.$$

Reading (6) and (7) together, we get

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Hence there exists $n_1\in\mathbb{N}$ such that $n\beta_n^k\leq 1$ for all $n\geq n_1$. That is, for all $n\geq n_1$,

$$\beta_n \leq \frac{1}{n^{\frac{1}{k}}}, \tag{8}$$

or

$$\rho(x_n-x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{9}$$

Similarly, there exists $n_2\in\mathbb{N}$ such that

$$\begin{aligned} \rho(x_n-x_{n+2}) &\leq \omega(2)[\rho(x_n-x_{n+1})+\rho(x_{n+1}-x_{n+2})] \\ &\leq \omega(2)\left(\frac{1}{n^{\frac{1}{k}}}+\frac{1}{(n+1)^{\frac{1}{k}}}\right) \\ &\leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \end{aligned}$$

This implies that

$$\rho\left(x_n - x_{n+2}\right) \leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \tag{10}$$

Now we have the following two cases.

CASE 1: If $m > 2$ is odd, then $m = 2L+1$, $L \geq 1$, using (9) for all $n \geq h$, $h = \max(n_0, n_1)$

$$\begin{aligned} \rho\left(x_n - x_{n+m}\right) &\leq \omega(2L+1)\left[\rho\left(x_n - x_{n+1}\right) + \rho\left(x_{n+1} - x_{n+2}\right) + \dots + \rho\left(x_{n+2L} - x_{n+2L+1}\right)\right] \\ &\leq \omega(2L+1)\left[\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \dots + \frac{1}{(n+2L)^{\frac{1}{k}}}\right] \\ &\leq \omega(2L+1)\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

CASE 2: If $m > 2$ is even, then $m = 2L$, $L \geq 2$, using (9) and (10) for all $n \geq h$, $h = \max(n_0, n_1)$

$$\begin{aligned} \rho\left(x_n - x_{n+m}\right) &\leq \omega(2L)\left[\rho\left(x_n - x_{n+2}\right) + \rho\left(x_{n+2} - x_{n+3}\right) + \dots + \rho\left(x_{n+2L-1} - x_{n+2L}\right)\right] \\ &\leq \omega(2L)\left[\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+2)^{\frac{1}{k}}} + \dots + \frac{1}{(n+2L-1)^{\frac{1}{k}}}\right] \\ &\leq \omega(2L)\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Combining these two cases, we have

$$\rho\left(x_n - x_{n+m}\right) \leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}} \text{ for all } n \geq h, m \in \mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent $\left(\text{as } \frac{1}{k} > 1\right)$, we deduce that $\{x_n\}$ is a Cauchy sequence. Now X_ρ is complete and A is a ρ-closed subset of X_ρ, there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is ρ-continuous, Tx_n is ρ-convergent to Tx^*. Hence the continuity of the modular ρ implies that $\rho\left(x_{n+1} - Tx_n\right)$ $\rho-$converges to $\rho\left(x^* - Tx^*\right)$ and by (2),we have

$$\rho\left(x^* - Tx^*\right) = d_\rho(A,B).$$

That is, x^* is a best proximity point of T.

Next, we show the uniqueness of the best proximity point. Let us suppose that T has two best proximity points x_1 and $x_2 \in A$, such that $x_1 \neq x_2$ and $\rho\left(x_1 - Tx_1\right) = \rho\left(x_2 - Tx_2\right) = d_\rho(A,B)$.

Then by the $P-$property of (A,B), we have $\rho(x_1-x_2)=\rho(Tx_1-Tx_2)$. Note that $\rho(x_1-x_2)>0$ as $x_1\neq x_2$, T is F_ρ-contraction and ρ is an increasing function, thus

$$\begin{aligned} F(\rho(c(x_1-x_2))) &= F(\rho(c(Tx_1-Tx_2)))\leq F(\rho(l(x_1-x_2)))-\tau \\ &\leq F(\rho(c(x_1-x_2)))-\tau<F(\rho(c(x_1-x_2))), \end{aligned}$$

which is a contradiction. Hence the best proximity point is unique. □

In support of our above theorem $(Theorem\ 3.1)$,we give the following example.

Example 3.1. Let $X_\rho=\mathbb{R}$, $A=\left\{x\in X_\rho:-1\leq x\leq 1\right\}=A_0$, $B=\left\{x\in X_\rho:\ -2\leq x\leq 2\right\}=B_0$, $T:A\rightarrow B$ be defined by $T(x)=\frac{|x|}{2}+\frac{1}{2}$, $\rho(x)=|x|$, $F_\rho(\alpha)=\ln\alpha$, $\tau=\ln(2)$. Then there exists a unique best proximity point $x=1\in A$, such that $\rho(1-T(1))=d_\rho(A,B)$ .Clearly$T(A_0)\subseteq B_0$. Next

$$\begin{aligned} \rho(Tx-Ty) &= |Tx-Ty| \\ &= \left|\frac{|x|}{2}-\frac{|y|}{2}\right|\leq\frac{1}{2}|x-y|=\frac{1}{2}\rho(x-y). \end{aligned}$$

Taking ln on both side, we have $\ln(\rho(Tx-Ty))\leq\ln\left(\frac{1}{2}\rho(x-y)\right)=\ln\left(\frac{1}{2}\right)+\ln(\rho(x-y))$. This yields $F_\rho(\rho(Tx-Ty))\leq F_\rho(\rho(x-y))-\tau$, showing that T is $F_\rho-$contraction

Further note that $d_\rho(A,B)=0$.To verify property P, let $x_1=-1\in A_0$, $y_1=-1\in B_0$. Then $\rho(x_1-y_1)=|-1+1|=0=d_\rho(A,B)$. Next let $x_2=1\in A_0$ and $y_2=1\in B_0$.Then $\rho(x_2-y_2)=|1-1|=0=d_\rho(A,B)$ and $\rho(x-Tx)=|1-T(1)|=\left|1-\left(\frac{|1|}{2}+\frac{1}{2}\right)\right|=0=d_\rho(A,B)$. That is $\rho(x_1-x_2)=|-1-1|=2=\rho(y_1-y_2)=|-1-1|=2$. Implies that $\rho(x_1-x_2)=\rho(y_1-y_2)$.

If we define $T(f)$ in a different way, we will have a different unique best proximity point such as for $T(f)=\frac{|f|}{2}+1$, $T(f)=\frac{|f|}{2}+\frac{3}{4}$, best proximity points are 0 and $\frac{1}{4}$ respectively.

If we put $A=B$ in Theorem 3.1, we get the following important corollary where the best proximity points become fixed points.

Corollary 3.1. Let X_ρ be a modular function space, and A a nonempty ρ-closed subset of X_ρ. Let $T:A\rightarrow A$ be $F_\rho-$contractive self map. Then T has a unique fixed point x^*in A.

We give the following example to validate the above corollary.

Example 3.2. Let $X_\rho=\mathbb{R}$,$A=[-2,3]$, $\rho(x)=\sqrt{x}$, $T(x)=\frac{|x|+1}{2}$, $\tau=\frac{1}{2}\ln 2$, $F(\alpha)=\ln\alpha$, then $\rho(Tx-Ty)=\sqrt{(Tx-Ty)}=\sqrt{\left(\frac{|x|}{2}-\frac{|y|}{2}\right)}\leq\sqrt{\left(\frac{1}{2}|x-y|\right)}$. That is, $\rho(Tx-Ty)\leq\sqrt{\frac{1}{2}}\rho(x-y)$.

By taking ln both sides, we have $\ln(\rho(Tx-Ty)) \leq \ln\left(\sqrt{\tfrac{1}{2}}\rho(x-y)\right) = \ln\left(\sqrt{\tfrac{1}{2}}\right) + \ln\left(\sqrt{}\rho(x-y)\right).$
That is, $\ln(\rho(Tx-Ty)) \leq \ln(\rho(x-y)) - \frac{1}{2}\ln(2)$. This implies that
$F(\rho(Tx-Ty)) \leq F(\rho(x-y)) - \tau$. Next, $1 \in A$ is the unique fixed point of T.

If we put $F(\alpha) = \ln\alpha$ in Corollary 3.1, we will get

Theorem(5.7) on page187 of Almezel et al [**?**].as special case in the form of the following corollary.

Corollary 3.2. Consider a modular, ρ and take $C \subseteq X_\rho$ such that C be nonempty, ρ-closed and ρ-bounded. Let $T: C \to C$ be a ρ-contraction. Then, T has a unique fixed point $\widetilde{x} \in C$. Moreover, for any $x \in C$, $\rho(T^n(x) - \widetilde{x}) \to 0$ as $n \to \infty$, where T^n is the n-th iterate of T.

3.2. F_ρ-proximal contractions of first kind.

In this part of the paper, we present a result using F_ρ-proximal contractions of first kind. It actually deals with the coincidence best proximity point of two mappings.

Theorem 3.2. Let A and B be nonempty ρ-closed subsets of a modular function spaces X_ρ. Also, it is supposed that A_0 is nonempty. Let $T: A \to B$ be a ρ-continuous F_ρ-proximal contraction of first kind such that $T(A_0) \subseteq B_0$. Let $g: A \to A$ be an isometry with $A_0 \subseteq g(A_0)$. Then there exists a unique element $x \in A$ such that $\rho(gx - Tx) = d_\rho(A,B)$.That is, x is the coincidence best proximity point of T and g.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, there exists x_1 in A_0 such that $\rho(gx_1 - Tx_0) = d_\rho(A,B)$. If $x_0 = x_1$ then put $x_n := x_1$ for all $n \geq 2$. Also, since $Tx_1 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, there exists x_2 in A_0 such that $\rho(gx_2 - Tx_1) = d_\rho(A,B)$. If $x_1 = x_2$, then put $x_n := x_2$ for all $n \geq 3$. Going on in this way, we get a sequence $\{x_n\}$ in A_0 such that

$$\rho(gx_{n+1} - Tx_n) = d_\rho(A,B) \text{ for all } n \in \mathbb{N}. \tag{11}$$

We now prove that the sequence $\{x_n\}$ is ρ-convergent in A_0. Without loss of real generality, we can assume that $\rho(gx_n - gx_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Since T is a ρ-continuous F_ρ-proximal contraction of the first kind, for any positive integer n, by(11), we have

$$\tau + F(\rho(c(gx_n - gx_{n+1}))) \leq F(\rho(l(x_{n-1} - x_n))),$$

or

$$\begin{aligned} F(\rho(c(x_n - x_{n+1}))) &\leq F(\rho(l(x_{n-1} - x_n))) - \tau \\ &\leq F(\rho(c(x_{n-1} - x_n))) - \tau \\ &\leq F(\rho(l(x_{n-1} - x_n))) - 2\tau. \end{aligned}$$

Inductively, we reach at

$$F(\rho(c(x_n - x_{n+1}))) \leq F(\rho(l(x_0 - x_1))) - n\tau. \tag{12}$$

Following the techniques similar to Theorem 3.1, it follows that $\{x_n\}$ is a $\rho-$Cauchy sequence in A. Thus $\lim_{n\to\infty} x_n = x$ for some $x \in A$ from the assumptions on X_ρ and A. Now continuity of ρ, T and g implies that $\rho(gx_{n+1} - Tx_n)$ $\rho-$converges to $\rho(gx - Tx)$. Thus from (11), we achieve

$$\rho(gx - Tx) = d_\rho(A, B).$$

That is, x is the coincidence best proximity point of T and g.

To show the uniqueness of the coincidence best proximity point, suppose that T and g has two coincidence best proximity points x_1 and $x_2 \in A$. Let $x_1 \neq x_2$ so $\rho(x_1 - x_2) > 0$. Exploiting the facts that T is an F_ρ-proximal contraction of first kind and g is an isometry, we can write

$$F(\rho(x_1 - x_2)) = F(\rho(gx_1 - gx_2)) \leq F(\rho(x_1 - x_2)) - \tau < F(\rho(x_1 - x_2)).$$

This is a contradiction. Hence the coincidence best proximity point of T and g is unique. □

If g is the identity mapping, then we get the following corollary.

Corollary 3.3. Let A and B be nonempty ρ-closed subsets of a modular function spaces X_ρ such that A is approximately ρ-compact with respect to B. Further, suppose that A_0 is nonempty. If $T : A \to B$ is a ρ-continuous F_ρ-proximal contraction with $T(A_0) \subseteq B_0$. Then T has a unique best proximity point in A.

3.3. Ciric type generalized F_ρ-proximal contractions.

Here we extend our idea of F_ρ-proximal contraction to Ciric type generalized F_ρ-proximal contraction and prove some related results. Our first result in this direction reads as follows.

Theorem 3.3. Let X_ρ be a complete modular function space and A,B nonempty ρ-closed subset of X_ρ. Further, let $T : A \to B$ be a continuous Ciric type generalized F_ρ-proximal contraction with $T(A_0) \subseteq B_0$. Let $g : A \to A$ be an isometry with $A_0 \subseteq g(A_0)$. Then, there exists a unique element x in A such that $\rho(gx - Tx) = d_\rho(A,B)$.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $\rho(gx_1 - Tx_0) = d_\rho(A,B)$. Put $x_n := x_1$ for all $n \geq 2$ if $x_0 = x_1$. Since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $\rho(gx_2 - Tx_1) = d_\rho(A,B)$. If $x_1 = x_2$, then put $x_n := x_2$ for all $n \geq 3$. A sequence $\{x_n\}$ in A_0 can be obtained in this way such that

$$\rho(gx_{n+1} - Tx_n) = d_\rho(A,B) \text{ for all } n \in \mathbb{N}. \tag{13}$$

If $\rho(gx_{n_0} - gx_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$ such that, then it is clear that the sequence $\{x_n\}$ is ρ-convergent. We thus prove that the sequence $\{x_n\}$ is ρ-convergent in A_0 when $\rho(gx_n - gx_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. As g is an isometry, T is a ρ-continuous generalized F_ρ-proximal contraction, by (13), we get

$$\begin{aligned}\tau &+ F(\rho(c(x_n - x_{n+1}))) = \tau + F(\rho(c(gx_n - gx_{n+1}))) \\ &\leq F\left(\max\left\{\begin{array}{c}\rho(l(x_{n-1} - x_n)), \rho(l(x_{n-1} - x_n)), \rho(l(x_n - x_{n+1})), \\ \frac{1}{2}\left\{\rho\left(\frac{l}{2}(x_{n-1} - x_{n+1})\right) + \rho\left(\frac{l}{2}(x_n - u_n)\right)\right\}\end{array}\right\}\right),\end{aligned}$$

for all $n \in \mathbb{N}$. That is

$$\begin{aligned}&\tau + F(\rho(c(x_n - x_{n+1}))) \\ \leq\ & F\left(\max\left\{\begin{array}{c}\rho(l(x_{n-1} - x_n)), \rho(l(x_n - x_{n+1})), \\ \frac{1}{2}\left\{\rho\left(\frac{l}{2}(x_{n-1} - x_{n+1})\right)\right\}\end{array}\right\}\right),\end{aligned} \tag{14}$$

for all $n \in \mathbb{N}$. Noting

$$\frac{1}{2}\left\{\rho\left(\frac{l}{2}(x_{n-1} - x_{n+1})\right)\right\} \leq \frac{1}{2}\{\rho(l(x_{n-1} - x_n) + \rho(l(x_n - x_{n+1})))\} \tag{15}$$

and using (14), we can write

$$\begin{aligned}&\tau + F(\rho(c(x_n - x_{n+1}))) \\ \leq\ & F\left(\max\left\{\begin{array}{c}\rho(l(x_{n-1} - x_n)), \rho(l(x_n - x_{n+1})), \\ \frac{1}{2}\{\rho(l(x_{n-1} - x_n) + \rho(l(x_n - x_{n+1})))\}\end{array}\right\}\right).\end{aligned}$$

That is to say

$$\tau + F(\rho(c(x_n - x_{n+1}))) \leq F(\rho(l(x_{n-1} - x_n))).$$

This implies that

$$\begin{aligned} F(\rho(c(x_n - x_{n+1}))) &\leq F(\rho(l(x_{n-1} - x_n))) - \tau \\ &\leq F(\rho(c(x_{n-1} - x_n))) - \tau \\ &\leq F(\rho(l(x_{n-2} - x_{n-1}))) - 2\tau \\ &\leq F(\rho(c(x_{n-2} - x_{n-1}))) - 2\tau. \end{aligned}$$

Inductively, we get

$$F(\rho(c(x_n - x_{n+1}))) \leq F(\rho(l(x_0 - x_1))) - n\tau.$$

Now walking on the foot steps of Theorem (3.1), we can reach the fact that $\{x_n\}$ is a $\rho-$Cauchy sequence in A. Since X_ρ is ρ-complete and A is a ρ-closed subset of X_ρ, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$. Now making use of continuity of ρ, T and g, we can say $\rho(gx_{n+1} - Tx_n)$ ρ-converges to $\rho(gx - Tx)$. In turn, by (13), we have

$$\rho(gx - Tx) = d_\rho(A, B).$$

Hence x is the coincidence best proximity point of T and g.

Next let us suppose that T and g has two coincidence best proximity point x_1 and $x_2 \in A$ with $x_1 \neq x_2$ so that $\rho(x_1 - x_2) > 0$. Since T is a generalized F_ρ-proximal contraction, we have

$$\begin{aligned} F(\rho(x_1 - x_2)) &= F(\rho(gx_1 - gx_2)) \\ &\leq F\left(\max\left(\begin{array}{c} \rho(l(x_1 - x_2)), \rho(l(x_1 - x_1)), \rho(l(x_2 - x_2)), \\ \frac{1}{2}\left\{\rho\left(\frac{l}{2}(x_1 - x_2) + \rho\left(\frac{l}{2}(x_2 - x_1)\right)\right)\right\} \end{array}\right)\right) \\ -\tau &< F(\rho(x_1 - x_2)), \end{aligned}$$

which is a contradiction. Hence the coincidence best proximity point of T and g is unique. □

If g is the identity mapping, the following is immediate.

Corollary 3.4. Let X_ρ be a complete modular function space and A, B be nonempty ρ-closed subset of X_ρ. Further, suppose that there exists $a \in A_0$. Let $T : A \to B$ be a continuous generalized

F_ρ-proximal contraction such that $T(A_0) \subseteq B_0$. Then, there exists a unique element x in A such that

$$\rho(x - Tx) = d_\rho(A,B).$$

If we choose $F(\alpha) = \ln(\alpha)$ in Theorem 3.3, then Ciric type generalized F_ρ-proximal contraction becomes a Ciric type generalized ρ-proximal contraction and we get the following corollary

Corollary 3.5. Let X_ρ be a complete modular function space and A, B be nonempty ρ-closed subset of X_ρ. Further, let $T : A \to B$ be continuous Ciric type generalized $\rho-$proximal contraction with $T(A_0) \subseteq B_0$, and $g : A \to A$ be an isometry with $A_0 \subseteq g(A_0)$. Then there exists a unique $x \in A$ such that

$$\rho(gx - Tx) = d_\rho(A,B).$$

3.4. F_ρ-proximal-quasi contractions.

This portion of the paper is dedicated to the best proximity point results of F_ρ-proximal-quasi contractions in modular function spaces. Before proceeding to our main target, we need to gather some basics as follows.

Lemma 3.1. [11] Let $T : A \to B$ be a nonself mapping with $T(A) \subseteq B_0$. Suppose that $A_0 \neq \phi$.Then, for any $a \in A_0$, there exists a sequence $\{x_n\} \subset A_0$ such that $x_0 = a$ and

$$\rho(x_{n+1} - Tx_n) = dist_\rho(A,B) \text{ for all } n \in \mathbb{N}. \tag{16}$$

Proof. Let $a \in A_0$, then $T(a) \in B_0$. Thus there exists $x_1 \in A_0$ such that $\rho(x_1 - T(a)) = dist_\rho(A,B)$ owing to B_0. Again, we have $T(x_1) \in B_0$, which implies that there exists $x_2 \in A_0$ such that $\rho(x_2 - T(x_1)) = dist_\rho(A,B)$. Continuing this process, we obtain a sequence $\{x_n\} \subset A_0$ satisfying $\rho(x_{n+1} - Tx_n) = dist_\rho(A,B)$ for all $n \in \mathbb{N}$.The sequence $\{x_n\} \subset A_0$ satisfying (16) is called a proximal Picard sequence associated to $a \in A_0$. We denote by $PP(a)$ the set of all proximal sequences associated to $a \in A_0$. The set A_0 is called proximal $T-$orbitally $\rho-$complete if every $\rho-$Cauchy sequence $\{x_n\} \in PP(a)$ for some $a \in A_0$ ρ-convergence to an element in A_0. Let $a \in A_0$ and $\{x_n\} \in PP(a)$. For all $n \in \mathbb{N}$, we denote

$$\delta_\rho(x_n) := \sup\{\rho(x_{n+s} - x_{n+r}) : r, s \in \mathbb{N}\}. \tag{17}$$

Since $x_0 = a$, we have

$$\delta_\rho(a) := \sup\{\rho(x_s - x_r) : r, s \in \mathbb{N}\}. \tag{18}$$

□

Theorem 3.4. Let X_ρ be a complete modular function space and A, B be nonempty ρ-closed subset of X_ρ. Further, let $T : A \to B$ be a continuous F_ρ-proximal-quasicontraction with $T(A_0) \subseteq B_0$. Let $g : A \to A$ be an isometry with $A_0 \subseteq g(A_0)$. Suppose further that there exists $a \in A_0$ such that $\delta_\rho(a) < \infty$. Then there exists a unique element x in A such that

$$\rho(gx - Tx) = d_\rho(A, B).$$

Proof. Let $x_n \in PP(a)$ and $(s, r) \in \mathbb{N}^2$. From the definition of $PP(a)$, for all $n \geq 1$, we have

$$\rho(x_{n+s} - Tx_{n-1+s}) = \rho(gx_{n+s} - Tx_{n-1+s}) = dist_\rho(A, B),$$

and

$$\rho(x_{n+r} - Tx_{n-1+r}) = \rho(gx_{n+r} - Tx_{n-1+r}) = dist_\rho(A, B).$$

Because of the fact that T is F_ρ-proximal-quasi contraction, we have for all $n \geq 1$ and $(s, r) \in \mathbb{N}^2$,

$$\begin{aligned} F(\rho(x_{n+s} - x_{n+r})) - \tau &= F(\rho(gx_{n+s} - gx_{n+r})) - \tau \\ &\leq F(\max\{\rho(x_{n-1+s} - x_{n-1+r}), \rho(x_{n-1+s} - x_{n+s}), \\ &\quad \rho(x_{n-1+r} - x_{n+r}), \rho(x_{n-1+s} - x_{n+r}) + \rho(x_{n-1+r} - x_{n+s})\}). \end{aligned}$$

Since F is strictly increasing, we deduce that

$$\begin{aligned} \rho(x_{n+s} - x_{n+r}) &\leq \max\{\rho(x_{n-1+s} - x_{n-1+r}), \rho(x_{n-1+s} - x_{n+s}), \\ &\quad \rho(x_{n-1+r} - x_{n+r}), \rho(x_{n-1+s} - x_{n+r}) + \rho(x_{n-1+r} - x_{n+s})\}. \end{aligned}$$

This implies that

$$\rho(x_{n+s} - x_{n+r}) \leq \delta_\rho(x_{n-1}),$$

and consequently

$$\tau + F(\delta_\rho(x_n)) \leq F(\delta_\rho(x_{n-1})), \text{ for all } n \geq 1.$$

That is,

$$F\left(\delta_\rho\left(x_n\right)\right) \leq F\left(\delta_\rho\left(x_{n-1}\right)\right) - \tau, \text{ for all } n \geq 1.$$

Hence for any $n \in \mathbb{N}$,we have

$$F\left(\delta_\rho\left(x_n\right)\right) \leq F\left(\delta_\rho\left(a\right)\right) - n\tau$$

and so

$$\lim_{n\to+\infty} F\left(\delta_\rho\left(x_n\right)\right) = -\infty.$$

By property (C_1) of Definition 2,we get that $\lim_{n\to+\infty}\left(\delta_\rho\left(x_n\right)\right) = 0$. It is not hard now to prove that $\{x_n\}$ is a $\rho-$Cauchy sequence in A on the lines similar to Theorem 3.1. Since T is continuous, we have $Tx_n \to x$. Also by continuity of ρ and g,

$$\rho\left(gx_{n+s} - Tx_{n-1+s}\right) \to \rho\left(gx - Tx\right).$$

ans so by 13, we have

$$\rho\left(gx - Tx\right) = dist_\rho(A,B).$$

Uniqueness: Let us suppose that $x^* \in A$ is another best proximity point of T such that

$$\rho\left(gx^* - Tx^*\right) = dist_\rho\left(A - B\right),$$

Since $x^* \neq x$ so $\rho\left(x - x^*\right) \neq 0$. Now g is an isometry, T is F_ρ-proximal-quasi contraction, so we obtain the following:

$$\begin{aligned} F\left(\rho\left(x - x^*\right)\right) &= F\left(\rho\left(gx - gx^*\right)\right) \\ &\leq F\left(\max\left(\rho\left(x - x^*\right), \rho\left(x - x\right), \rho\left(x^* - x^*\right), \rho\left(x - x^*\right) + \rho\left(x^* - x\right)\right)\right) - \tau. \end{aligned}$$

This yields a contradiction on rewriting it as

$$F\left(\rho\left(x - x^*\right)\right) \leq F\left(\rho\left(x - x^*\right)\right) - \tau.$$

Hence the uniqueness. $\square$

If we take g as the identity map in Theorem 3.4, we get the following corollary.

Corollary 3.6. Let X_ρ be a complete modular function space and A, B be nonempty ρ-closed subset of X_ρ. Suppose that there exists $a \in A_0$ such that $\delta_\rho(a) < \infty$. Further, let $T : A \to B$ be a continuous F_ρ-proximal-quasi contraction satisfying $T(A_0) \subseteq B_0$. Then, there exists a unique element x in A such that

$$\rho(x - Tx) = d_\rho(A, B).$$

If we put $F(\alpha) = \ln \alpha$ in Corollary 3.4, then we get the following corollary which is Theorem 10 of Jleli et al. [11]

Corollary 3.7. Let X_ρ be a complete modular function space and A, B be nonempty ρ-closed subset of X_ρ. Suppose that there exists $a \in A_0$ such that $\delta_\rho(a) < \infty$. Further, let $T : A \to B$ be a continuous ρ-proximal-quasi contraction satisfying $T(A_0) \subseteq B_0$. Then, there exists a unique element x in A such that

$$\rho(x - Tx) = d_\rho(A, B).$$

Remark 3.1 In case $A = B$, a best proximity point of $T : A \to B$ becomes a fixed point of the self-mapping T. If we put $A = B$ in Corollary 3.4, we get Khamsi [16] result. This means that our results are more general than Khamsi [16], Jleli et al. [11], Omidvari et at. [9], etc. Both Khamsi [16] and Jleli et al. [11] have used Fatou property, but we have successfully avoided this property by involving F_ρ-contractions

Conflict of Interests

The authors declare that there is no conflict of interests.

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