Available online at http://scik.org
Adv. Fixed Point Theory, 9 (2019), No. 2, 178-205
https://doi.org/10.28919/afpt/4039
ISSN: 1927-6303

STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY k-STRICTLY PSEUDOCONTRACTIVE MAPS

E.E. CHIMA[1,*], M.O. OSILIKE[2], S.E. ODIBO[2], R.Z. NWOHA[2], P.U. NWOKORO[2], D.F. AGBEBAKU[2]

[1]Department of Mathematics, Bingham University, Karu, Nigeria Fixed Point Theory and Applications Research Group (FPTA-RG), Department of Mathematics, University of Nigeria, Nsukka, Nigeria

[2]Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Copyright © 2019 the authors. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. Let C be a nonempty closed convex subset of a real Hilbert space, H and let $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping with a nonempty fixed-point set, $F(T) = \{x \in C : Tx = x\}$. Let $\{t_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$. We consider the sequence $\{x_n\}$, generated from an arbitrary $x_1 \in C$, by either

I. $$x_{n+1} = P_C[(1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n],\ n \geq 1, \text{ or}$$

II. $$\begin{cases} v_n = P_C((1-t_n)x_n) \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n v_n,\ n \geq 1 \end{cases}$$

We prove that under some mild conditions on the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by I converges strongly to a fixed point of T. Furthermore, under some mild conditions on the sequences $\{t_n\}$ and $\{\alpha_n\}$, the sequence generated by II converges strongly to the least norm element of the fixed point set of T. Some examples are used to compare the convergence rates of these two iteration schemes. Our results compliment and extend several strong convergence results in the literature to the class of mappings considered in our work.

Keywords: asymptotically nonexpansive maps; asymptotically k-strictly pseudocontractive maps; fixed points; strong convergence; Hilbert spaces.

2010 AMS Subject Classification: 47H09, 47J25, 65J15.

---

*Corresponding author
E-mail address: epukeezinneuk@yahoo.com
Received February 27, 2019

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||.||$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0, 1)$ and T is said to be *nonexpansive* if $L = 1$ (see for example [1],[17]). T is said to be *asymptotically nonexpansive* (see for example [7],[10-11]) if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n||x - y||, \ \forall x, y \in C. \tag{1.2}$$

It is well known (see for example [7]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.3}$$

T is said to be demiclosed at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$.

Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that $z = P_C(x)$ if and only if $\langle x - z, z - y \rangle \geq 0$, $\forall y \in C$, and that P_C is nonexpansive.

A mapping $T : C \to C$ is said to be *k-strictly asymptotically pseudocontractive* if there exist $k \in [0, 1)$ and a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y||^2 \leq k_n||x - y||^2 + k||(x - T^n x) - (y - T^n y)||^2. \tag{1.4}$$

The class of k-strictly asymptotically pseudocontractive maps is more general than the class of asymptotically nonexpansive maps and each k-strictly asymptotically pseudocontractive map is uniformly L-Lipschitzian with $L = \sup_{n\geq 1} \frac{\sqrt{k_n}+\sqrt{k}}{1-\sqrt{k}}$. It is proved in [19] that the class of k-strictly asymptotically pseudocontractive maps and the class of k-stricly pseudocontractive maps (i.e., mappings satisfying $||Tx - Ty||^2 \leq ||x - y||^2 + k||x - Tx - (y - Ty)||^2$, $\forall x, y \in C$ and for some $k \in [0, 1)$, see for example [2]) are independent.

In the iterative approximation of fixed points of asymptotically nonexpansive maps, the modified averaging iterative scheme of Mann:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1; \tag{1.5}$$

and Ishikawa:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n[(1-\beta_n)x_n + \beta_n T^n x_n],\ n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are suitable sequence in $[0,1]$ have played pivotal role. These schemes were first studied by Schu ([24-25]) in 1991 and the schemes have played pivotal roles in approximation of fixed points of maps with asymptotic type behaviours (see for example [1],[4-6],[11],[18-22],[24-26]). However, these two iteration schemes yield only weak convergence usually obtained mostly from $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$; and require strong "compactness" assumption either on the operator or the domain of the operator or even both to yield strong convergence. Even for nonexpansive maps, k-strictly pseudocontractive maps and other generalizations that do not exhibit asymptotic behaviours, sometimes very strong condition are imposed on the fixed-point set, $F(T)$ to obtain strong convergence using the usual Mann or the Ishikawa iteration process (see for example [1],[6],[23],[30]) . For instance in [23], the author required that $F(T)$ is finite where T is a continuous pseudocontractive-type self-mapping of a nonempty convex compact of a Hilbert space, and in [30] the authors required that the interior of $F(T)$ is nonempty where T is a Lipschitz pseudocontractive self-mapping of a nonempty closed convex subset of a Hilbert space. Thus many other schemes have been recently studied by several authors to achieve strong convergence with mild assumptions on the operator, its domain, its set of fixed points and other necessary components (see for example [8-9],[12-16],[28],[29],[31]). It is our purpose in this paper to consider the following modified averaging iteration scheme of Krasnoselskii-Mann generated for arbitrary $x_1 \in C$ by either

$$x_{n+1} = P_C[(1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n] \quad n \geq 1, \tag{1.7}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1), \{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, are suitable real sequences in $[0,1]$ or

$$\begin{cases} v_n = P_C\Big((1-t_n)x_n\Big) \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n v_n, \end{cases} \tag{1.8}$$

where $\{t_n\}_{n=1}^{\infty} \subset (0,1), \{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, are suitable real sequences in $(0,1)$.
We prove that under some mild conditions on the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by I converges strongly to a fixed point of T. Furthermore, under some mild conditions on the sequences $\{t_n\}$ and $\{\alpha_n\}$, the sequence generated by II converges strongly to the least norm element of the fixed point set of T. Some examples are used to compare the convergence rates of these two iteration schemes. Our results compliment and extend several strong convergence results in the literature to the class of mappings considered here.

2. Preliminaries

We shall need the following results: **Lemma 2.1. ([27])** Let H be a real Hilbert space. Then,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,\ \forall x,y \in H.$$

Lemma 2.2. ([19]) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \longrightarrow C$ be a k-strictly asymptotically pseudocontractive mapping. Then $I-T$ is demiclosed at 0. i.e, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.
Lemma 2.3. ([1],[6]) Let H be a real Hilbert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \forall\, y \in H.$$

Lemma 2.4 ([12]). Let $\{\Gamma_n\}$ be a sequence of real numbers. Assume $\{\Gamma_n\}$ does not decrease at infinity, that is , there exists at least a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_k} \leq \Gamma_{n_k+1}$ for all $k \geq 0$. For every $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$, and for all $n \geq n_0$,

$$\max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

3. Main results

We begin with the following Lemma which will be useful in the proof of our results.

Lemma 3.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n + \sigma_n, n \geq 1,$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of real numbers which is bounded above, $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ satisfies $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\sum_{n=1}^{\infty}\sigma_n < \infty$. Then, $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}\gamma_n$.

Proof. Since $\{\gamma_n\}$ is bounded above, then for arbitrary but fixed $m \in N$, we define

$$\lambda_m = \sup_{n\geq m}\gamma_n.$$

Then for $n \geq m$, we obtain

$$a_{n+1} - a_n + \alpha_n(a_n - \lambda_m) \leq \sigma_n.$$

Let

$$b_n = a_n - \lambda_m \tag{3.1}.$$

Then

$$\begin{aligned}
b_{n+1} &= a_{n+1} - \lambda_m \\
&\leq (1-\alpha_n)a_n + \alpha_n\gamma_n + \sigma_n - \lambda_m \\
&= a_n - \lambda_m - \alpha_n a_n + \alpha_n\gamma_n + \sigma_n \\
&\leq a_n - \lambda_m - \alpha_n a_n + \alpha_n\lambda_m + \sigma_n \\
&= (1-\alpha_n)b_n + \sigma_n.
\end{aligned}$$

This implies that $\{b_n\}$ is bounded and so $\{a_n\}$ is also bounded. Moreover,

$$b_{n+1} \leq \Big[\prod_{j=m}^{n}(1-\alpha_j)\Big]b_m + \sum_{j=m}^{n}\sigma_j. \tag{3.2}$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, we have that $\lim_{n\to\infty}\prod_{j=m}^{n}(1-\alpha_j) = 0$. Thus, it follows from (3.2) that

$$\limsup_{n\to\infty} b_n = \limsup_{n\to\infty} b_{n+1} \leq 0,$$

which is equivalent to

$$\limsup_{n\to\infty} a_n \leq \lambda_m,$$

and letting $m \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n.$$

We now prove the following strong convergence theorems:

Theorem 3.1 Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences $\in (0,1)$. Assume that the following conditions are satisfied:

$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$

$(C2)$ $\sum_{n=1}^{\infty} (k_n - 1) < \infty$

b $(C3)$ $\beta_n \in (0, (1-k)(1-\alpha_n))$, with $\lim\limits_{n\to\infty} \beta_n = \delta > 0$

(C4) $\lim\limits_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a fixed point of T.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded.

Take $p \in F(T)$. From (1.7) , we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|P_C[(1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n] - p\| \\
&\leq \|(1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n - p\| \\
&= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n (T^n x_n - p) - \alpha_n p\| \\
&\leq \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n (T^n x_n - p)\| + \alpha_n \|p\| \qquad (3.3)
\end{aligned}$$

From (1.4) we obtain

$$\begin{aligned}
\|T^n x - T^n y\|^2 &= \|x-y\|^2 + \|x - T^n x - (y - T^n y)\|^2 \\
&\quad -2\langle x - T^n x - (y - T^n y), x - y\rangle \\
&\leq k_n\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}\langle x-T^n x-(y-T^n y),x-y\rangle &\geq \frac{(1-k)}{2}\|x-T^n x-(y-T^n y)\|^2\\ &\quad -\frac{(k_n-1)}{2}\|x-y\|^2, \end{aligned}\tag{3.4}$$

or equivalently,

$$\begin{aligned}\langle T^n x-T^n y,x-y\rangle &\leq \frac{(1+k_n)}{2}\|x-y\|^2\\ &\quad -\frac{(1-k)}{2}\|x-T^n x-(y-T^n y)\|^2. \end{aligned}\tag{3.5}$$

Using (3.3) we obtain

$$\begin{aligned}
&\|\,(1-\alpha_n-\beta_n)\,(x_n-p)+\beta_n\,(T^n x_n-p)\,\|^2\\
=\;& (1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|T^n x_n-p\|^2\\
&+2(1-\alpha_n-\beta_n)\beta_n\langle T^n x_n-p,x_n-p\rangle\\
\leq\;& (1-\alpha_n-\beta_n)^2\,\|x_n-p\|^2+\beta_n^2\left[k_n\|x_n-p\|^2+k\|x_n-T^n x_n\|^2\right]\\
&+2(1-\alpha_n-\beta_n)\beta_n\left[\frac{(1+k_n)}{2}\|x_n-p\|^2-\frac{(1-k)}{2}\|x_n-T^n x_n\|^2\right]\\
=\;& (1-\alpha_n)^2\|x_n-p\|^2+\beta_n^2\|x_n-p\|^2-2\beta_n(1-\alpha_n)\|x_n-p\|^2\\
&+\beta_n^2 k_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\beta_n(1+k_n)\|x_n-p\|^2\\
&+\beta_n^2 k\|x_n-T^n x_n\|^2-(1-k)(1-\alpha_n-\beta_n)\beta_n\|x_n-T^n x_n\|^2\\
=\;& (1-\alpha_n)^2\|x_n-p\|^2+\beta_n\left[(1-\alpha_n)(k_n-1)\right]\|x_n-p\|^2\\
&-\beta_n\left[(1-k)(1-\alpha_n)-\beta_n\right]\|x_n-T^n x_n\|^2\\
\leq\;& (1-\alpha_n)^2\|x_n-p\|^2+\beta_n\left[(1-\alpha_n)(k_n-1)\right]\|x_n-p\|^2\\
\leq\;& (1-\alpha_n)^2\|x_n-p\|^2+(1-\alpha_n)\left[(1-\alpha_n)(k_n-1)\right]\|x_n-p\|^2\\
=\;& (1-\alpha_n)^2 k_n\|x_n-p\|^2\\
\leq\;& (1-\alpha_n)^2 k_n^2\|x_n-p\|^2
\end{aligned}$$

which implies that

$$\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(T^n x_n-p)\|\leq(1-\alpha_n)k_n\|x_n-p\|. \tag{3.6}$$

It follows from (3.6) and (3.3) that

$$\begin{aligned}\|x_{n+1}-p\| &\leq k_n(1-\alpha_n)\|x_n-p\|+\alpha_n\|p\| \\ &\leq (1-\alpha_n)k_n\|x_n-p\|+\alpha_n k_n\|p\| \\ &\leq k_n\max\{\|x_n-p\|,\|p\|\}\end{aligned}$$

Hence

$$\|x_{n+1}-p\| \leq e^{k_n-1}\max\{\|x_n-p\|,\|p\|\}, \tag{3.7}$$

and it follows by induction that

$\|x_{n+1}-p\| \leq e^{\sum_{j=1}^{\infty}(k_j-1)}\max\{\|x_1-p\|,\|p\|\}$.

This implies that $\{x_n\}$ is bounded.

Using (1.7), (3.4) and Lemma 2.1, we obtain,

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|P_C[(1-\alpha_n-\beta_n)x_n+\beta_n T^n x_n]-p\|^2 \\ &\leq \|(1-\alpha_n-\beta_n)x_n+\beta_n T^n x_n-p\|^2 \\ &= \|(x_n-p)-\beta_n(x_n-T^n x_n)-\alpha_n x_n\|^2 \\ &\leq \|(x_n-p)-\beta_n(x_n-T^n x_n)\|^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle \\ &= \|(x_n-p)\|^2-2\beta_n\langle x_n-T^n x_n,x_n-p\rangle+\beta_n^2\|x_n-T^n x_n\|^2 \\ &\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle \\ &\leq \|(x_n-p)\|^2-\beta_n(1-k)\|x_n-T^n x_n)\|^2+(k_n-1)\|x_n-p\|^2 \\ &\quad +\beta_n^2\|(x_n-T^n x_n)\|^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle \text{ (using (3.4))} \\ &= [1+(k_n-1)]\|x_n-p\|^2-\beta_n\left[(1-k)-\beta_n\right]\|x_n-T^n x_n\|^2 \\ &\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle. \qquad (3.8)\end{aligned}$$

Since $\{x_n\}$ is bounded $\exists$ constants $M>0$, $D>0$ such that

$-2\langle x_n,x_{n+1}-p\rangle \leq M$; $\|x_n-p\|^2 \leq D$ for all $n \geq 0$.

Consequently from (3.8), we get

$$\|x_{n+1}-p\|^2-\|x_n-p\|^2+\beta_n\left[(1-k)-\beta_n\right]\|x_n-T^n x_n\|^2 \leq M\alpha_n+(k_n-1)D. \tag{3.9}$$

We now prove that $\{x_n\}$ converges strongly to a point in $F(T)$. We divide the proof into two cases:

Case I :- Assume that the sequence $\{\|x_n - p\|\}$ is a monotonically decreasing sequence. Then, $\{\|x_n - p\|\}$ is convergent and we have, $\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \longrightarrow 0$ as $n \to \infty$. This together with (C1) and (3.9) imply that

$\beta_n [(1-k) - \beta_n] \|x_n - T^n x_n\|^2 \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.10}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n - \beta_n)x_n + \beta_n T^n x_n - x_n\| \\
&= \|\alpha_n x_n + \beta_n (x_n - T^n x_n)\| \\
&\leq \beta_n \|x_n - T^n x_n\| + \alpha_n \|x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\
&\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\
&\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - T^{n-1}x_{n-1}\| \\
&\quad + L\|T^{n-1}x_{n-1} - x_n\| \\
&\leq \|x_n - T^n x_n\| + L^2\|x_n - x_{n-1}\| \\
&\quad + L\|T^{n-1}x_{n-1} - x_{n-1}\| + L\|x_n - x_{n-1}\| \\
&= \|x_n - T^n x_n\| + L(1+L)\|x_n - x_{n-1}\| \\
&\quad + L\|T^{n-1}x_{n-1} - x_{n-1}\| \to 0 \text{ as } n \to \infty. \qquad (3.11)
\end{aligned}$$

By Lemma 2.2 and (3.11), we see that if we define,

$w_\omega(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ the weak w-limit set of $\{x_n\}$. Then $w_\omega(x_n) \subset F(T)$. If we take x^* , $\bar{x} \in w_\omega(x_{n_i})$ and let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$ and $x_{m_j} \rightharpoonup \bar{x}$ respectively, since $\lim_{n\to\infty}\|x_n - z\|$ exists for $z \in F(T)$, then it follows from Lemma 2.3 that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - x^*\|^2 &= \lim_{j\to\infty}\|x_{m_j} - x^*\|^2 \\
&= \lim_{j\to\infty}\|x_{m_j} - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 \\
&= \lim_{i\to\infty}\|x_{n_i} - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 \\
&= \lim_{n\to\infty}\|x_n - x^*\|^2 + 2\|\bar{x} - x^*\|^2
\end{aligned}$$

Hence $\bar{x} = x^*$ and this implies that $w_\omega(x_n)$ is singleton so that $\{x_n\}$ converges weakly to a fixed point x^* in $F(T)$.

Next we prove that $\{x_n\}$ converges strongly to x^*.

Setting $y_n = (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 0$ then we can rewrite (1.7) as

$x_{n+1} = P_C(y_n - \alpha_n x_n)\ , n \geq 0$

It follows that

$$\begin{aligned}
x_{n+1} &= P_C[(1-\alpha_n)y_n - \alpha_n(x_n - y_n)] \\
&= P_C[(1-\alpha_n)y_n - \alpha_n x_n + \alpha_n[(1-\beta_n)x_n + \beta_n T^n x_n]] \\
&= P_C[(1-\alpha_n)y_n - \alpha_n\beta_n(x_n - T^n x_n)] \qquad (3.12)
\end{aligned}$$

Furthermore, we note that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n(x_n - T^n x_n)\|^2 \\
&= \|x_n - x^*\|^2 - 2\beta_n\langle x_n - T^n x_n, x_n - x^*\rangle + \beta_n^2\|x_n - T^n x_n\|^2 \\
&\leq \|x_n - x^*\|^2 + \beta_n\left[(k_n - 1)\|x_n - x^*\|^2 - (1-k)\|x_n - T^n x_n\|^2\right] \\
&\quad + \beta_n^2\|x_n - T^n x_n\|^2 \\
&\leq [1 + \beta_n(k_n - 1)]\|x_n - x^*\|^2 - \beta_n[1 - k - \beta_n]\|x_n - T^n x_n\|^2 \\
&\leq [1 + (k_n - 1)]\|x_n - x^*\|^2.
\end{aligned}$$

Applying Lemma 2.1 to (3.12), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|P_C[(1-\alpha_n)y_n-\alpha_n\beta_n(x_n-T^nx_n)]-x^*\|^2\\
&\leq \|(1-\alpha_n)y_n-\alpha_n\beta_n(x_n-T^nx_n)-x^*\|^2\\
&= \|(1-\alpha_n)y_n-(1-\alpha_n)x^*-\alpha_n\beta_n(x_n-T^nx_n)-\alpha_nx^*\|^2\\
&= \|(1-\alpha_n)(y_n-x^*)-\alpha_n\beta_n(x_n-T^nx_n)-\alpha_nx^*\|^2\\
&\leq \|(1-\alpha_n)(y_n-x^*)\|^2-2\langle\alpha_n\beta_n(x_n-T^nx_n)-\alpha_nx^*,x_{n+1}-x^*\rangle\\
&= (1-\alpha_n)^2\|y_n-x^*\|^2-2\alpha_n\beta_n\langle x_n-T^nx_n,x_{n+1}-x^*\rangle\\
&\quad -2\alpha_n\langle x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)^2\|x_n-x^*\|^2-2\alpha_n\beta_n\langle x_n-T^nx_n,x_{n+1}-x^*\rangle\\
&\quad -2\alpha_n\langle x^*,x_{n+1}-x^*\rangle+(1-\alpha_n)^2(k_n-1)\|x_n-x^*\|^2. \qquad (3.13)
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\langle x_n-T^nx_n,x_{n+1}-x^*\rangle=0$, $\lim\limits_{n\to\infty}\langle x^*,x_{n+1}-x^*\rangle=0$ and $\sum_{n=1}^{\infty}(k_n-1)<\infty$, it follows from Lemma 3.1 that $\lim\limits_{n\to\infty}\|x_n-x^*\|=0$

Thus, $x_n\longrightarrow x^*$ as $n\to\infty$.

Case 2: Assume that $\{\|x_n-p\|\}$ is not a monotone decreasing sequence.

Set $\Gamma_n=\|x_n-p\|^2$ and let $\tau:N\longrightarrow N$ be a mapping for all $n\geq n_0$ for some n_0 large enough by $\tau(n)=max\{k\in N:k\leq n,\ \Gamma_k\leq\Gamma_{k+1}\}$

Clearly, τ is a non decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ for $n\geq n_0$

From (3.9) we see that

$$\begin{aligned}
&\|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2+\beta_{\tau(n)}[(1-k)-\beta_{\tau(n)}]\|x_{\tau(n)}-T^{\tau(n)}x_{\tau(n)}\|^2\\
&\leq M\alpha_{\tau(n)}+(k_{\tau(n)}-1)D.
\end{aligned}$$

It follows that,

$$\|x_{\tau(n)}-T^{\tau(n)}x_{\tau(n)}\|^2\leq\frac{M\alpha_{\tau(n)}+(k_{\tau(n)}-1)D}{\beta_{\tau(n)}\left[(1-k)-\beta_{\tau(n)}\right]}\longrightarrow 0.$$

By the argument similar to case 1, we conclude that $x_{\tau(n)}$ converges weakly to x^* as $\tau(n)\longrightarrow\infty$.

Furthermore, we note that for all $n\geq n_0$

$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2$

Observe from (3.13) that,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \alpha_n[2\beta_n\langle x_n - T^n x_n, x^* - x_{n+1}\rangle \\ &+2\langle x^*, x^* - x_{n+1}\rangle - \|x_n - x^*\|^2] \\ &+(1-\alpha_n)^2(k_n - 1)\|x_n - x^*\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 &\leq \alpha_n[2\beta_n\langle x_n - T^n x_n, x^* - x_{n+1}\rangle \\ &+2\langle x^*, x^* - x_{n+1}\rangle - \|x_n - x^*\|^2] \\ &+(1-\alpha_n)^2(k_n - 1)\|x_n - x^*\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}0 &\leq \alpha_{\tau(n)}[2\beta_{\tau(n)}\langle x_{\tau(n)} - T^{\tau(n)} x_{\tau(n)}, x^* - x_{\tau(n)+1}\rangle \\ &+2\langle x^*, x^* - x_{\tau(n)+1}\rangle - \|x_{\tau(n)} - x^*\|^2] \\ &+(1-\alpha_{\tau(n)})^2(k_{\tau(n)} - 1)\|x_{\tau(n)} - x^*\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{\tau(n)} - x^*\|^2 &\leq 2\beta_{\tau(n)}\langle x_{\tau(n)} - T^{\tau(n)} x_{\tau(n)}, x^* - x_{\tau(n)+1}\rangle \\ &+2\langle x^*, x^* - x_{\tau(n)+1}\rangle \\ &+\frac{(1-\alpha_{\tau(n)})^2(k_{\tau(n)} - 1)}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x^*\|^2.\end{aligned}$$

Hence we deduce that $\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0$, and hence $\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$

Hence, $\lim_{n\to\infty}\Gamma_n = 0$, this is, $\{x_n\}$ converges strongly to x^* . This completes the proof.

Corollary 3.1. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H with zero in C. Let $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences $\in (0,1)$. Assume that the following conditions are satisfied:

$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0;$ $\sum_{n=1}^{\infty} \alpha_n = \infty$

$(C2)$ $\sum_{n=1}^{\infty} (k_n - 1) < \infty$

$(C3)$ $\beta_n \in (0,(1-k)(1-\alpha_n))$, with $\lim\limits_{n\to\infty} \beta_n = \delta > 0$

(C4) $\lim\limits_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0.$

Then the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1$$

converges strongly to a fixed point of T.

Remark 3.1. If zero is not necessarily in C in Corollary 3.1, one can still dispense with P_C by generating the iteration sequence $\{x_n\}$ from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1-\alpha_n-\beta_n)x_n + \beta_n T^n x_n + \alpha_n x_1,\ n \geq 1.$$

Corollary 3.2. Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. Assume that the following conditions are satisfied:

$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0;$ $\sum_{n=1}^{\infty} \alpha_n = \infty$

$(C2)$ $\sum_{n=1}^{\infty} (k_n - 1) < \infty$

$(C3)$ $\beta_n \in [\varepsilon,(1-\alpha_n)]$ $for\ some\ \varepsilon\ >\ 0$

(C4) $\lim\limits_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0.$

Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a fixed point of T.

Prototype for our real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are:
$\alpha_n = \sqrt{k_n - 1} + \frac{1}{n+1}$ and $\beta_n = \frac{1}{2}(1-k)(1-\alpha_n)$, $n \geq 1$.

Theorem 3.2. Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be k-strictly asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$ and a sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}_{n=1}^{\infty}$ by (1.8) where $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions given below:

i $\lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$

ii $\{\alpha_n\}_{n=1}^{\infty} \subset (0,b) \subset (0,1)$ with $0 < b < 1-k$, where k is the constant appearing in (1.4)

iii $\lim_{n\to\infty} \frac{t_n}{\alpha_n} \to 0$ as $n \to \infty$.

Then the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of the fixed point set of T.

Proof: Let $p \in F(T)$. Then

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||v_n - \alpha_n v_n + \alpha_n T^n v_n - p||^2 \\ &= ||v_n - p - \alpha_n(v_n - T^n v_n)||^2 \\ &= ||v_n - p||^2 + \alpha_n^2 ||v_n - T^n v_n||^2 \\ &\quad -2\alpha_n \langle v_n - T^n v_n, v_n - p \rangle . \qquad (3.14) \end{aligned}$$

Using (3.4) in (3.14) yields:

$$\begin{aligned} ||x_{n+1} - p||^2 &\le ||v_n - p||^2 + \alpha_n^2 ||v_n - T^n v_n||^2 \\ &\quad -\alpha_n(1-k)||v_n - T^n v_n||^2 + \alpha_n(k_n^2 - 1)||v_n - p||^2 \\ &= [1 + \alpha_n(k_n^2 - 1)]||v_n - p||^2 - \alpha_n(1 - k - \alpha_n)||v_n - T^n v_n||^2 \\ \text{But } [1 + \alpha_n(k_n^2 - 1)] &\le (1 + k_n^2 - 1) = k_n^2, \quad (\text{since } \{\alpha_n\} < 1 \ \forall \ n \in \mathbb{N}.) \end{aligned}$$

So,

$$||x_{n+1} - p||^2 \le k_n^2 ||v_n - p||^2 - \alpha_n(1 - k - \alpha_n)||v_n - T^n v_n||^2. \qquad (3.15)$$

From (1.8)

$$v_n - T^n v_n = \frac{1}{\alpha_n}(v_n - x_{n+1}), \qquad (3.16)$$

and using (3.16) in (3.15), we obtain

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq k_n^2||v_n-p||^2-\alpha_n(1-k-\alpha_n)\frac{1}{\alpha_n^2}||v_n-x_{n+1}||^2\\
&= k_n^2||v_n-p||^2-(1-k-\alpha_n)\frac{1}{\alpha_n}||v_n-x_{n+1}||^2 \qquad (3.17)
\end{aligned}
$$

since $0<\alpha_n\leq b<1-k$, we have that $(1-k-\alpha_n)\frac{1}{\alpha_n}\geq(1-k-b)\frac{1}{\alpha_n}>0$ and hence it follows from (3.17) that

$$
||x_{n+1}-p||\leq k_n||v_n-p||. \tag{3.18}
$$

Furthermore,

$$
\begin{aligned}
||v_n-p|| &= ||P_C[(1-t_n)x_n]-p||\\
&\leq ||(1-t_n)x_n-p||\\
&= ||(1-t_n)(x_n-p)-t_np||\\
&\leq (1-t_n)||x_n-p||+t_n||p||. \qquad (3.19)
\end{aligned}
$$

Using (3.19) in (3.18), we obtain:

$$
\begin{aligned}
||x_{n+1}-p|| &\leq k_n(1-t_n)||x_n-p||+k_nt_n||p||\\
&\leq k_n(1-t_n)\max\{||x_n-p||,||p||\}+k_nt_n\max\{||x_n-p||,||p||\}\\
&= k_n\max\{||x_n-p||,||p||\}\\
&\leq k_nk_{n-1}\max\{\|x_{n-1}-p\|,\|p\|\}\\
&\vdots\\
&\leq \prod_{i=1}^{n}k_i\max\{||x_1-p||,||p||\}<\infty,
\end{aligned}
$$

This implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Furthermore,

$$
\begin{aligned}
||v_n-p||^2 &\leq ||(1-t_n)(x_n-p)-t_np||^2\\
&= (1-t_n)^2||x_n-p||^2+t_n^2||p||^2+2t_n(1-t_n)\langle p-x_n,p\rangle. \qquad (3.20)
\end{aligned}
$$

Using (3.20) in (3.17) gives

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq k_n^2\Big[(1-t_n)^2||x_n-p||^2+t_n^2||p||^2\\
&\quad +2t_n(1-t_n)\langle p-x_n,p\rangle\Big]-(1-k-\alpha_n)\frac{1}{\alpha_n}||v_n-x_{n+1}||^2\\
&= (1-t_n)^2||x_n-p||^2+(k_n^2-1)(1-t_n)^2||x_n-p||^2+k_n^2t_n^2||p||^2\\
&\quad +2k_n^2t_n(1-t_n)\langle p-x_n,p\rangle-(1-k-\alpha_n)\frac{1}{\alpha_n}||v_n-x_{n+1}||^2\\
&= (1-t_n)^2||x_n-p||^2-t_n\Big[-2k_n^2(1-t_n)\langle p-x_n,p\rangle\\
&\quad -k_n^2t_n||p||^2+(1-k-\alpha_n)\frac{1}{\alpha_nt_n}||v_n-x_{n+1}||^2\Big]\\
&\quad +(k_n^2-1)(1-t_n)^2||x_n-p||^2.
\end{aligned}$$

Hence

$$||x_{n+1}-p||^2\leq(1-t_n)^2||x_n-p||^2-t_nY_n+\sigma_n \tag{3.21}$$

where

$$Y_n=-2k_n^2(1-t_n)\langle p-x_n,p\rangle-k_n^2t_n||p||^2+(1-k-\alpha_n)\frac{1}{\alpha_nt_n}||v_n-x_{n+1}||^2 \qquad (3.22),$$

and $\sigma_n=(k_n^2-1)(1-t_n)^2||x_n-p||^2$. Since $\{(1-k-\alpha_n)\frac{1}{\alpha_nt_n}||v_n-x_{n+1}||^2\}$ is bounded below, and $\{-2k_n^2(1-t_n)\langle p-x_n,p\rangle\}$ and $\{k_n^2t_n||p||^2\}$ are bounded, then Y_n is bounded below. So, using (3.22) and the condition $\sum\limits_{n=1}^{\infty}t_n=\infty$, it follows from Lemma 3.1 that

$$\limsup_{n\to\infty}||x_n-p||^2\leq\limsup_{n\to\infty}(-Y_n)=-\liminf_{n\to\infty}Y_n. \tag{3.23}$$

This implies that

$$\liminf_{n\to\infty}Y_n\leq-\limsup_{n\to\infty}||x_n-p||^2<\infty. \tag{3.24}$$

But

$$\liminf_{n\to\infty}Y_n=\liminf_{n\to\infty}[-2\langle p-x_n,p\rangle+(1-k-\alpha_n)\frac{1}{\alpha_nt_n}\|v_n-x_{n+1}\|^2]. \tag{3.25}$$

Thus, by the property of liminf, there exists a subsequence $\{Y_{n_j}\}_{j=1}^{\infty}$ of $\{Y_n\}_{n=1}^{\infty}$ such that

$$\liminf_{n\to\infty}Y_n \;=\; \liminf_{j\to\infty}Y_{n_j}.$$

This implies that

$$\liminf_{n\to\infty} Y_n = \lim_{j\to\infty}\left[-2\left\langle p-x_{n_j},p\right\rangle+\frac{(1-k-\alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j}-x_{n_j+1}||^2\right]. \tag{3.26}$$

Since $\liminf_{n\to\infty} Y_n<\infty$, it follows that

$$\lim_{j\to\infty}\left[-2\left\langle p-x_{n_j},p\right\rangle+\frac{(1-k-\alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j}-x_{n_j+1}||^2\right]<\infty.$$

Thus, since $\{x_n\}$ is bounded we have that

$$\left\{\frac{(1-k-\alpha_{n_j})}{\alpha_{n_j}}\frac{1}{t_{n_j}}||v_{n_j}-x_{n_j+1}||^2\right\}_{n=1}^{\infty}$$

is bounded. Furthermore, from condition (ii) we have $-\alpha_n>-b>-(1-k)$ and hence

$$1-k-\alpha_n>1-k-b>0.$$

Thus

$$\frac{(1-k-\alpha_n)}{\alpha_n t_n}||v_n-x_{n+1}||^2>\frac{(1-k-b)}{\alpha_n t_n}||v_n-x_{n+1}||^2,$$

and hence

$$\frac{1}{\alpha_{n_j}t_{n_j}}||v_{n_j}-x_{n_j+1}||^2<\frac{(1-k-\alpha_{n_j})}{(1-k-b)\alpha_{n_j}t_{n_j}}||v_{n_j}-x_{n_j+1}||^2.$$

It follows that $\{\frac{1}{\alpha_{n_j}t_{n_j}}||v_{n_j}-x_{n_j+1}||^2\}$ is bounded. From (3.16) we have

$$\begin{aligned}||v_{n_j}-T^{n_j}v_{n_j}||^2 &= \frac{1}{\alpha_{n_j}^2}||v_{n_j}-x_{n_j+1}||^2\\ &= \frac{t_{n_j}}{\alpha_{n_j}}\frac{||v_{n_j}-x_{n_j+1}||^2}{\alpha_{n_j}t_{n_j}}\to 0 \text{ as } j\to\infty,\end{aligned}$$

since $\frac{t_{n_j}}{\alpha_{n_j}}\to 0$ as $j\to\infty$, and $\{\frac{1}{\alpha_{n_j}t_{n_j}}||v_{n_j}-x_{n_j+1}||^2\}$ is bounded. Observe that

$$\begin{aligned}||x_{n+1}-x_n|| &= ||(1-\alpha_n)v_n+\alpha_n T^n v_n-x_n||\\ &= ||v_n-x_n-\alpha_n(v_n-T^n v_n)||\\ &\le t_n||x_n||+\alpha_n||v_n-T^n v_n||.\end{aligned}$$

Thus,

$$||x_{n_j+1}-x_{n_j}||\le t_{n_j}||x_{n_j}||+\alpha_{n_j}||v_{n_j}-T^{n_j}v_{n_j}||\to 0 \text{ as } j\to\infty.$$

Also

$$
\begin{aligned}
||v_{n_j} - x_{n_j+1}|| &\leq ||v_{n_j} - x_{n_j}|| + ||x_{n_j} - x_{n_j+1}|| \\
&\leq t_{n_j}||x_{n_j}|| + ||x_{n_j} - x_{n_j+1}|| \to 0 \text{ as } j \to \infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|v_{n_j-1} - T^{n_j-1}v_{n_j}\| &\leq \|v_{n_j-1} - T^{n_j-1}v_{n_j-1}\| + \|T^{n_j-1}v_{n_j-1} - T^{n_j-1}v_{n_j}\|^2 \\
&\leq \|v_{n_j-1} - T^{n_j-1}v_{n_j-1}\| + L\|v_{n_j-1} - v_{n_j}\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|v_{n_j-1} - v_{n_j}\| &\leq \|(1-t_{n_j-1})x_{n_j-1} - (1-t_{n_j})x_{n_j}\| \\
&\leq \|x_{n_j-1} - x_{n_j}\| + \|t_{n_j-1}x_{n_j-1} - t_{n_j}x_{n_j}\| \\
&= \|x_{n_j-1} - x_{n_j}\| + \|t_{n_j-1}x_{n_j-1} - t_{n_j-1}x_{n_j} + t_{n_j-1}x_{n_j} - t_{n_j}x_{n_j}\| \\
&\leq \|x_{n_j-1} - x_{n_j}\| + t_{n_j-1}\|x_{n_j-1} - x_{n_j}\| + |t_{n_j-1} - t_{n_j}|\|x_{n_j}\| \\
&\to 0 \text{ as } j \to \infty.
\end{aligned}
$$

Thus,

$$||v_{n_j-1} - T^{n_j-1}v_{n_j}||^2 \to 0 \text{ as } n \to \infty.$$

Hence,

$$
\begin{aligned}
||v_{n_j} - Tv_{n_j}|| &\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + ||T^{n_j}v_{n_j} - Tv_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j} - v_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j} - T^{n_j-1}v_{n_j-1}|| + L||T^{n_j-1}v_{n_j-1} - v_{n_j}|| \\
&\leq ||v_{n_j} - T^{n_j}v_{n_j}|| + L||T^{n_j-1}v_{n_j-1} - v_{n_j-1}|| \\
&\quad + L(1+L)||v_{n_j} - v_{n_j-1}|| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Since $(I-T)$ is demiclosed at 0, it follows that any weak cluster point of $\{v_{n_j}\}$ is a fixed point of T. Furthermore, since $F(T)$ is closed and convex, $P_{F(T)} : H \to F(T)$ is well defined. Thus, for $p = P_{F(T)}(0)$, we obtain from (3.26) and a property of $P_{F(T)}$ that

$$\begin{aligned} \liminf_{n\to\infty} Y_n &= -2\liminf_{j\to\infty}[\langle x_{n_j} - P_{F(T)}(0), P_{F(T)}(0)\rangle] \leq 0. \\ -\liminf_{n\to\infty} Y_n &= 2\liminf_{j\to\infty}[\langle x_{n_j} - P_{F(T)}(0), P_{F(T)}(0)\rangle] \geq 0. \end{aligned}$$

Consequently, from (3.23) we have that

$$0 \leq \limsup_{n\to\infty} ||x_n - P_{F(T)}(0)||^2 \leq -\liminf_{n\to\infty} Y_n \leq 0$$

$$\text{i.e.,} \quad \limsup_{n\to\infty} ||x_n - P_{F(T)}(0)|| = 0$$

Clearly $\|P_{F(T)}(0)\| = \|p\| \leq \|y\| \;\forall\, y \in F(T)$.

Thus, the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of fixed point set of T.

Corollary 3.3. Let H be a real Hilbert space. Let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$, $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. For arbitrary $x_1 \in C$ define the sequence $\{x_n\}_{n=1}^{\infty}$ by (1.8), where $\{t_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions given below:

- i $\quad \lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$
- ii $\quad \{\alpha_n\}_{n=1}^{\infty} \subset (0,b) \subset (0,1)$
- iii $\quad \frac{t_n}{\alpha_n} \to 0 \; as \; n \to \infty.$

Then the generated sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the least norm element of the fixed point set of T.

Remark 3.2. If zero is in C (in particular if $C = H$), then P_C can be dispensed with in the iteration scheme (1.8).

If zero is not necessarily in C, one can still dispense with P_C by generating the iteration sequence $\{x_n\}$ from an arbitrary $x_1 \in C$ by

$$\begin{aligned} v_n &= (1-t_n)x_n + t_n x_1 \\ x_{n+1} &= (1-\alpha_n)v_n + \alpha_n T^n v_n \quad n \geq 1. \end{aligned}$$

Prototype for the sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are respectively given by

$$\begin{aligned} t_n &= \frac{1}{n+1} \\ \alpha_n &= \frac{1-k}{\sqrt{n+1}} \end{aligned}$$

4. Numerical Examples

Example 4.1. Let $\Re$ denote the reals with the usual norm and define $T:\Re\to\Re$ by

$$Tx = \begin{cases} -3x, \ x\in(-\infty,0] \\ x, \ x\in(0,\infty) \end{cases}$$

$\forall x,y\in(-\infty,0]$, we obtain $|Tx-Ty|^2 = 9|x-y|^2,\ |x-Tx-(y-Ty)|^2 = 16|x-y|^2$, and hence

$$|Tx-Ty|^2 = 9|x-y|^2 = |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2.$$

Observe also that $\forall x,y\in(0,\infty)$ we have

$$|Tx-Ty|^2 = |x-y|^2 \leq |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2.$$

Furthermore, for all $x\in(-\infty,0]$ and $y\in(0,\infty)$ we have
$|Tx-Ty|^2 = |-3x-y|^2 = 9x^2+6xy+y^2$ and

$$\begin{aligned} |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2 &= |x-y|^2+\frac{1}{2}|4x|^2 \\ &= x^2-2xy+y^2+8x^2 \\ &= 9x^2+6xy+y^2-8xy \geq 9x^2+6xy+y^2 = |Tx-Ty|^2. \end{aligned}$$

Thus

$$|Tx-Ty|^2 \leq |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2,\ \forall x,y\in\Re.$$

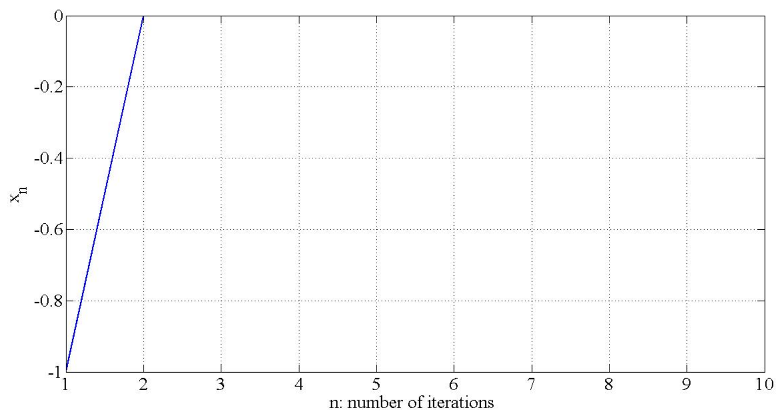
Observe that for all integer $n\geq 2$ we have

$$T^n x = \begin{cases} -3x, \ x\in(-\infty,0] \\ x, \ x\in(0,\infty). \end{cases}$$

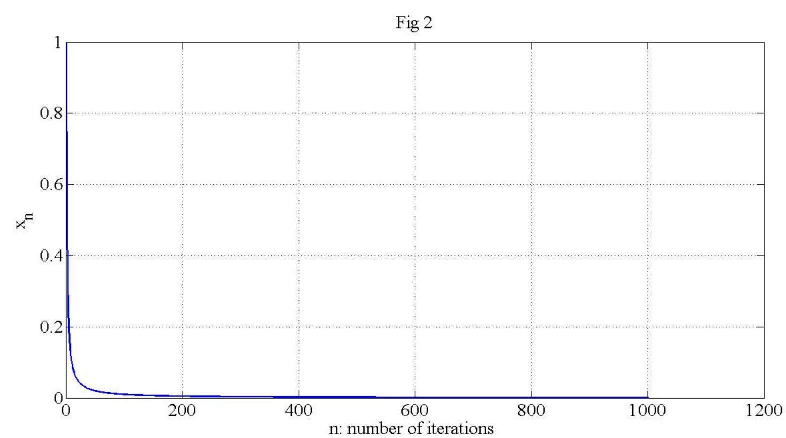
Thus for all $x,y\in\Re$, $n\in\mathbb{N}$ we have

$$|T^n x-T^n y|^2 \leq |x-y|^2+\frac{1}{2}|x-T^n x-(y-T^n y)|^2,\ \forall x,y\in\Re.$$

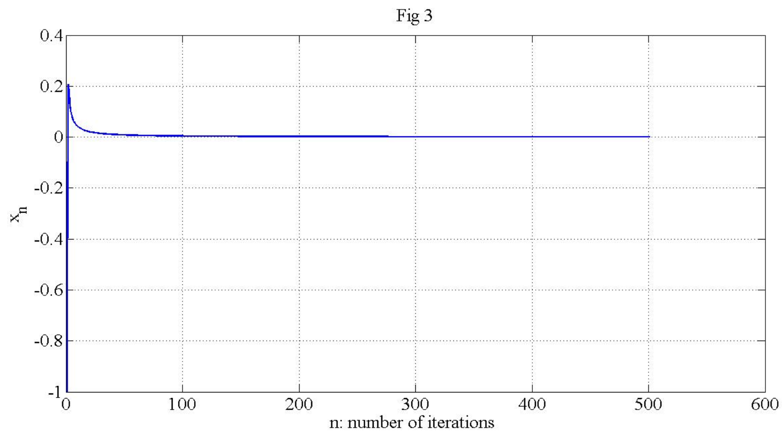
It follows that T is strictly asymptotically pseudocontractive. Observe that T is not asymptotically nonexpansive; $F(T) = [0,\infty)$; $k_n = 1$, $\forall n \geq 1$; $k = \frac{1}{2}$ and for the iteration (1.7), we can take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}(1-k)(1-\alpha_n) = \frac{n}{4(n+1)}$, $\forall n \geq 1$, $x_1 = -1$ and the sequence $\{x_n\}$ converges to zero as shown in Figure 1 below.



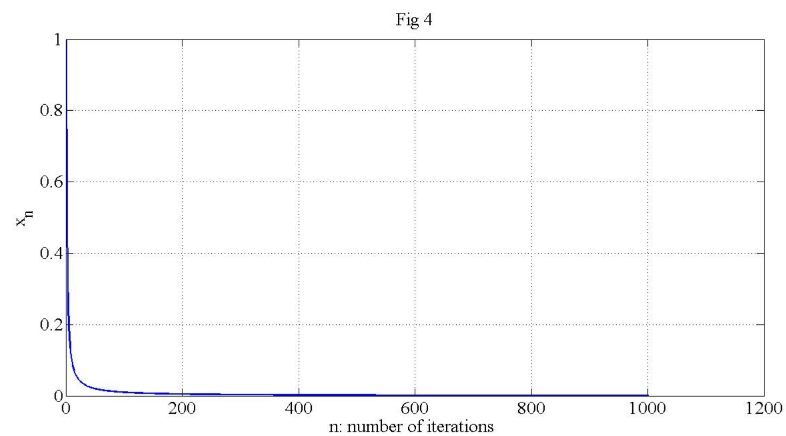
Starting from $x = 1$ yields convergence as in Figure 2 below.



For the iteration scheme 1.8 we can take $t_n = \frac{1}{n+1}$; $\alpha_n = \frac{1-k}{\sqrt{n+1}}$ $\forall n \geq 1, x_1 = -1$ and the sequence $\{x_n\}$ converges to zero as shown in Figure 3 below.



Starting from $x = 1$ yields convergence as in Figure 4 below.



Example 4.2. Let $\Re$ denote the reals with the usual norm and define $T : \Re \to \Re$ by

$$Tx = \begin{cases} -3x+1, \ x \in (-\infty, 0] \\ \frac{1}{2}(x+2), \ x \in (0, \infty) \end{cases}$$

$\forall x, y \in (-\infty, 0]$, we obtain $|Tx - Ty|^2 = 9|x-y|^2, \ |x - Tx - (y - Ty)|^2 = 16|x-y|^2$, and hence

$$|Tx - Ty|^2 = 9|x-y|^2 = |x-y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2.$$

Observe also that $\forall x, y \in (0, \infty)$ we have

$$|Tx - Ty|^2 = \frac{1}{4}|x-y|^2 \leq |x-y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2.$$

Furthermore, for all $x \in (-\infty, 0]$ and $y \in (0, \infty)$ we have

$|Tx - Ty|^2 = |-3x + 1 - \frac{1}{2}(y+2)|^2 = \frac{1}{4}|-6x - y|^2 = \frac{1}{4}(36x^2 + 12xy + y^2) = 9x^2 + 3xy + \frac{1}{4}y^2,$

and

$$\begin{aligned}
|x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2 &= |x-y|^2+\frac{1}{2}|4x-1-(y-\frac{1}{2}(y+2))|^2 \\
&= x^2-2xy+y^2+\frac{1}{8}|8x-y|^2 \\
&= x^2-2xy+y^2+\frac{1}{8}(64x^2-16xy+y^2) \\
&= 9x^2-4xy+y^2+\frac{1}{8}y^2 \\
&= 9x^2+3xy+\frac{1}{4}y^2-7xy+\frac{7}{8}y^2 \\
&\geq 9x^2+3xy+\frac{1}{4}y^2=|Tx-Ty|^2.
\end{aligned}$$

Thus

$$|Tx-Ty|^2 \leq |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2,\ \forall x,y\in\Re.$$

Observe that for all integer $n\geq 2$, we have

$$T^n x=\begin{cases} \frac{1}{2^{n-1}}(-3x+2^n-1),\ x\in(-\infty,0] \\ \frac{1}{2^n}(x+2(2^n-1)),\ x\in(0,\infty). \end{cases}$$

Observe that $\forall x,y\in(-\infty,0]$, we obtain

$$|T^n x-T^n y|^2=\frac{3^2}{2^{2(n-1)}}|x-y|^2.$$

Furthermore,

$$|x-T^n x-(y-T^n y)|^2=(1+\frac{3}{2^{n-1}})^2|x-y|^2.$$

Thus

$$\begin{aligned}
|x-y|^2+\frac{1}{2}|x-T^n x-(y-T^n y)|^2 &= [1+\frac{1}{2}(1+\frac{3}{2^{n-1}})^2]|x-y|^2 \\
&= \frac{3^2}{2^{2(n-1)}}|x-y|^2+[1+\frac{1}{2}(1+\frac{3}{2^{n-1}})^2-\frac{3^2}{2^{2(n-1)}}]|x-y|^2 \\
&= \frac{3^2}{2^{2(n-1)}}|x-y|^2+[\frac{3}{2}+\frac{3}{2^{n-1}}(1-\frac{3}{2^n})]|x-y|^2 \\
&\geq \frac{3^2}{2^{2(n-1)}}|x-y|^2=|T^n x-T^n y|^2.
\end{aligned}$$

Next for all $x, y \in (0, \infty)$ we have

$$|T^n x - T^n y|^2 = \frac{1}{2^{2n}}|x-y|^2 \leq |x-y|^2 + \frac{1}{2}|x - T^n x - (y - T^n y)|^2.$$

If we now consider all $x \in (-\infty, 0]$ and all $y \in (0, \infty)$ we obtain

$$|T^n x - T^n y|^2 = \frac{1}{2^{2(n-1)}}[9x^2 + 3xy + \frac{y^2}{4}],$$

and

$$|x - T^n x - (y - T^n y)|^2 = (1 + \frac{3}{2^{n-1}})^2 x^2 + (1 - \frac{1}{2^n})^2 y^2 - 2(1 + \frac{3}{2^{n-1}})(1 - \frac{1}{2^n})xy.$$

Hence

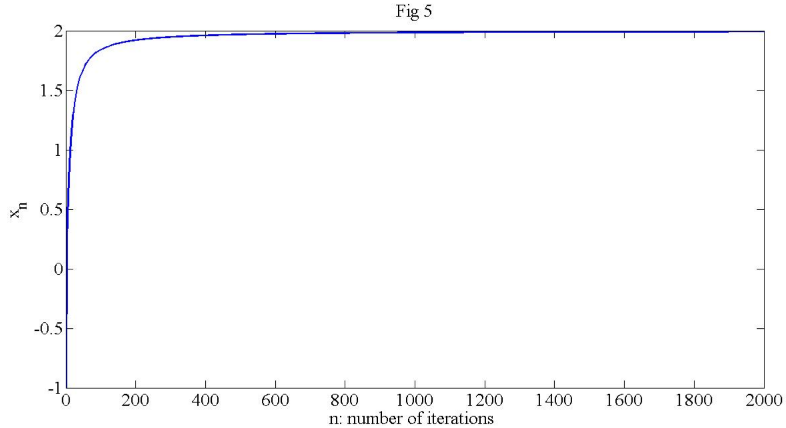
$$\begin{aligned}
|x-y|^2 + \frac{1}{2}|x - T^n x - (y - T^n y)|^2 &= [1 + \frac{1}{2}(1 + \frac{3}{2^{n-1}})^2]x^2 + [1 + \frac{1}{2}(1 - \frac{1}{2^n})^2]y^2 \\
&\quad -[2 + 2(1 + \frac{3}{2^{n-1}})(1 - \frac{1}{2^n})]xy \\
&= \frac{1}{2^{2(n-1)}}[9x^2 + 3xy + \frac{y^2}{4}] + [1 + \frac{1}{2}(1 + \frac{3}{2^{n-1}})^2 - \frac{9}{2^{2(n-1)}}]x^2 \\
&\quad +[1 + \frac{1}{2}(1 - \frac{1}{2^n})^2 - \frac{1}{2^{2n}}]y^2 \\
&\quad -[2 + 2(1 + \frac{3}{2^{n-1}})(1 - \frac{1}{2^n}) + \frac{3}{2^{2(n-1)}}]xy \\
&= |T^n x - T^n y|^2 + [\frac{3}{2} + \frac{3}{2^{n-1}}(1 - \frac{3}{2^n})]x^2 + \frac{3}{2}(1 - \frac{1}{2^n})y^2 \\
&\quad -[4 + \frac{5}{2^{(n-1)}}]xy \\
&\geq |T^n x - T^n y|^2.
\end{aligned}$$

Thus for all $x, y \in \Re$, $n \in \mathbb{N}$ we have

$$|T^n x - T^n y|^2 \leq |x-y|^2 + \frac{1}{2}|x - T^n x - (y - T^n y)|^2, \ \forall x, y \in \Re.$$

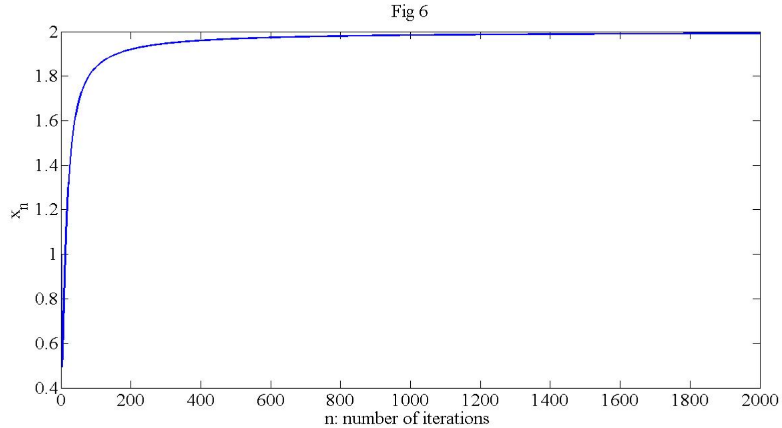
It follows that T is strictly asymptotically pseudocontractive and $F(T) = \{2\}$. $k_n = 1$, $\forall n \geq 1$; $k = \frac{1}{2}$ and for the iteration (1.7), we can take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}(1-k)(1-\alpha_n) = \frac{n}{4(n+1)}$, $\forall n \geq 1$, $x_1 = -1$ and the sequence $\{x_n\}$ converges to 2 as shown in Figure 5. below

Fig 5



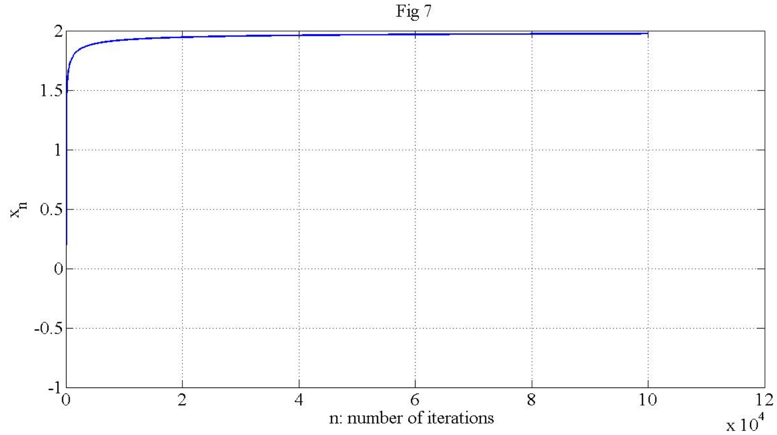
Starting from $x = 1$ yields convergence as in Figure 6 below.

Fig 6

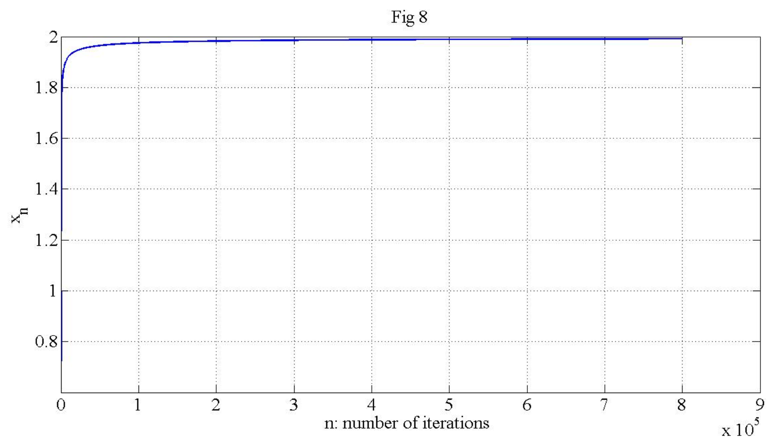


For the iteration scheme 1.8 we can take $t_n = \frac{1}{n+1}$; $\alpha_n = \frac{1-k}{\sqrt{n+1}}$ $\forall n \geq 1, x_1 = -1$ and the sequence $\{x_n\}$ converges to 2 as shown in Figure 7 below.

Fig 7



Starting from $x = 1$ yields convergence as in Figure 8 below.



Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The authors used the facilities of the Office of Pastor E.A. Adeboye Professorial Chair in Mathematics at University of Nigeria, Nsukka. We are grateful to the Office.

REFERENCES

[1] V. Berinde, Iterative Approximation of Fixed Points, Lectures Notes 1912 Springer (2002).

[2] F.E. Browder and W.V.Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20(1967), 197-228.

[3] F. E. Browder, Nonexpansive Nonlinear Operators in Banach Spaces,*Proc. Nat. Acad. Sci,* 54 (1965), 1041-1044.

[4] S-S. Chang, Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings,Proc. Amer. Math. Mech. 129(2000), 845-853.

[5] S-S. Chang,Y.J Cho, and H.Zhou,Demi-Closed principle and weak convergence problems for asymptotically nonexpansive mappings,J. Korean Math. Soc. 38(2001), No.6. 1245-1260.

[6] C.E. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, Lecture Notes in Mathematics 1965, Springer (2009).

[7] K. Goebel and W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35(1972), 171-174.

[8] N. Hussain, K. Ullah, M. Arshad, Fixed point approximation of Suzuki generalized nonexpansive mapping via new faster iteration process, arXiv:1802.09888v1[math.FA] (2018).

[9] T.H. Kim and H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. TMA, 61(2005), 51-60.

[10] W.A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, Isreal J. Math. 17(1974), 339-346.

[11] T.C. Lim and H.K. Xu, Fixed point point theorems for asymptotically nonexpansive mappings, Nonlinear Anal. TMA, 22(1994),1345-1355.

[12] P.E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (7) (2008), 899-912.

[13] P.E. Mainge and S. Maruster, Convergence in Norm of modified Krasnoselkii- Mann iterations for fixed points of demicontractive mappings, Appl. Math. Comput., 217 (2011), 9864-9874.

[14] P. Majee and C. Nahak, A modified iterative method for a finite collection of non-self mappings and family of variational inequality problems, Med. J. Math. 15(2018), 58

[15] G. Marino and H.K. Xu, Weak and strong convergence theorems for strict pseudocontrations in Hilbert spaces, J. Math. Anal. Appl., 329(2007), 336-349.

[16] M. Li and Y. Yao, Strong convergence of an Iterative algorithm for λ-strictly pseudocontractive mappings in Hilbert spaces, An. Şt. Univ. Ovidius Constanţa 18(1) (2010), 219-228.

[17] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[18] M.O. Osilike S.C. Aniagbasor, B.G. Akuchu,Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, Panamer. Math. J. 12(2)(2002), 77-88.

[19] M.O. Osilike, A. Udomene, D.I. Igbokwe, B.G. Akuchu, Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps, J. Math. Anal. Appl. 326 (2007), 1334-1345.

[20] M.O. Osilike, Iterative approximations of fixed points of asymptotically demicontractive mappings, Indian J. Pure Appl. Math. 29(12) (1998), 1291-1300.

[21] M.O. Osilike and A. Udomene, Demiclosedness principle and convergence results for strictly pseudocontractive mappings of Browder-Petryshyn type, J. Math. Anal. Appl. 256(2001), 431-445.

[22] L.Qihou, Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings, Nonlinear Anal. 26(11)(1996), 1835-1842.

[23] L. Qihou, The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings, J. Math. Anal. Appl. 148 (1990), 55-62.

[24] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings,Bull. Austral. Math.Soc. 43(1991), 153-159.

[25] J. Schu, Iterative construction of fixed point of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158(1991), 407-413.

[26] K. Ullah, M. Arshad, New three step iteration process and fixed point approximation in Banach spaces, J. Linear Topol. Algebra 7 (2) (2018), 87-100.

[27] H.K. Xu, Inequalities in Banach spaces with applications. Nonlinear Anal. 16(2) (1991) 1127-1138.

[28] L. Yang, Strong convergence theorem of an iterative scheme for strictly pseudocontractive mappings in Banach spaces, Optimization, in press.

[29] Y. Yao, H. Zhou and Y-C. Liou, Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings, J. Appl. Math. Comput. 29 (2009), 383-389.

[30] H. Zegeye, N. Shahzad and M. A. Alghamdi, Convergence of Ishikawa's iteration method for pseudocontractive mappings, Nonlinear Anal. TMA, 74 (2011) 7304-7311.

[31] H. Zegeye and A.R. Tufa, Halpern-Ishikawa type iteration method for approximating fixed points of non-self pseudocontractive mappings, Fixed Point Theory Appl. 2018 (2018), 15.