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APPROXIMATING COMMON FIXED POINTS UNDER GENERALIZED RATIONAL CONTRACTIONS IN 2-BANACH SPACES WITH APPLICATIONS

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Abstract. In this manuscript, the existence and uniqueness of common fixed points of a family of self-mappings under more generalized rational contractive conditions in 2-Banach spaces are obtained with an illustrative example. Then the common fixed points are approximated by means of Krasnoselskii, Mann and Picard type iteration schemes. Later, an application to well-posedness of the common fixed point problem is provided. The obtained results generalize and improve many well-known results in the literature.

Keywords: common fixed point; Krasnoselskii iteration; Mann iteration; Picard iteration; well-posedness; 2-Banach space.

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1. INTRODUCTION

Since the famous Banach Contraction Principle [5], metric fixed point theory has had a rapid development. A huge number of papers was appeared, which represented a generalization of

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this important result in various generalized metric spaces, (see, for example, [17, 18, 19, 20, 21, 22, 23]). Some of such generalizations are provided satisfying rational contractive conditions. Das and Gupta [7] extended the Banach contraction using rational inequality and proved the following result:

Theorem 1.1. *Let f be a mapping of a complete metric space X into itself such that*

(i)
$$d(f(x), f(y)) \leq \alpha \frac{d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta d(x, y),$$
for all $x, y \in X, \alpha > 0, \beta > 0, \alpha + \beta < 1$ and

(ii) *for some $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ has a subsequence $\{f_k^n(x_0)\}$ with $\xi = \lim_{n\to\infty} f_k^n(x_0)$.*

Then ξ is a unique fixed point of f.

Subsequently, Pachpatte [14] extended it to a pair of self mappings. Recently, Azam et al. [4] and Nashine et al.[12] obtained fixed point theorems for a pair of contractive mappings using generalized rational inequalities of [7] in a complex-valued metric space setting. In [25], Shahkoohi and Razani proved the existence of fixed point of a self mapping under rational Geraghty contractive conditions in partially ordered b-metric spaces, see [3, 24].

In the 1960's, Gähler [8, 9, 10] introduced a new theory of 2-metric spaces. Since then, many authors have focused on these spaces and presented papers that dealt with fixed point theory for single-valued and multi-valued operators in 2-Banach spaces (see [1, 2, 6, 13, 11, 26]). Recently, Pitchaimani and Ramesh Kumar [15] obtained common fixed points under implicit relation in 2-Banach spaces and proved some common and coincidence fixed point theorems for asymptotically regular mappings in [16].

In the light of these developments, in this paper we intend to establish the existence and uniqueness of common fixed point of a family of self mappings satisfying the generalized rational contractive condition in 2-Banach spaces with supportive example. Then, the approximation of the common fixed point by means of Krasnoselskii, Mann and Picard iteration method is given. Finally, the well-posedness of the common fixed point problem is obtained.

2. PRELIMINARIES

First of all, let us recall the basic definitions and concepts that will be required in the sequel. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

Definition 2.1. *Let X be a real linear space and $\|\cdot,\cdot\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions :*

(i) $\|x,y\| = 0$ *if and only if x and y are linearly dependent;*

(ii) $\|x,y\| = \|y,x\|$, *for all $x,y \in X$;*

(iii) $\|x,ay\| = |a|\|x,y\|$, *for all $x,y \in X$ and $a \in \mathbb{R}$;*

(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$, *for all $x,y,z \in X$;*

Then $\|\cdot,\cdot\|$ is called a 2 - norm and the pair $(X,\|\cdot,\cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x,y+ax\| = \|x,y\|$, for all $x,y \in X$ and $a \in \mathbb{R}$.

Definition 2.2. *A sequence $\{x_n\}$ in a linear 2-normed space $(X,\|\cdot,\cdot\|)$ is called a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_m - x_n, y\| = 0$ for all $y \in X$.*

Definition 2.3. *A sequence $\{x_n\}$ in a linear 2-normed space $(X,\|\cdot,\cdot\|)$ is said to converge to a point $x \in X$ if $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in X$.*

Definition 2.4. *A linear 2-normed space $(X,\|\cdot,\cdot\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.*

Example 2.5. *Let $X = \mathbb{R}^3$ and a 2-norm $\|\cdot,\cdot\|$ be defined as follows:*

$$\|x,y\| = |x \times y| = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

where $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in X$ and $\vec{i},\vec{j},\vec{k}$ are the unit vectors along the axes. Note that $(X,\|\cdot,\cdot\|)$ is a 2-Banach space.

Definition 2.6. *A sequence $\{x_n\}$ in a 2-Banach space X is said to be asymptotically T- regular if $\lim_{n\to\infty} \|x_n - Tx_n, y\| = 0$ for all $y \in X$.*

Definition 2.7. *Let X be a nonempty set and $S, T : X \to X$ be self mappings. Then*

(i) *an element $x \in X$ is said to be a fixed point of T if $x = Tx$.*

(ii) *If $x = Sx = Tx$ then x is called a common fixed point of S and T.*

Let C be a nonempty convex subset of a 2-Banach space $(X, \|\cdot,\cdot\|)$ and $T : X \to X$ be a mapping then

(i) the sequence $\{x_n\}$ defined by

$$x_0 \in C,\ x_{n+1} = (1-\delta)x_n + \delta T(x_n), \quad \forall n \geq 0,$$

where $0 < \delta < 1$ is called the Krasnoselskii iteration scheme.

(ii) the sequence $\{x_n\}$ defined by

$$x_0 \in C,\ x_{n+1} = (1-\beta_n)x_n + \beta_n T(x_n), \quad \forall n \geq 0,$$

where $\{\beta_n\}$ satisfies $0 < \beta_n \leq 1, \forall n$ and $\sum_{n=0}^{\infty} \beta_n = \infty$, is called the Mann iteration scheme.

(iii) the sequence $\{x_n\}$ defined by

$$x_0 \in C,\ x_{n+1} = T(x_n),\ \ \forall n \geq 0,$$

is called the Picard iteration scheme which is particular case of the Mann iteration scheme.

3. Main Results

In this section, we first prove the following result which establishes the existence and uniqueness of common fixed point of a pair of self mappings.

Theorem 3.1. *Let $(X,\|\cdot,\cdot\|)$ be a 2-Banach space and $S,T:X\to X$ be two self mappings such that*

$$\begin{aligned}\|Sx-Ty,u\| \leq{}& \lambda\max\Big\{\frac{\|y-Ty,u\|\left[1+\|x-Sx,u\|\right]}{1+\|x-y,u\|},\frac{\|x-Sx,u\|\left[1+\|y-Ty,u\|\right]}{1+\|x-y,u\|},\\ &\frac{\|x-Ty,u\|\left[1+\|y-Sx,u\|\right]}{1+\|x-y,u\|},\frac{\|y-Sx,u\|\left[1+\|x-Ty,u\|\right]}{1+\|x-y,u\|},\\ &e\|x-y,u\|\Big\}, \qquad (1)\end{aligned}$$

for all $x,y,u\in X$, $x\neq y$, where $0\leq\lambda<1$. Then S and T have a unique common fixed point in X.

Proof. Let $x_0\in X$ be arbitrary. Let us define a sequence $\{x_n\}$ such that

$$x_{2n+1}=Sx_{2n},x_{2n+2}=Tx_{2n+1},n=0,1,2,\ldots$$

Applying (1) for all $u\in X$, we have

$$\begin{aligned}\|x_{2n+1}-x_{2n},u\| ={}& \|Sx_{2n}-Tx_{2n-1},u\|\\ \leq{}& \lambda\max\Big\{\frac{\|x_{2n-1}-x_{2n},u\|[1+\|x_{2n}-x_{2n+1},u\|]}{1+\|x_{2n}-x_{2n-1},u\|},\\ &\frac{\|x_{2n}-x_{2n+1},u\|[1+\|x_{2n}-x_{2n-1},u\|]}{1+\|x_{2n}-x_{2n-1},u\|},\\ &\frac{\|x_{2n}-x_{2n},u\|[1+\|x_{2n-1}-x_{2n+1},u\|]}{1+\|x_{2n}-x_{2n-1},u\|},\\ &\frac{\|x_{2n-1}-x_{2n+1},u\|[1+\|x_{2n}-x_{2n},u\|]}{1+\|x_{2n}-x_{2n-1},u\|},\|x_{2n}-x_{2n-1},u\|\Big\}.\end{aligned}$$

Case 1: If the maximum is $\frac{\|x_{2n-1}-x_{2n},u\|[1+\|x_{2n}-x_{2n+1},u\|]}{1+\|x_{2n}-x_{2n-1},u\|}$, then

$$\|x_{2n+1}-x_{2n},u\|=\lambda\frac{\|x_{2n-1}-x_{2n},u\|[1+\|x_{2n}-x_{2n+1},u\|]}{1+\|x_{2n}-x_{2n-1},u\|},$$

which yields that

$$\|x_{2n+1}-x_{2n},u\|\leq l_n\|x_{2n}-x_{2n-1},u\|$$

where $l_n=\dfrac{\lambda}{1+\|x_{2n}-x_{2n-1},u\|-\lambda\|x_{2n}-x_{2n-1},u\|}<1$. Retracing the same steps again and again, we get

$$\|x_{2n+1}-x_{2n},u\|\leq(l_n)^{2n}\|x_1-x_0,u\|$$

For $n > m$, we have

$$\begin{aligned}\|x_n - x_m, u\| &\leq \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \cdots\cdots + \|x_{m+1} - x_m, u\| \\ &\leq ((l_n)^{n-1} + (l_n)^{n-2} + \cdots\cdots + (l_n)^m)\|x_1 - x_0, u\| \\ &\leq \frac{(l_n)^m}{1-l_n}\|x_1 - x_0, u\|.\end{aligned}$$

Therefore, $\|x_n - x_m, u\| \to 0$ as $m, n \to \infty$, since $\frac{(l_n)^m}{1-l_n} \to 0$ as $m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X.

Case 2: If the maximum is $\frac{\|x_{2n} - x_{2n+1}, u\|[1 + \|x_{2n} - x_{2n-1}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|}$, then

$$\|x_{2n+1} - x_{2n}, u\| \leq \lambda \frac{\|x_{2n} - x_{2n+1}, u\|[1 + \|x_{2n} - x_{2n-1}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|}$$

which implies that

$$\|x_{2n+1} - x_{2n}, u\|\big(1 + \|x_{2n} - x_{2n-1}, u\| - \lambda - \lambda\|x_{2n} - x_{2n-1}, u\|\big) \leq 0.$$

It has the only possibility that $\|x_{2n+1} - x_{2n}, u\| = 0$ which shows that $\{x_n\}$ is a Cauchy sequence in X.

Case 3: If the maximum is $\frac{\|x_{2n-1} - x_{2n+1}, u\|[1 + \|x_{2n} - x_{2n}, u\|]}{1 + \|x_{2n} - x_{2n-1}, u\|}$, then

$$\|x_{2n+1} - x_{2n}, u\| \leq \lambda \frac{\|x_{2n-1} - x_{2n}, u\|\|x_{2n} - x_{2n+1}, u\|}{1 + \|x_{2n} - x_{2n-1}, u\|}$$

which follows that

$$\|x_{2n+1} - x_{2n}, u\|\big(1 + \|x_{2n} - x_{2n-1}, u\| - \lambda\|x_{2n} - x_{2n-1}, u\|\big) \leq 0.$$

This shows that $\|x_{2n+1} - x_{2n}, u\| = 0$.

Case 4: If the maximum is $\|x_{2n} - x_{2n-1}, u\|$, then we have

$$\begin{aligned}\|x_{2n+1} - x_{2n}, u\| &\leq \lambda\|x_{2n} - x_{2n-1}, u\| \\ &\vdots \\ &\vdots \\ &\leq \lambda\|x_1 - x_0, u\|.\end{aligned}$$

Hence, it is clear that $\{x_n\}$ is a Cauchy sequence in X from all four cases. Since X is a 2-Banach space, there exists a member v of X such that $x_n \to v$ as $n \to \infty$. Now, we have

$$\begin{aligned}
\|w - Tw, u\| &\leq \|w - x_{2n+1}, u\| + \|x_{2n+1} - Tw, u\| \\
&= \|w - x_{2n+1}, u\| + \|Sx_{2n} - Tw, u\| \\
&\leq \|w - x_{2n+1}, u\| + \lambda \max \left\{ \frac{\|w - Tw, u\|[1 + \|x_{2n} - x_{2n+1}, u\|]}{1 + \|x_{2n} - w, u\|}, \right. \\
&\quad \frac{\|x_{2n} - x_{2n+1}, u\|[1 + \|w - Tw, u\|]}{1 + \|x_{2n} - w, u\|}, \\
&\quad \frac{\|x_{2n} - Tw, u\|[1 + \|w - x_{2n+1}\|]}{1 + \|x_{2n} - w, u\|}, \\
&\quad \left. \frac{\|w - x_{2n+1}, u\|[1 + \|x_{2n} - Tw, u\|]}{1 + \|x_{2n} - w, u\|}, \|x_{2n} - w, u\| \right\}.
\end{aligned}$$

Taking limit $n \to \infty$, we obtain

$$\|w - Tw, u\| \leq \lambda \|w - Tw, u\|,$$

which shows that $Tw = w$ since $\lambda < 1$ for all $u \in X$. Likewise, it is easy to see that $Sw = w$. Hence w is a common fixed point of S and T. In order to prove the uniqueness, let us take $v_0 \in X$ is another common fixed point of S and T, that is, $v_0 = Sv_0 = Tv_0$. Then

$$\begin{aligned}
\|v_0 - v, u\| &= \|Sv_0 - Tv, u\| \\
&\leq \lambda \max \left\{ \frac{\|v - Tv, u\|[1 + \|v_0 - Sv_0, u\|]}{1 + \|v_0 - v, u\|} \right. \\
&\quad \frac{\|v_0 - Sv_0, u\|[1 + \|v - Tv, u\|]}{1 + \|v_0 - v, u\|}, \\
&\quad \frac{\|v_0 - Tv, u\|[1 + \|v - Sv_0, u\|]}{1 + \|v_0 - v, u\|}, \\
&\quad \left. \frac{\|v - Sv_0, u\|[1 + \|v_0 - Tv, u\|]}{1 + \|v_0 - v, u\|}, \|v_0 - v, u\|, \right\}, \\
&= \lambda \max \left\{ \frac{\|v_0 - v, u\|[1 + \|v_0 - v, u\|]}{1 + \|v_0 - v, u\|}, \|v_0 - v, u\|, \right\}, \\
&= \lambda \|v_0 - v, u\|.
\end{aligned}$$

which yields a contradiction as $\lambda < 1$. Thus, S and T has a unique common fixed point v in X. $\square$

The following result is obtained by putting $S=T$ in Theorem 3.1 which generalizes Theorem 1.1.

Corollary 3.2. *Let* $(X,\|\cdot,\cdot\|)$ *be a 2-Banach space and* $T:X\to X$ *be a self mapping such that*

$$\begin{aligned}\|Tx-Ty,u\| \;\le\; & \lambda\max\left\{\frac{\|y-Ty,u\|\left[1+\|x-Tx,u\|\right]}{1+\|x-y,u\|},\frac{\|x-Tx,u\|\left[1+\|y-Ty,u\|\right]}{1+\|x-y,u\|},\right.\\ & \frac{\|x-Ty,u\|\left[1+\|y-Tx,u\|\right]}{1+\|x-y,u\|},\frac{\|y-Tx,u\|\left[1+\|x-Ty,u\|\right]}{1+\|x-y,u\|},\\ & \left.\|x-y,u\|\right\}, \qquad (2)\end{aligned}$$

for all $x,y,u\in X$, $x\neq y$, *where* $0\le\lambda<1$. *Then* T *have a unique fixed point in* X.

Proof. We omit the proof as it is immediate from Theorem 3.1. □

The following result is an extension of Theorem 3.1 to the case of pair of mappings S^p and T^q where p and q are some positive integers.

Theorem 3.3. *Let* $(X,\|\cdot,\cdot\|)$ *be a 2-Banach space and* $S,T:X\to X$ *be two self mappings such that*

$$\begin{aligned}\|S^px-T^qy,u\| \;\le\; & \lambda\max\left\{\frac{\|y-T^qy,u\|\left[1+\|x-S^px,u\|\right]}{1+\|x-y,u\|},\frac{\|x-S^px,u\|\left[1+\|y-T^qy,u\|\right]}{1+\|x-y,u\|},\right.\\ & \frac{\|x-T^qy,u\|\left[1+\|y-S^px,u\|\right]}{1+\|x-y,u\|},\frac{\|y-S^px,u\|\left[1+\|x-T^qy,u\|\right]}{1+\|x-y,u\|},\\ & \left.\|x-y,u\|\right\}, \qquad (3)\end{aligned}$$

for all $x,y,u\in X$, $x\neq y$, *where* p *and* q *are some positive integers and* $0\le\lambda<1$. *Then* S *and* T *have a unique common fixed point in* X.

Proof. Since S^p and T^q satisfy the conditions of Theorem 3.1, S^p and T^q have a unique common fixed point, w (say). Now

$$\begin{aligned} S^pw=w \quad \Rightarrow \quad & S(S^pw)=Sw,\\ S^p(Sw) \quad = \quad & Sw,\end{aligned}$$

which implies that Sw is a fixed point of S^p. Likewise, we get $T^q(Tw) = Tw$. Then

$$\begin{aligned}
\|w - Tw, u\| &= \|S^p w - T^q(Tw), u\| \\
&\leq \lambda \max\left\{ \frac{\|Tw - T^q(Tw), u\|[1 + \|w - S^p w, u\|]}{1 + \|w - Tw, u\|}, \right. \\
&\quad \frac{\|w - S^p w, u\|[1 + \|Tw - T^q(Tw), u\|]}{1 + \|w - Tw, u\|}, \\
&\quad \frac{\|w - T^q(Tw), u\|[1 + \|Tw - S^p w, u\|]}{1 + \|w - Tw, u\|}, \\
&\quad \frac{\|Tw - S^p w, u\|[1 + \|w - T^q(Tw), u\|]}{1 + \|w - Tw, u\|}, \\
&\quad \left. \|w - Tw, u\| \right\},
\end{aligned}$$

which yields that

$$\|w - Tw, u\| \leq \lambda \|w - Tw, u\|$$

which is a contradiction since $\lambda < 1$. Therefore, $w = Tw$ for all $u \in X$. Similarly, we have $Sw = w$.

To prove the uniqueness of v, assume that v is another common fixed point of S and T. Then it can be viewed that w is also a common fixed point of S^p and T^q which shows $w = v$. Hence, S and T have a unique common fixed point. □

Remarks 3.4. *Note that if x_0 is a unique common fixed point of S^p and T^q, where p, q are some positive integers then x_0 is a unique common fixed point of S and T.*

Next, Theorem 3.1 is extended to a case of family of mappings satisfying the condition (1).

Theorem 3.5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $\{F_\alpha\}$ be a family of self mappings on X such that*

$$\begin{aligned}
\|F_\alpha x - F_\beta y, u\| &\leq \lambda \max\left\{ \frac{\|y - F_\beta y, u\| \left[1 + \|x - F_\alpha x, u\|\right]}{1 + \|x - y, u\|}, \frac{\|x - F_\alpha x, u\| \left[1 + \|y - F_\beta y, u\|\right]}{1 + \|x - y, u\|}, \right. \\
&\quad \frac{\|x - F_\beta y, u\| \left[1 + \|y - F_\alpha x, u\|\right]}{1 + \|x - y, u\|}, \frac{\|y - F_\alpha x, u\| \left[1 + \|x - F_\beta y, u\|\right]}{1 + \|x - y, u\|}, \\
&\quad \left. \|x - y, u\| \right\},
\end{aligned}$$

for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $x, y, u \in X$ with $x \neq y$, where $0 \leq \lambda < 1$. Then there exists a unique $w \in X$ satisfying $F_\alpha w = w$ for all $\alpha \in \Lambda$.

Proof. Replacing F_α and F_β for S and T respectively in Theorem 3.1, an application of which gives a unique $w \in X$ to satisfy $F_\alpha w = F_\beta w = w$. For any other member F_γ, uniqueness of w gives $F_\gamma w = w$ and this completes the proof. □

Remarks 3.6. *Theorem* 3.5 *generalizes and improves the main results of* [4, 7, 12, 14, 24] *in the framework of 2-Banach spaces.*

Example 3.7. *Let $X = \mathbb{R}^2$ and a 2-norm $\|\cdot,\cdot\|$ be defined by $\|x,y\| = |x_1y_2 - x_2y_1|$. Note that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space. Let $S, T : X \to X$ be two self mappings defined as follows:*

$$S(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$$

and

$$T(u,v) = \left(\frac{u}{2}, \frac{u}{2}\right).$$

For $\lambda \in [\frac{1}{2}, 1)$, it can be easily viewed that all the conditions of Theorem 3.1 *are satisfied. Therefore, by Theorem* 3.1*, S and T have a unique common fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$ in X.*

3.1. Approximations.

One of the most important iteration procedure is Picard iteration schemes in approximating the fixed points. To overcome the difficulty that Picard iteration does not converge to a fixed point of all kind of contractive mappings, some other iteration schemes came to existence namely Mann iteration, Ishikawa iteration, Krasnoselskii iteration, Jungck iteration etc.

In this section, we approximate the common fixed point of S and T by means of Jungck-Krasnoselskii, Jungck-Mann and Jungck-Picard type iteration schemes.

Theorem 3.8. *Let C be a nonempty closed convex subset of a 2-Banach space X and $S, T : C \to C$ be two self mappings such that $(1-k)S(C) + kT(C) \subset S(C)$ for $0 < k \leq 1$ and $S(C)$ is closed. Suppose that S and T satisfy all the conditions of Theorem* 3.1*, then S and T have a unique common fixed point. In addition, if, for arbitrary $y_0 \in C$ and any fixed number δ with*

$0 < \delta < 1$, *the sequence* $\{y_n\}$ *defined by*

$$S(y_{n+1}) = (1-\delta)S(y_n) + \delta T(y_n), \quad \forall n \geq 0, \tag{4}$$

is asymptotically T*-regular, then it converges to the unique common fixed point of* S *and* T*, with a rate estimated by*

$$\|(S(y_{n+1}) - w, u\| \leq \mu^{n+1} L,$$

where $\mu \in [0,1)$ *and* $L \geq 0$ *are some constants.*

Proof. It follows from Theorem 3.1 that S and T have a unique common fixed point $w \in X$. Let $y_0 \in C$ and the sequence $\{y_n\}$ be defined by (4). Now, for all $u \in X$ and $n \geq 0$, we have

$$\begin{aligned} \|S(y_{n+1}) - w, u\| &= \|(1-\delta)S(y_n) + \delta T(y_n) - w, u\| \\ &\leq (1-\delta)\|S(y_n) - w, u\| + \delta \|T(y_n) - w, u\| \end{aligned} \tag{5}$$

From (1), we obtain

$$\begin{aligned} \|T(y_n) - w, u\| &= \|T(y_n) - Sw, u\| \\ &\leq \lambda \max \left\{ \frac{\|w - Sw, u\|[1 + \|y_n - Ty_n, u\|]}{1 + \|y_n - w, u\|}, \right. \\ &\quad \frac{\|y_n - Ty_n, u\|[1 + \|w - Sw, u\|]}{1 + \|y_n - w, u\|}, \\ &\quad \frac{\|y_n - Sw, u\|[1 + \|w - Ty_n, u\|]}{1 + \|y_n - w, u\|}, \\ &\quad \left. \frac{\|w - Ty_n, u\|[1 + \|y_n - Sw, u\|]}{1 + \|y_n - w, u\|}, \|y_n - w, u\| \right\}. \end{aligned}$$

As $Sw = Tw = w$ and $\{y_n\}$ is asymptotically T-regular, letting $n \to \infty$, we get

$$\|Ty_n - w, u\| \leq \lambda \|Ty_n - w, u\|,$$

which brings a contradiction since $\lambda < 1$. Therefore, $\lim_{n\to\infty} \|T(y_n) - w, u\| = 0$. Now using (5), we have

$$\|S(y_{n+1}) - w, u\| \leq (1-\delta)\|S(y_n) - w, u\|.$$

Proceeding in this manner, we obtain

$$\|(S(y_{n+1}) - w, u\| \leq \mu^{n+1} L, \tag{6}$$

where $\mu = (1-\delta) \in [0,1)$ and $L = \|Sy_0 - w, u\| \geq 0$ are some constants. As $\mu \in [0,1)$, using (6) we get

$$\lim_{n\to\infty} \|(S(y_{n+1}) - w, u\| \to 0.$$

This completes the proof. □

Remarks 3.9. *Note that Theorem 3.8 gives the assurance of convergence of Jungck-Krasnoselskii iteration scheme satisfying* (1) *in the context of 2-Banach spaces.*

Corollary 3.10. *Let C be a nonempty closed convex subset of a 2-Banach space X and $T : C \to C$ be a mapping satisfying all the conditions of Corollary* 3.2*. Then T has a unique fixed point $w \in X$. Further, if for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1-\delta)x_n + \delta T(x_n, \ldots, x_n), \quad \forall n \geq 0, \tag{7}$$

is asymptotically T-regular, then it converges to the unique fixed point of T, with a rate estimated by

$$\|x_n - w, u\| \leq \mu^n L,$$

where $\mu \in [0,1)$ and $L \geq 0$ are some constants.

Remarks 3.11. *From Corollary* 3.10*, an approximation of fixed point of a self mapping by the Krasnoselskii iteration scheme in 2-Banach spaces is obtained. Note that the result holds even if $\{x_n\}$ is asymptotically S-regular.*

Theorem 3.12. *Let C be a nonempty closed convex subset of a 2-Banach space X and S, T : $C \to C$ be two self mappings such that $(1-k)S(C) + kT(C) \subset S(C)$ for $0 < k \leq 1$ and $S(C)$ is closed. Suppose that S and T satisfy all the conditions of Theorem* 3.1*, then S and T have a unique common fixed point. Moreover, if, for arbitrary $y_0 \in C$, the sequence $\{y_n\}$ defined by*

$$S(y_{n+1}) = (1-\beta_n)S(y_n) + \beta_n T(y_n), \quad \forall n \geq 0, \tag{8}$$

where $\{\beta_n\}$ satisfies $0 < \beta_n \leq 1$, $\forall n$ and $\sum_{n=0}^{\infty} \beta_n = \infty$, *is asymptotically T-regular, then it converges to the unique common fixed point of S and T, with a rate estimated by*

$$\|(S(y_{n+1}) - w, u\| \leq \mu^{n+1} L,$$

where $\mu \in [0,1)$ and $L \geq 0$ are some constants.

Remarks 3.13. *In Theorem 3.12, the common fixed point is approximated by Juncgk-Mann iteration scheme. When S is an identity mapping of X in Theorem 3.12, the approximation of fixed point under Mann iteration scheme can be proved.*

Theorem 3.14. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $S, T : X \to X$ be two mappings such that $T(X) \subset S(X)$ and $S(X)$ is a closed subset of X. Suppose all the conditions of Theorem 3.1 are satisfied, then S and T have a unique common fixed point $w \in X$. Further, if for arbitrary $x_0 \in C$, the sequence $\{y_n\}$ defined by*

$$y_n = S(x_n) = T(x_{n-1}), \quad \forall n \in \mathbb{N}$$

is asymptotically T-regular, then it converges to the unique common fixed point of S and T, with a rate estimated by

$$\|y_n - w, u\| \leq \lambda^n L,$$

where $\lambda \in [0,1)$ and $L \geq 0$ are some constants.

Corollary 3.15. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T : X \to X$ be a mapping such that all the conditions of Corollary 3.2 are satisfied. Then T has a unique fixed point X. In addition, if for arbitrary $x_0 \in C$, the sequence $\{z_n\}$ defined by*

$$z_n = T(x_{n-1}), \quad \forall n \in \mathbb{N},$$

is asymptotically T-regular, then it converges to the unique fixed point of T, with a rate estimated by

$$\|z_n - w, u\| \leq \lambda^n L,$$

where $\lambda \in [0,1)$ and $L \geq 0$ are some constants.

Remarks 3.16. *From Corollary 3.15, we obtain an approximation of fixed point of self mapping by the Picard iteration scheme in 2-Banach space.*

4. APPLICATIONS

In this section, we prove the well-posedness of the common fixed point problem obtained in our results.

Definition 4.1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mapping on X. Then the fixed point problem of T is said to be well-posed if*

(i) *T has a unique fixed point $x_0 \in X$*

(ii) *for any sequence $\{x_n\} \subset X$* and $\lim\limits_{n\to\infty} \|x_n - Tx_n, u\| = 0$*, we have* $\lim\limits_{n\to\infty} \|x_n - x_0, u\| = 0$.

Let $CFPP(S,T,X)$ denote a common fixed point problem of self mappings T and f on X and $CFP(S,T)$ denote the set of all common fixed points of T and f.

Definition 4.2. *$CFPP(S,T,X)$ is called well- posed if $CFP(S,T)$ is singleton and for any sequence $\{x_n\}$ in X with*

$$\hat{x} \in CFP(S,T) \text{ and } \lim_{n\to\infty} \|x_n - Sx_n, u\| = \lim_{n\to\infty} \|x_n - Tx_n, u\| = 0$$

implies $\hat{x} = \lim\limits_{n\to\infty} x_n$.

Theorem 4.3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, S and T be two self mappings on X as in* Theorem 3.1. *Then the common fixed point problem of S and T is well posed.*

Proof. By Theorem 3.1, the mappings S and T have a unique common fixed point, say $w \in X$. Let $\{x_n\}$ be a sequence in X and $\lim\limits_{n\to\infty} \|Sx_n - x_n, u\| = \lim\limits_{n\to\infty} \|Tx_n - x_n, u\| = 0$. Without loss of generality, let us assume that $w \neq x_n$ for any non-negative integer n. Applying (1) and $Sw =$

$Tw = w$, we have

$$\begin{aligned}
\|w-x_n,u\| &\leq \|Tv-Tx_n,u\|+\|Tx_n-x_n,u\| \\
&= \|Tx_n-x_n,u\|+\|Sw-Tx_n,u\| \\
&\leq \|Tx_n-x_n,u\|+\lambda \max\left\{\frac{\|x_n-Tx_n,u\|[1+\|w-Sw,u\|]}{1+\|w-x_n,u\|},\right. \\
&+b\frac{\|w-Sw,u\|[1+\|x_n-Tx_n,u\|]}{1+\|w-x_n,u\|}, \\
&+c\frac{\|w-Tx_n,u\|[1+\|x_n-Sw,u\|]}{1+\|w-x_n,u\|}, \\
&\left.+d\frac{\|x_n-Sw,u\|[1+\|w-Tx_n,u\|]}{1+\|w-x_n,u\|},\|w-x_n,u\|\right\}.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\|w-x_n,u\| \leq \lambda \|w-x_n,u\|,$$

which brings a contradiction since $\lambda < 1$. This completes the proof. □

Corollary 4.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mapping on X as in* Corollary 3.2. *Then the fixed point problem of T is well posed.*

Remarks 4.5. *Notice that well-posedness of the common fixed points obtained in Theorems 3.3 and 3.5 can easily be viewed.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] M. A. Ahmed, A common fixed point theorem for expansive mappings in 2-metric spaces and its application, Chaos Solitons Fractals, 42 (2009), 2914-2920.

[2] A. Aliouche, C. Simpson, Fixed points and lines in 2-metric spaces, Adv. Math. 229 (2012), 668-690.

[3] M. Arshad, E. Karapinar, J. Ahmad, Some unique fixed point theorems for rational contractions in partially ordered metric spaces, J. Inequal. Appl. 2013 (2013), 248.

[4] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 32 (3) (2011), 243-253.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133-181.

[6] K. Ciepliński, Approximate multi-additive mappings in 2-Banach spaces, Bull. Iranian Math. Soc. 41(3) (2015), 785-792.

[7] B. K. Dass, S. Gupta, An extension of Banach's contraction principle through rational expression, Indian J. Pure Appl. Math. 6(4) (1975), 1445-1458.

[8] S. Gähler, 2-metrische Räume and ihre topologische strucktur, Math. Nachr. 26 (1963), 115-148.

[9] S. Gähler, Uber die unifromisieberkeit 2-metrischer Räume, Math. Nachr. 28 (1965), 235-244.

[10] S. Gähler, Über 2-Banach-Räume, Math. Nachr. 42 (1969), 335-347.

[11] M. Gürdal, A. Şahiner, I. Açik, Approximation theory in 2-Banach spaces, Nonlinear Anal. 71 (2009), 1654-1661.

[12] H. K. Nashinea, M. Imdadb, M. Hasan, Common fixed point theorems under rational contractions in complex valued metric spaces, J. Nonlinear Sci. Appl. 7 (2014), 42-50.

[13] N. V. Dung, N. T. Hieu, N. T. Thanh Ly, V. D. Thinh, Remarks on the fixed point problem of 2-metric spaces, Fixed Point Theory Appl. 2013 (2013), 167.

[14] B. G. Pachpatte, Common fixed point theorems for mappings satisfying rational inequalities, Indian J. Pure appl. Math. 10(11) (1979), 1362-1368.

[15] M. Pitchaimani, D. Ramesh Kumar, Some common fixed point theorems using implicit relation in 2-Banach spaces, Surv. Math. Appl. 10 (2015), 159-168.

[16] M. Pitchaimani, D. Ramesh Kumar, Common and coincidence fixed point theorems for asymptotically regular mappings in 2-Banach Spaces, Nonlinear Funct. Anal. Appl. 21(1) (2016), 131-144.

[17] M. Pitchaimani, D. Ramesh Kumar, On construction of fixed point theory under implicit relation in Hilbert spaces, Nonlinear Funct. Anal. Appl. 21(3) (2016), 513-522.

[18] M. Pitchaimani, D. Ramesh Kumar, On Nadler type results in ultrametric spaces with application to well-posedness, Asian-Eur. J. Math. 10(4) (2017), 1750073.

[19] M. Pitchaimani, D. Ramesh Kumar, Generalized Nadler type results in ultrametric spaces with application to well-posedness, Afr. Mat. 28 (2017), 957-970.

[20] D. Ramesh Kumar, M. Pitchaimani, Set-valued contraction mappings of Prešić-Reich type in ultrametric spaces, Asian-Eur. J. Math. 10(4) (2017), 1750065.

[21] D. Ramesh Kumar, M. Pitchaimani, A generalization of set-valued Prešić-Reich type contractions in ultrametric spaces with applications, J. Fixed Point Theory Appl. 19(3) (2017), 1871-1887.

[22] D. Ramesh Kumar, M. Pitchaimani, Approximation and stability of common fixed points of Prešić type mappings in ultrametric spaces, J. Fixed Point Theory Appl. 20 (2018), 4.

[23] D. Ramesh Kumar, M. Pitchaimani, New coupled fixed point theorems in cone metric spaces with applications to integral equations and Markov process, Trans. A. Razmadze Math. Inst. 172 (2018), 409-419.

[24] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, J. Inequal. Appl. 2012 (2012), 84.

[25] R. J. Shahkoohi, A. Razani, Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces, J. Inequal. Appl. 2014 (2014), 373.

[26] S. Shukla, Prešić type results in 2-Banach spaces Afr. Mat. 25(4) (2014), 1043-1051.