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STABILITY OF FIXED POINTS FOR A DISCRETE GRAZER-PREDATOR MODEL AND OPTIMAL CONTROLS FOR THE ENVIRONMENTAL SUSTAINABILITY AND BIOECONOMY

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Abstract. In this paper, we devise a discrete-time trophic-halieutic model in the form of three discrete equations which describe grazer-predator dynamics in the presence of a variable vegetation biomass. Further, we study the stability of the fixed points of the proposed system, and seek two optimal control approaches that are associated to the environmental sustainability and bioeconomic cases and aim to minimize two fishing efforts presented as discrete control functions applied to grazers and predators respectively. The first optimization approach aims to find an optimal control strategy which focuses more on the harvest of only one type of fishes rather than overfishing of both grazers and predators populations, while the second optimization approach aims to maximize the profits of fishermen. The two sought optimal fishing efforts are characterized based on a discrete version of Pontryagin's maximum principle, while the two obtained two-point boundary value problems, are resolved based on discrete progressive-regressive iterative schemes which converge following a test related to the forward-backward sweep computational method.

Keywords: discrete model; trophic-halieutic model; prey-predator model; grazer-predator dynamics; vegetation; fish harvesting; environmental sustainability; bioeconomy; optimal control.

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1. Introduction

Since the inception of human civilization, human activities have caused adversarial impact on natural resources in the territorial and aquatic biosphere [1],[2]. Minimal emphasis on sustainable management of resources has led to extinction of various species and already posing a threat to many endangered species. This can be detrimental as depletion of resources introduce instability in biosphere and which in turn ruins the economy of a region [3].

Recent days, scientific research is being employed while building fisheries, wherein marine biologists, economists and applied mathematicians are working on improvement of the ancient fishing rules. Since it is advent 100 years back, significant development in the field of specialized fisheries have been conducted [4]. Most countries with sizeable fisheries possess laboratories where rigorous research is conducted. Fisheries research involves improving fish breeding techniques as well as enhancing and improving fishing gear and operations with the intent of increasing the profitability of fishing.

The mathematical modeling of the commercial exploitations of renewable resources, is a complex task as it requires to acknowledge the nonlinear interaction of biological, economic, social components in addition to degree of uncertainty. Several mathematical models in the field of marine biology, have been developed in the study of prey-predator fishes populations [5],[6],[7],[8],[9],[10],[11],[12]. Recently, we have worked on devising a system which considers aquatic plants population density as important variable that influences prey-predator evolution equations, and then, we start speaking about grazers rather than preys as explained in our work in [13]. In fact, availability of vegetation in hydraulic environments, is very essential on life-cycle and development of many fishes, but this has been ignored in most models. Nowadays, research in the area of fishing management, becomes more important than the past, due to the increase of human population, and if models take more into account the consideration of preys food resource, harvesting policies for the sustainability of the environment would be more effective. Most studies focused on describing prey-predator interactions using equations at continuous-time because of their mathematical tractability. However, as noted in [14],[15], observations of experiments in populations, are usually collected at discrete times as also reported in [16] in the case of prey-predator models framed in difference equations.

In similar pattern of compartmental modeling as in [18], we devise here, a discrete-time version of a food-chain model without considering a compartment associated to detritus, and with the introduction of two fishing efforts. In fact, the detritus variable has not an important impact on other compartments [18]. In such models, preys are specified as grazers which feed mainly from aquatic plants that are essential for providing them oxygen, food and also shelter. Thus, we focus on studying a hydraulic food chain activity where vegetation, grazers and predators biomasses represent the only components of our model. Our objective is to study dynamics of these populations when they are exploited by one fishing fleet or more with two different fishing efforts. Before all this, we study the stability of the fixed points using results obtained in paper [19]. We also suggest in this paper, two different harvesting policies based on optimal control theory. Since the proposed model here, will be discrete in time, the optimal control approach will require a discrete version of Pontryagin's maximum principle [17]. Both stability and control strategies aim also to show the oscillatory behavior exhibited by grazer-predator populations when the trophic-halieutic dynamics are being a discrete process, and which makes in general the discrete model less stable than its continuous version. In a first part of the optimal control framework, we try to find optimal harvesting control strategies for the sustainability of the marine environment, and in a second part, we seek an optimal control strategy which aims to minimize the fishing costs while maximizing the economic profits of fishermen.

The paper is organized as follows. In the next section, we present three discrete equations which we have devised for modeling trophic-halieutic dynamics. Section 3. is divided into two subsections. In the first subsection, we characterize the two sought optimal control functions introduced in the model, for the environmental sustainability case, and we illustrate the results in numerical simulations. The second subsection concerns with the analysis of the optimal harvesting strategy proposed for the bioeconomic case. We present numerical simulations in section 4. and finally, conclude our work in section 5.

2. THE DISCRETE-TIME TROPHIC-HALIEUTIC MODEL

We devise a trophic-halieutic model to study dynamics of a fish population during the fishing season. We designate densities of vegetation, grazers and predators at each instant k using discrete variables v_k, x_k and y_k respectively. A grazer in marine environment, represents the type of prey fishes that feed on plants such as grasses, or on other multicellular autotroph such as algaes.

It is important to understand that the literature which treats hydraulic food chain models that include vegetation population density and that are at the same time framed in discrete-time, is very limited. Based on hypotheses in [18], we propose here, a discrete-time food chain model where aquatic plants are renewed by logistic growth and decrease either due to grazers consumption and natural mortality or to other reasons such as moving towards detritus or non-grazing population. As regards grazer-predator evolution equations, they are developed using Lotka-Volterra-like equations which are well-known to describe such biological interactions in most mathematical food chain models cited in this paper.

The discrete-time system representing the evolution the considered hydraulic food chain over time, with presence of a harvesting activity, is described based on the following discrete equations at each instant k

$$(1) \qquad \begin{cases} v_{k+1} = v_k + r v_k \left(1 - \frac{v_k}{K}\right) - \gamma x_k v_k - m v_k \\ x_{k+1} = x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - h_{1_k} \\ y_{k+1} = y_k + \beta x_k y_k - m_2 y_k - h_{2_k} \end{cases}$$

with $v_0 \geqslant 0$, $x_0 \geqslant 0$ and $y_0 \geqslant 0$ as given initial conditions, and where all parameters are defined in table (1).

As for h_{1_k} and h_{2_k}, $k = 0,...,N-1$, they represent the harvesting functions associated to grazers and predators respectively, with q_1, q_2, e_1 and e_2 representing the catchability coefficients and fishing effort rates, and we have $h_{1_k} = q_1 e_1 x_k$, $h_{2_k} = q_2 e_2 y_k$. Thus, the discrete-time system (1) can be rewritten as

TABLE 1. List of all parameters in the system (1):

Parameter	Physical interpretation	Unit
r	intrinsic growth rate of vegetation	d^{-1}
K	vegetation carrying capacity	g/m^2
m	nongrazing mortality of vegetation	d^{-1}
m_1	natural mortality rate of grazers	d^{-1}
m_2	natural mortality rate of predators	d^{-1}
α	per capita predation rate on grazers	$m^2.g^{-1}.d^{-1}$
β	conversion efficiency for predators eating grazers $\times\alpha$	$m^2.g^{-1}.d^{-1}$
γ	per capita grazing rate on vegetation	$m^2.g^{-1}.d^{-1}$
δ	conversion efficiency for grazers eating vegetation $\times\gamma$	$m^2.g^{-1}.d^{-1}$

$$\begin{cases} v_{k+1} = v_k + rv_k\left(1-\frac{v_k}{K}\right) - \gamma x_k v_k - m v_k \\ x_{k+1} = x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - q_1 e_1 x_k \\ y_{k+1} = y_k + \beta x_k y_k - m_2 y_k - q_2 e_2 y_k \end{cases} \tag{2}$$

As usual, the results of study or analysis of the stability of differential systems, are based on the characteristics of their equilibrium points. In fact, in the continuous-time case, the derivatives of these points, are zero for all continuous time t. Here, since the mathematical model (2) is in the form of three difference equations, we choose to seek the fixed point vector $(v^{eq}, x^{eq}, y^{eq})^T$ such that for all discrete time or instant k, we have

$$\begin{cases} v_k^{eq} = v_k^{eq} + rv_k^{eq}\left(1-\frac{v_k^{eq}}{K}\right) - \gamma x_k^{eq} v_k^{eq} - m v_k^{eq} \\ x_k^{eq} = x_k^{eq} + \delta x_k^{eq} v_k^{eq} - \alpha x_k^{eq} y_k^{eq} - m_1 x_k^{eq} - q_1 e_1 x_k^{eq} \\ y_k^{eq} = y_k^{eq} + \beta x_k^{eq} y_k^{eq} - m_2 y_k^{eq} - q_2 e_2 y_k^{eq} \end{cases}$$

It is often difficult and not obvious to study the stability of discrete-time systems. These systems are characterized by the periodicity of their solutions for some values of their parameters. For instance, in our present case in the model (2), periodicity of solutions and the start of chaos in the sense of $Li-Yorke$ [19] could occur for some values of the parameter r. Such phenomena are detailed and can be clearly observed in section 4. from where we can deduce this is sometimes due to logistic terms as in the first equation in (2) and as more r approaches 1, as more the effect of the term $\frac{v_k}{K}$ on solutions of (2), is reduced.

Based on this last assumption, we consider a change of variables as $V_k = \frac{v_k}{K}$, $X_k = x_k$ and $Y_k = y_k$. Thus, (2) becomes

$$\begin{cases} V_{k+1} = V_k + rV_k\left(1-V_k\right) - \gamma X_k V_k - mV_k \\ X_{k+1} = X_k + \delta' X_k V_k - \alpha X_k Y_k - m_1 X_k - q_1 e_1 X_k \\ Y_{k+1} = Y_k + \beta X_k Y_k - m_2 Y_k - q_2 e_2 Y_k \end{cases} \tag{3}$$

with $\delta' = \delta K$.

Then, we seek the fixed point vector $B^{eq} = (V^{eq}, X^{eq}, Y^{eq})^T$ of (3) such that for all discrete time or instant k, we have

$$\begin{cases} V_k^{eq} = V_k^{eq} + rV_k^{eq}\left(1-V_k^{eq}\right) - \gamma X_k^{eq} V_k^{eq} - mV_k^{eq} \\ X_k^{eq} = X_k^{eq} + \delta' X_k^{eq} V_k^{eq} - \alpha X_k^{eq} Y_k^{eq} - m_1 X_k^{eq} - q_1 e_1 X_k^{eq} \\ Y_k^{eq} = Y_k^{eq} + \beta X_k^{eq} Y_k^{eq} - m_2 Y_k^{eq} - q_2 e_2 Y_k^{eq} \end{cases}$$

In the following, we announce a theorem in which we provide conditions of stability of the fixed point vector B^{eq} in three different situations.

Theorem 2.1.

The fixed point vector B^{eq} is asymptotically stable if the following conditions on parameters in the discrete-time model (3) are satisfied

- $r < m-1$, $e_1 > \frac{1-m_1}{q_1}$ and $e_2 > \frac{1-m_2}{q_2}$, when B^{eq} is the trivial fixed point.

- $\frac{\gamma}{\beta}(m_1+q_1e_1-1)(m_2+q_2e_2-1) > 0$,
and $1+r-m-\frac{2r}{\delta'}(m_1+q_1e_1-1)-\frac{\gamma}{\beta}(m_2+q_2e_2-1) < 0$, when B^{eq} is the axial fixed point.

- $1+A+B+C>0$, $1-A+B-C>0$, $1-B+AC-C^2>0$, $B<3$, when B^{eq} is the interior positive fixed point. *(A, B and C to be determined thereafter)*

Proof. After an algebraic calculation, we derive the equilibria of system (3) as follows

(i) Trivial fixed point

$B^{eq^0} = (0,0,0)$

(ii) Axial fixed point in the absence of predator

$B^{eq^1} = (\frac{1}{\delta'}(m_1+q_1e_1-1), \frac{1}{\beta}(m_2+q_2e_2-1), 0)$

(iii) Interior positive fixed point

$B^{eq^*} = (V^{eq^*}, X^{eq^*}, Y^{eq^*})$ with

$$\begin{cases} V^{eq^*} = 1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1) \\ X^{eq^*} = \frac{m_2+q_2e_2-1}{\beta} \\ Y^{eq^*} = \frac{1}{\alpha}\left(1+\delta'\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right)-m_1-q_1e_1\right) \end{cases}$$

We define a jacobian matrix M associated to the system (3) as

$$M = M(V_k, X_k, Y_k)$$

$$(4) \quad = \begin{pmatrix} 1+r(1-2V_k)-m-\gamma X_k & -\gamma V_k & 0 \\ \delta' X_k & 1+\delta' V_k-\alpha Y_k-m_1-q_1e_1 & -\alpha X_k \\ 0 & \beta Y_k & 1+\beta X_k-m_2-q_2e_2 \end{pmatrix}$$

- The jacobian matrices M^0, M^1 and M^* of the system (3) associated to fixed points B^{eq^0}, B^{eq^1} and B^{eq^*} respectively, are then defined as

$$M^0 = \begin{pmatrix} 1+r-m & 0 & 0 \\ 0 & 1-m_1-q_1e_1 & 0 \\ 0 & 0 & 1-m_2-q_2e_2 \end{pmatrix}$$

As we have obtained large matrices for M^1 and M^*, here are their components

$$\begin{aligned}
M^1_{11} &= 1+r-m-\frac{2r}{\delta'}(m_1+q_1e_1-1)-\frac{\gamma}{\beta}(m_2+q_2e_2-1) \\
M^1_{12} &= -\frac{\gamma}{\delta'}(m_1+q_1e_1-1) \\
M^1_{13} &= 0 \\
M^1_{21} &= \frac{\delta'}{\beta}(m_2+q_2e_2-1) \\
M^1_{22} &= 0 \\
M^1_{23} &= -\frac{\alpha}{\beta}(m_2+q_2e_2-1) \\
M^1_{3j} &= 0, j=1,2,3
\end{aligned}$$

and

$$\begin{aligned}
M^*_{11} &= m-r-1+\frac{\gamma}{\beta}(m_2+q_2e_2-1) \\
M^*_{12} &= -\gamma\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right) \\
M^*_{13} &= 0 \\
M^*_{21} &= \frac{\delta'}{\beta}(m_2+q_2e_2-1) \\
M^*_{22} &= 0 \\
M^*_{23} &= -\frac{\alpha}{\beta}(m_2+q_2e_2-1) \\
M^*_{31} &= 0 \\
M^*_{32} &= \frac{\beta}{\alpha}\left(1+\delta'\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right)-m_1-q_1e_1\right) \\
M^*_{33} &= 0
\end{aligned}$$

The characteristic equation of the jacobian matrix (4) can be written as

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{5}$$

which is a cubic equation with one variable.

We note that after calculations, A, B and C are defined explicitly by

$A = -3 - r + m + (2r - \delta')V_k + (\gamma - \beta)X_k + \alpha Y_k + m_1 + q_1e_1 + m_2 + q_2e_2)$

$B = (1 + r(1 - 2V_k) - m - \gamma X_k)(2 + \delta' V_k + \beta X_k - \alpha Y_k - m_1 - q_1e_1 - m_2 - q_2e_2)$
$+(1 + \delta' V_k - \alpha Y_k - m_1 - q_1e_1)(1 + \beta X_k - m_2 - q_2e_2) + \alpha\beta Y_k X_k + \gamma\delta' X_k V_k$

$C = (1 + r(1 - 2V_k) - m - \gamma X_k)$
$\times(-\alpha\beta Y_k X_k + (1 + \delta' V_k - \alpha Y_k - m_1 - q_1e_1)(1 + \beta X_k - m_2 - q_2e_2))$
$-\gamma\delta' X_k V_k(1 + \beta X_k - m_2 - q_2e_2)$

- For the matrix M_0, (5) becomes
$\lambda^3 - (3 + r - m - (m_1 + q_1e_1) - (m_2 + q_2e_2))\lambda^2$
$+((1 + r - m)(1 - m_1 - q_1e_1 + 1 - m_2 - q_2e_2) + (1 - m_1 - q_1e_1)(1 - m_2 - q_2e_2))\lambda$
$+(1 + r - m)(1 - m_1 - q_1e_1)(1 - m_2 - q_2e_2)$
$= 0$

whose roots are $\lambda_1 = 1 + r - m$, $\lambda_2 = 1 - m_1 - q_1e_1$ and $\lambda_3 = 1 - m_2 - q_2e_2$.
Then, the trivial equilibrium B^{eq^0} is is asymptotically stable if
$r < m - 1$, $e_1 > \frac{1-m_1}{q_1}$ and $e_2 > \frac{1-m_2}{q_2}$
For the matrix M_1, (5) becomes
$\lambda^3 - (1 + r - m - \frac{2r}{\delta'}(m_1 + q_1e_1 - 1) - \frac{\gamma}{\beta}(m_2 + q_2e_2 - 1))\lambda^2 + \frac{\gamma}{\beta}(m_1 + q_1e_1 - 1)$
$(m_2 + q_2e_2 - 1)\lambda = 0$

whose first root is $\lambda_1 = 0$ and for obtaining the other roots, we solve the quadratic equation

$$\lambda^2 - tr(M_1)\lambda + det(M_1) = 0 \tag{6}$$

with $tr(M_1) = 1 + r - m - \frac{2r}{\delta'}(m_1 + q_1 e_1 - 1) - \frac{\gamma}{\beta}(m_2 + q_2 e_2 - 1)$

and $det(M_1) = \frac{\gamma}{\beta}(m_1 + q_1 e_1 - 1)(m_2 + q_2 e_2 - 1)$

The discriminant of the quadratic equation (6) is $\Delta = tr(M_1)^2 - 4det(M_1)$

- if $\Delta > 0$, then, $\lambda_2 = \frac{tr(M_1)+\sqrt{\Delta}}{2}$ and $\lambda_3 = \frac{tr(M_1)-\sqrt{\Delta}}{2}$

- if $\Delta = 0$, then, $\lambda_2 = \lambda_3 = \frac{tr(M_1)}{2}$

- if $\Delta < 0$, then, $\lambda_2 = \frac{tr(M_1)+i\sqrt{|\Delta|}}{2}$ and $\lambda_3 = \frac{tr(M_1)-i\sqrt{|\Delta|}}{2}$

Then, the axial equilibrium B^{eq^1} is asymptotically stable if

$det(M_1) = \frac{\gamma}{\beta}(m_1 + q_1 e_1 - 1)(m_2 + q_2 e_2 - 1) > 0$

and

$tr(M_1) = 1 + r - m - \frac{2r}{\delta'}(m_1 + q_1 e_1 - 1) - \frac{\gamma}{\beta}(m_2 + q_2 e_2 - 1) < 0$

- For the matrix M^*, we have

$$A = 1 + r - m - \frac{\gamma}{\beta}(m_2 + q_2 e_2 - 1)$$

$$B = \left(1+\delta'\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right)-m_1-q_1e_1\right)$$
$$\times(m_2+q_2e_2-1)$$
$$+\frac{\delta'\gamma}{\beta}(m_2+q_2e_2-1)\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right)$$

$$C = (m-r-1+\frac{\gamma}{\beta}(m_2+q_2e_2-1))$$
$$\times\left(\frac{\beta}{\alpha}\left(1+\delta'\left(1+\frac{1-m}{r}-\frac{\gamma}{r\beta}(m_2+q_2e_2-1)\right)-m_1-q_1e_1\right)\right)$$
$$\times\left(-\frac{\alpha}{\beta}(m_2+q_2e_2-1)\right)$$

The local stability of positive fixed point B^{eq^*} is determined by the modulus of eigenvalues of the characteristic equation (5) at this fixed point. In fact, the modulus of all roots of (5), is less than 1 if and only if the conditions

(i) $1+A+B+C>0$

(ii) $1-A+B-C>0$

(iii) $1-B+AC-C^2>0$

(iv) $B<3$

hold [20].

Thus, based on conditions (i), (ii), (iii) and (iv), we seek conditions on all parameters in M^* under which the fixed point B^{eq^*} is asymptotically stable. We note that analytic conditions on the parameters for the local stability are not suitable in this case, to easy algebraic manipulations. □

In order to reduce the conservatism, we may refer to [21],[11],[23],[24]. All the mentioned approaches agree that Gershgorin circle theorem [25] is important to find a region that contains all roots of a characteristic polynomial in the complex plane. In fact, this helps us to find a circle that contains all the eigenvalues and therefore, to deduce which one has the minimal radius. This is important because the tighter the found bound, the higher the stability margin will be, which implies a more important stability of our system in (3) [21]. For instance, the matrix $M=(m_{ij})_{1\leq i,j\leq 3}$ (4), will have eigenvalues contained in the union of the three disks

$$D_i=\{\lambda|\ |\lambda-m_{ii}|\leq\sum_{j=1,j\neq i}^{3}|m_{ij}|\}$$

respectively, in

$$D_j=\{\lambda|\ |\lambda-m_{jj}|\leq\sum_{i=1,i\neq j}^{3}|m_{ij}|\}$$

or in the intersection of above row-wise and column-wise disks.

Based on the approach in [11], an improved bound can be obtained by applying Gershgorin theorem to square of the companion matrix which is defined in our case by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -C & -B & -A \end{pmatrix}$$

with A, B and C defined below equation (5).

The square of the companion matrix is equal to

$$\begin{pmatrix} 0 & 0 & 1 \\ -C & -B & -A \\ AC & -C+AB & -B+A^2 \end{pmatrix}$$

After applying Gershgorin theorem to this matrix, solutions of equation (5) will be contained in the three Gershgorin circles

$|\lambda^2|\leq 1, |\lambda^2+B|\leq|A|+|C|, |\lambda^2+B-A^2|\leq|AC|+|AB-C|$

Using triangle inequality, we have

$|\lambda^2|\leq 1, |\lambda^2|\leq|A|+|B|+|C|, |\lambda^2+B-A^2|\leq|AC|+|AB-C|+|A^2-B|$

Thus, the circle which contains all solutions of equation (5), has a radius equal to $\max\{1,|A|+|B|+|C|,|AC|+|AB-C|+|A^2-B|\}$

Hote et al. in [23] reported that tighter bounds can be obtained if Gershgorin theorem is applied to higher powers of companion matrix. For this, they suggested at first, to calculate the powers of companion matrix using generalized result in [12]. In our case, the rows of the matrices to be formed can be written as

$$\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
-C & -B & -A \\
AC & AB-C & A^2-B \\
-A^2C+BC & B^2-A^2B+AC & -A^3+2AB-C \\
A^3-2ABC+C^2 & -A^3B-2AB^2+2BC-A^2C & A^4+2AC-3A^2B+B^2 \\
\cdots & \cdots & \cdots
\end{array}$$

Now, the companion matrix with power $p=3$ for example, is obtained by deleting the first two rows and forming it with three successive rows, and so on for other powers. Hence, by applying Gershgorin theorem to the power of a companion matrix $(c_{ij})_{1\leq i,j\leq 3}$ with $p\geq 2$, the solutions of (5) will be obtained in at least one of the union of the three disks

$$D_i = \{\lambda |\ |\lambda - c_{ii}| \leq \max_{1\leq i\leq 3} \sqrt[p]{\sum_{j=1}^{3} |c_{ij}|}\}$$

respectively, in

$$D_j = \{\lambda |\ |\lambda - c_{jj}| \leq \max_{1\leq j\leq 3} \sqrt[p]{\sum_{i=1}^{3} |c_{ij}|}\}$$

By following these results, Hote et al. in [23], came back to the work of Yau in [21], for proving that a tighter bound could be obtained in the intersection of the above sets.

3. DISCRETE OPTIMAL HARVESTING POLICIES

3.1. The environmental sustainability case. In the following, we introduce two controls e_1 and e_2 as functions of time now into the model (2), and which are associated to h_{1_k} and h_{2_k}

respectively. Then, (1) and (2) become

$$\begin{cases} v_{k+1} = v_k + r v_k \left(1 - \frac{v_k}{K}\right) - \gamma x_k v_k - m v_k \\ x_{k+1} = x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - q_1 e_{1_k} x_k \\ y_{k+1} = y_k + \beta x_k y_k - m_2 y_k - q_2 e_{2_k} y_k \end{cases} \tag{7}$$

Our objective is to suggest an optimal harvesting policy subject to system (7), and which concerns the maximization of the two harvesting functions h_{1_k} and h_{2_k} while minimizing the fishing effort functions e_{1_k} and e_{2_k}, related to an objective function J which takes one of the two following forms

$$J(e_1, e_2) = \sum_{k=0}^{N-1} (a_0 h_{1_k} + b_1 h_{2_k} - A e_{1_k}^2 - B e_{2_k}^2) \tag{8}$$

with a_0 a positive grazers harvesting severity weight which is very close to 0, and b_1; a predators harvesting severity weight which is very close to 1.

Or,

$$J(e_1, e_2) = \sum_{k=0}^{N-1} (a_1 h_{1_k} + b_0 h_{2_k} - A e_{1_k}^2 - B e_{2_k}^2) \tag{9}$$

with a_1 a grazers harvesting severity weight which is very close to 1, and b_0; a positive predators harvesting severity weight which is very close to 0.

Thus, we seek two optimal control functions e_1^* and e_2^* satisfying

$$\max_{(e_1,e_2)\in E^2} J(e_1, e_2) = J(e_1^*, e_2^*) \tag{10}$$

with

$E = \{(e_1, e_2) \in \mathbb{R}^{2N} | 0 \leqslant e_{1_k}, e_{2_k} \leqslant 1,\ k = 0, .., N-1\}$ the set of admissible controls.

We note that from (8)-(9), we aim to minimize both fishing costs but with focusing on preservation of grazers or predators populations, and that is beneficial for fishermen since the two different optimal harvesting strategies will suggest minimal fishing efforts while maximizing one of the harvesting functions. An example which shows the importance of focusing sometimes on fishing preys more than predators as it is associated to the objective (9), can be found in [27]. A practical example related to the objective (8) can be found in details in [13].

Without loss of generality, we choose in this paper, to study the most frequent case for the sustainability of the grazers population when the objective function takes the form of (8).

In the following, we announce two theorems of sufficient and necessary conditions with the characterization of the two sought optimal controls $e^*_{1_k}$ and $e^*_{2_k}$.

Theorem 3.1.1. *(Sufficient conditions)*
For the discrete optimal control problem (10), subject to the discrete-time system (7), we have a verified existence of optimal controls e^*_1 and e^*_2) in E such that

$$\max_{(e_1,e_2)\in E} J(e_1,e_2) = J(e^*_1,e^*_2) \tag{11}$$

Proof: Let (e^*_1,e^*_2) be a control pair in E, and let try to verify the existence of $max_{(e_1,e_2)\in E}J(e_1,e_2)$. We have a finite number of time steps N, and discrete state equations in system (7) with bounded coefficients $r,\gamma,m,\delta,m_1,q_1,\beta,m_2$ and q_2, then for all (e_1,e_2) in the control set E, the n-component state variables
$v = (v_0,v_1,...,v_k,...,v_{N-1})$,
$x = (x_0,x_1,...,x_k,...,x_{N-1})$,
and $y = (y_0,y_1,...,y_k,...,y_{N-1})$ $\forall k = 0,...,N-1$ are uniformly bounded, which implies that $\forall(e_1,e_2)\in E$, $J(e_1,e_2)$ is uniformly bounded.

We can deduce then that $sup_{(e_1,e_2)\in E}J(e_1,e_2)$ is finite since $J(e_1,e_2)$ is bounded, and there exists a finite number j of uniformly bounded pair sequences $(e^j_1,e^j_2)\in E$ such that $lim_{j\to\infty}J(e^j_1,e^j_2) = sup_{(e_1,e_2)\in E}J(e_1,e_2)$ and corresponding sequences of states v^j, x^j and y^j.

Thus, there exists $(e_1^*, e_2^*) \in E$ and $v^*, x^*, y^* \in \mathbb{R}^N$ such that on a subsequence, we have
$(e_1^j, e_2^j) \to (e_1^*, e_2^*)$,
$v^j \to v^*$,
$x^j \to x^*$
and $y^j \to y^*$.

Finally, due to the finite dimensional structure of the system (7) and the objective function $J(e_1, e_2)$, (e_1^*, e_2^*) is an optimal control pair with corresponding states v^*, x^* and y^*. Therefore, taking into account the structure of J being a convex function, $sup_{(e_1,e_2)\in E} J(e_1, e_2)$ is achieved [30],[31].

Theorem 3.1.2. *(Necessary conditions and Characterization)*
Let e_1^* and e_2^* be two optimal controls with corresponding solutions v^*, x^* and y^* of the corresponding state system (7), there exist adjoint variables ψ_1, ψ_2 and ψ_3 satisfying

$$\triangle\psi_{1,k} = \psi_{1,k+1}\left(-1 + r\left(\frac{2}{K}v_k - 1\right) + \gamma x_k + m\right) - \psi_{2,k+1}\delta x_k$$

$$\triangle\psi_{2,k} = -a_0 q_1 e_{1_k} + \psi_{1,k+1}\gamma v_k - \psi_{2,k+1}\left(1 + \delta v_k - \alpha y_k - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1}\beta y_k$$

$$\triangle\psi_{3,k} = -b_1 q_2 e_{2_k} + \psi_{2,k+1}\alpha x_k - \psi_{3,k+1}(\beta x_k + 1 - m_2 - q_2 e_{2_k})$$

with $\triangle\psi_{l,k} = \psi_{l,k+1} - \psi_{l,k},\ l = 1,...,3,\ k = 0,...,N-1$ the difference operator and $\psi_{1,N} = \psi_{2,N} = \psi_{3,N} = 0$ are the transversality conditions.
Furthermore,

$$e_{1_k}^* = min(max(0, \frac{q_1 x_k}{2A}\left(a_0 - \psi_{2,k+1}\right), 1) \tag{12}$$

$$e_{2_k}^* = min(max(0, \frac{q_2 y_k}{2B}\left(b_1 - \psi_{3,k+1}\right), 1) \tag{13}$$

Proof: Let H be the hamiltonian function defined as the sum of the integrand term of J in (8) and the term $\psi_{1,k+1}v_{k+1} + \psi_{2,k+1}x_{k+1} + \psi_{3,k+1}y_{k+1}$, where ψ_1, ψ_2, ψ_3 represent the adjoint variables, corresponding to variables v, x and y respectively.
Then, we have

$H(v_k, x_k, y_k, \psi_{1,k+1}, \psi_{2,k+1}, \psi_{3,k+1}, e_{1_k}, e_{2_k})$
$= a_0 q_1 e_{1_k} x_k + b_1 q_2 e_{2_k} y_k - A e_{1_k}^2 - B e_{2_k}^2$
$+\psi_{1,k+1}(v_k + r v_k(1 - \frac{v_k}{K}) - m v_k - \gamma x_k v_k)$
$+\psi_{2,k+1}(x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - h_{1_k})$
$+\psi_{3,k+1}(y_k + \beta x_k y_k - m_2 y_k - h_{2_k})$

Thus, based on a discrete version of Pontryagin's maximum principle [32],[31], we obtain the existence of adjoint variables defined by
$\triangle\psi_{1,k} = -\dfrac{\partial H_k}{\partial v_k}$, $\triangle\psi_{2,k} = -\dfrac{\partial H_k}{\partial x_k}$ and $\triangle\psi_{3,k} = -\dfrac{\partial H_k}{\partial y_k}$

with H_k a brief notation of $H(v_k, x_k, y_k, \psi_{1,k+1}, \psi_{2,k+1}, \psi_{3,k+1}, e_{1_k}, e_{2_k})$.

We recall that in the formulation of H, $h_{1_k} = q_1 e_{1_k} x_k$ and $h_{2_k} = q_2 e_{2_k} y_k$.
Since the logistic term is explicitly given by $r v_k - r\dfrac{v_k^2}{K}$ which implies its derivative is equal to

$$r - 2\frac{r v_k}{K} = r\left(1 - \frac{2 v_k}{K}\right)$$

Thus,
$\triangle\psi_{1,k} = -\psi_{1,k+1}\left(1 + r\left(1 - \frac{2}{K} v_k\right) - \gamma\, x_k - m\right) - \psi_{2,k+1}\delta x_k$

which implies
$\triangle\psi_{1,k} = \psi_{1,k+1}\left(-1 + r\left(\frac{2}{K} v_k - 1\right) + \gamma\, x_k + m\right) - \psi_{2,k+1}\delta x_k$
$\triangle\psi_{2,k} = -a_0 q_1 e_{1_k} + \psi_{1,k+1}\gamma v_k - \psi_{2,k+1}\left(\delta v_k - \alpha y_k + 1 - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1}\beta y_k$
$\triangle\psi_{3,k} = -b_1 q_2 e_{2_k} + \psi_{2,k+1}\alpha x_k - \psi_{3,k+1}(\beta x_k + 1 - m_2 - q_2 e_{2_k})$

with the final conditions $\psi_{1,N}$, $\psi_{2,N}$ and $\psi_{3,N}$ defined as the derivatives of the terminal cost with respect to v_k, x_k and y_k at instant $k = N$ respectively. Since the function in the terminal cost term is zero in (8), then $\psi_{1,N} = \psi_{2,N} = \psi_{3,N} = 0$.

As regards to the characterization of the two sought optimal controls e_1^* and e_2^*, we differentiate H_k with respect to e_{1_k} and e_{2_k}.

Then, we have,

$\dfrac{\partial H_k}{\partial e_{1_k}} = a_0 q_1 x_k - 2Ae_{1_k} - \psi_{2,k+1} q_1 x_k = 0$ at $e_{1_k} = e^*_{1_k}$

$\dfrac{\partial H_k}{\partial e_{2_k}} = b_1 q_2 y_k - 2Be_{2_k} - \psi_{3,k+1} q_2 y_k = 0$ at $e_{2_k} = e^*_{2_k}$

which implies

$e_1^*(t) = \frac{q_1 x_k}{2A}\left(a_0 - \psi_{2,k+1}\right)$

and

$e_2^*(t) = \frac{q_2 y_k}{2B}\left(b_1 - \psi_{3,k+1}\right)$

Therefore, by using the bounds $0 \leqslant e_{1_k} \leqslant 1$, $0 \leqslant e_{2_k} \leqslant 1$, we obtain (12)-(13) which both verify the condition of maximum

$\max_{(e_1,e_2)\in E} H(v_k, x_k, y_k, \psi_{1,k+1}, \psi_{2,k+1}, \psi_{3,k+1}, e_{1_k}, e_{2_k})$

$= H(v_k^*, x_k^*, y_k^*, \psi_{1,k+1}, \psi_{2,k+1}, \psi_{3,k+1}, e^*_{1_k}, e^*_{2_k})$

equivalent to the condition (11).

3.2. The bioeconomic case.

3.2.1. *The profit function using the bioeconomic equilibrium model.* Based on Schaefer's model [33], Gordon developed in [34], one of the earliest bioeconomic models. Gordon's model showed that the benefit π_i (or net income) to fisherman i, is defined as the difference between total revenue R_i and total cost C_i (i.e. $\pi_i = R_i - C_i$).

Using details and explanations from [13], we deduce directly that the benefit or profit function applied to a fishing effort e_i, is defined as

$$\pi_i(e_i) = p_i q_i e_i - c_i e_i \tag{14}$$

where p_i, q_i and c_i represent the price of the harvested fish population, catchability coefficient and harvesting cost per fishing effort associated to fisherman i respectively.

In the following, we suggest a discrete-time optimal control strategy for the bioeconomic case.

3.2.2. *Discrete optimal bioeconomic control approach.* We consider the two same fishing effort controls e_1 and e_2 in (2), and we associate them now to the two profits functions $\pi_1(e_1)$ and $\pi_2(e_2)$. Then, we suggest a discrete optimal control problem associated to system (7) using the definition of the following objective function J

$$J(e_1,e_2) = \sum_{k=0}^{\infty} e^{-\theta k}\left(\pi_1(e_{1_k}) + \pi_2(e_{2_k})\right) \tag{15}$$

where θ is the annual discount rate.

Explicitly, J is defined as

$$J(e_1,e_2) = \sum_{k=0}^{\infty} e^{-\theta k}\left(p_1q_1e_{1_k}x_k + p_2q_2e_{2_k}y_k - c_1e_{1_k} - c_2e_{2k}\right) \tag{16}$$

Now, we aim to provide a characterization of e_1^* and e_2^* such that

$$\max_{0\leq e_1,e_2\leq 1} J(e_1,e_2) = J(e_1^*,e_2^*),$$

that leads to their minimization along with a maximization of $\pi_1(e_1)$ and $\pi_2(e_2)$.

Again, using theorem 3.1.2 announced earlier, the proof to the existence of an optimal control pair $\vec{e}^* = (e_1^*,e_2^*)$ satisfying (11), can easily be reached. In the following, we announce the theorem of necessary conditions and characterization associated to the bioeconomic case.

Theorem 3.3.2.1. *(Characterization)*

Given two optimal controls e_1^* and e_2^* and solutions v^*, x^* and y^* of the corresponding state system (7), there exist adjoint variables ψ_1, ψ_2 and ψ_3 defined by

$$\triangle\psi_{1,k} = -\psi_{1,k+1}\left(1 + r\left(1 - \frac{2v_k}{K}\right) - m - \gamma x_k\right) - \psi_{2,k+1}\delta x_k \tag{17}$$

$$\triangle\psi_{2,k} = -e^{-\theta k}p_1q_1e_{1_k} + \psi_{1,k+1}\gamma v_k - \psi_{2,k+1}\left(\delta v_k - \alpha y_k + 1 - m_1 - q_1e_{1_k}\right) - \psi_{3,k+1}\beta y_k \tag{18}$$

$$\triangle\psi_{3,k} = -e^{-\theta k}p_2q_2e_{2k} + \psi_{2,k+1}\alpha x_k - \psi_{3,k+1}\left(\beta x_k + 1 - m_2 - q_2e_{2_k}\right) \tag{19}$$

with $\triangle\psi_{l,k} = \psi_{l,k+1} - \psi_{l,k},\ l = 1,\dots,3,\ k = 0,\dots,N-1$ the difference operator and $\psi_{1,N} = \psi_{2,N} = \psi_{3,N} = 0$ are the transversality conditions.

The optimal controls $(e_{i_k}^*)_{i=1,2}$ and $k = 0,\dots,N-1$, are singular with

$\psi_{2,k} = e^{-\theta(k-1)}\left(p_1 - \frac{c_1}{q_1 x_{k-1}}\right)$

$\psi_{3,k} = e^{-\theta(k-1)}\left(p_2 - \frac{c_2}{q_2 y_{k-1}}\right)$

which imply that $e^*_{1_k}$ and $e^*_{2_k}$ are defined by the following analytical formulations

(20)

$$e^*_{1_k} = \frac{x^*_k}{c_1}\left[e^{\theta}\left(p_1 - \frac{c_1}{q_1 x^*_{k-1}}\right) - \left(p_1 - \frac{c_1}{q_1 x^*_k}\right)(2+\delta v^*_k - \alpha y^*_k - m_1) + \gamma v^*_k e^{\theta k}\psi_{1,k+1} - \beta\left(y^*_k p_2 - \frac{c_2}{q_2}\right)\right]$$

and

$$e^*_{2_k} = \frac{y^*_k}{c_2}\left[e^{\theta}\left(p_2 - \frac{c_2}{q_2 y^*_{k-1}}\right) - \left(p_2 - \frac{c_2}{q_2 y^*_k}\right)(2+\beta x^*_k - m_2) + \alpha\left(x^*_k p_1 - \frac{c_1}{q_1}\right)\right] \tag{21}$$

At a biological equilibrium (v_{eq}, x_{eq}, y_{eq}), $\psi_{1,k+1}$ is defined by

$\psi_{1,k+1} = \frac{1}{2 - \frac{rv_{eq}}{K}}\left(\psi_{1,k} - \delta x_{eq} e^{-\theta k}\left(p_1 - \frac{c_1}{q_1 x_{eq}}\right)\right)$

the optimal fishing effort $e^*_{1_k}$ becomes $e^*_{1_{eq_k}}$ defined by

$$\begin{aligned} e^*_{1_{eq_k}} &= \frac{x_{eq}}{c_1}[e^{\theta}\left(p_1 - \frac{c_1}{q_1 x_{eq}}\right) - \left(p_1 - \frac{c_1}{q_1 x_{eq}}\right)(2+\delta v_{eq} - \alpha y_{eq} - m_1) \\ &+ \frac{\gamma v_{eq} e^{\theta k}}{2 - \frac{rv_{eq}}{K}}\left(\psi_{1,k} - \delta x_{eq} e^{-\theta k}\left(p_1 - \frac{c_1}{q_1 x_{eq}}\right)\right) - \beta\left(y_{eq} p_2 - \frac{c_2}{q_2}\right)] \end{aligned} \tag{22}$$

Proof: As done in the environment sustainability case, we define here, a hamiltonian function H_k as a sum of the integrand term of J in (16) and the term $\psi_{1,k+1}v_{k+1} + \psi_{2,k+1}x_{k+1} + \psi_{3,k+1}y_{k+1}$, where ψ_1, ψ_2, ψ_3 represent the adjoint variables, corresponding to variables v, x and y respectively.

Then, we have

$H_k = e^{-\theta k}[p_1 q_1 e_{1_k} x_k + p_2 q_2 e_{2_k} y_k - c_1 e_{1_k}$

$-c_2 e_{2_k}] + \psi_{1,k+1}(v_k + rv_k(1 - \frac{v_k}{K}) - mv_k - \gamma x_k v_k)$

$+\psi_{2,k+1}(x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - q_1 e_{1_k} x_k)$

$+\psi_{3,k+1}(y_k + \beta x_k y_k - m_2 y_k - q_2 e_{2_k} y_k)$

By passage to the derivatives of H_k with respect to e_{1_k} and e_{2_k}, we have,

$\frac{\partial H_k}{\partial e_{1_k}} = e^{-\theta k}[p_1 q_1 x_k - c_1] - \psi_{2,k+1} q_1 x_k = \sigma_{1,k}$

and

$$\frac{\partial H_k}{\partial e_{2_k}} = e^{-\theta k}\left[p_2 q_2 y_k - c_2\right] - \psi_{3,k+1} q_2 y_k = \sigma_{2,k}$$

The optimal control e_{i_k} must satisfy

$$(e_{i_k})_{i=1,2} = \begin{cases} e_{i_k}^{\max} & \text{when } \sigma_{i,k} > 0 \\ 0 & \text{when } \sigma_{i,k} < 0 \end{cases} \tag{23}$$

with $e_{i_k}^{\max} \leqslant 1$

If the switching function $\sigma_{i,k} = 0, i \in \{1,2\}$ then, from the derivatives of H_k, we obtain
$\psi_{2,k+1} = e^{-\theta k}\left[p_1 - \frac{c_1}{q_1 x_k}\right]$
and
$\psi_{3,k+1} = e^{-\theta k}\left[p_2 - \frac{c_2}{q_2 y_k}\right]$
then,
$\psi_{2,k} = e^{-\theta(k-1)}\left[p_1 - \frac{c_1}{q_1 x_{k-1}}\right]$
and
$\psi_{3,k} = e^{-\theta(k-1)}\left[p_2 - \frac{c_2}{q_2 y_{k-1}}\right]$

On the other part, and based on a discrete version of Pontryagin's maximum principle above, the adjoint equations, are expressed as follows

$$\triangle\psi_{1,k} = -\frac{\partial H_k}{\partial v_k} = -\psi_{1,k+1}\left(1 + r\left(1 - \frac{2v_k}{K}\right) - m - \gamma x_k\right) - \psi_{2,k+1}\delta x_k \tag{24}$$

(25)
$$\triangle\psi_{2,k} = -\frac{\partial H_k}{\partial x_k} = -e^{-\theta k} p_1 q_1 e_{1_k} + \psi_{1,k+1}\gamma v_k - \psi_{2,k+1}\left(1 + \delta v_k - \alpha y_k - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1}\beta y_k$$

$$\triangle\psi_{3,k} = -\frac{\partial H_k}{\partial y_k} = -e^{-\theta k} p_2 q_2 e_{2_k} + \psi_{2,k+1}\alpha x_k - \psi_{3,k+1}\left(1 + \beta x_k - m_2 - q_2 e_{2_k}\right) \tag{26}$$

At a biological equilibrium (v_{eq}, x_{eq}, y_{eq}) which converts the difference operator $\triangle v_k = v_{k+1} - v_k$ to zero. Then, from the first equation of the discrete-time system (7), (24) becomes

$\triangle\psi_{1,k} = -\dfrac{\partial H_k}{\partial v_k} = \psi_{1,k+1}\dfrac{rv_{eq}}{K} - \psi_{2,k+1}\delta x_{eq} = \psi_{1,k+1}\dfrac{rv_{eq}}{K} - \delta x_{eq}e^{-\theta k}\left(p_1 - \dfrac{c_1}{q_1 x_{eq}}\right)$ which implies

$\psi_{1,k+1} = \dfrac{1}{2-\frac{rv_{eq}}{K}}(\psi_{1,k} - \delta x_{eq}e^{-\theta k}\left(p_1 - \dfrac{c_1}{q_1 x_{eq}}\right)$

Now, using the expressions of ψ_2 and ψ_3 from (25) and (26), we obtain

$$\begin{aligned}&e^{-\theta k}\left[p_1 - \tfrac{c_1}{q_1 x_k}\right] - e^{-\theta(k-1)}\left[p_1 - \tfrac{c_1}{q_1 x_{k-1}}\right]\\ = &-e^{-\theta k}p_1 q_1 e_{1_k} + \psi_{1,k+1}\gamma v_k - \psi_{2,k+1}\left(1+\delta v_k - \alpha y_k - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1}\beta y_k\end{aligned}$$

and

$$\begin{aligned}&e^{-\theta k}\left[p_2 - \tfrac{c_2}{q_2 y_k}\right] - e^{-\theta(k-1)}\left[p_2 - \tfrac{c_2}{q_2 y_{k-1}}\right]\\ = &-e^{-\theta k}p_2 q_2 e_{2_k} + \psi_{2,k+1}\alpha x_k - \psi_{3,k+1}\left(1+\beta x_k - m_2 - q_2 e_{2_k}\right)\end{aligned}$$

which implies, using the explicit formulations of ψ_2 and ψ_3 that

$e^*_{1_k} = \frac{x^*_k}{c_1}\left[e^{\theta}\left(p_1 - \frac{c_1}{q_1 x^*_{k-1}}\right) - \left(p_1 - \frac{c_1}{q_1 x^*_k}\right)\left(2+\delta v^*_k - \alpha y^*_k - m_1\right) + \gamma v^*_k e^{\theta k}\psi_{1,k+1} - \beta\left(y^*_k p_2 - \frac{c_2}{q_2}\right)\right]$

and

$e^*_{2_k} = \frac{y^*_k}{c_2}\left[e^{\theta}\left(p_2 - \frac{c_2}{q_2 y^*_{k-1}}\right) - \left(p_2 - \frac{c_2}{q_2 y^*_k}\right)\left(2+\beta x^*_k - m_2\right) + \alpha\left(x^*_k p_1 - \frac{c_1}{q_1}\right)\right]$

With taking into account that at equilibrium conditions, $x_{eq} = x_k = x_{k-1}$. Then, by formulation of $\psi_{1,k+1}$, we can also deduce that the optimal fishing effort $e^*_{1_k}$ converts to the function $e^*_{1_{eq_k}}$ defined by

$$\begin{aligned}e^*_{1_{eq_k}} \quad &= \frac{x_{eq}}{c_1}[e^{\theta}\left(p_1 - \frac{c_1}{q_1 x_{eq}}\right) - \left(p_1 - \frac{c_1}{q_1 x_{eq}}\right)\left(2+\delta v_{eq} - \alpha y_{eq} - m_1\right)\\ &+ \frac{\gamma v_{eq}e^{\theta k}}{2-\dfrac{rv_{eq}}{K}}(\psi_{1,k} - \delta x_{eq}e^{-\theta k}\left(p_1 - \frac{c_1}{q_1 x_{eq}}\right)) - \beta\left(y_{eq}p_2 - \frac{c_2}{q_2}\right)]\end{aligned}$$

4. NUMERICAL RESULTS

In order to provide numerical simulations associated to the environmental sustainability and bioeconomic cases, we resolve the two following discrete two-point value problems

$$\begin{cases} v_{k+1} = v_k + r v_k \left(1 - \frac{v_k}{K}\right) - \gamma x_k v_k - m v_k \\ x_{k+1} = x_k + \delta x_k v_k - \alpha x_k y_k - m_1 x_k - q_1 e_{1_k} x_k \\ y_{k+1} = y_k + \beta x_k y_k - m_2 y_k - q_2 e_{2_k} y_k \\ v_0 \geqslant 0,\ x_0 \geqslant 0\ y_0 \geqslant 0\ \textit{given intial conditions} \\ \\ \textit{First case} \\ \psi_{1,k+1} = \psi_{1,k} + \psi_{1,k+1} \left(-1 + r\left(\frac{2}{K} v_k - 1\right) + \gamma\, x_k + m\right) - \psi_{2,k+1} \delta x_k \\ \psi_{2,k+1} = \psi_{2,k} - a_0 q_1 e_{1_k} + \psi_{1,k+1} \gamma v_k - \psi_{2,k+1} \left(1 + \delta v_k - \alpha y_k - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1} \beta y_k \\ \psi_{3,k+1} = \psi_{3,k} - b_1 q_2 e_{2_k} + \psi_{2,k+1} \alpha x_k - \psi_{3,k+1} (\beta x_k + 1 - m_2 - q_2 e_{2_k}) \\ \psi_{1,N} = \psi_{2,N} = \psi_{3,N} = 0\ \textit{the transversality conditions} \\ \\ \textit{Second case} \\ \psi_{1,k+1} = \psi_{1,k} - \psi_{1,k+1} \left(1 + r\left(1 - \frac{2v_k}{K}\right) - m - \gamma x_k\right) - \psi_{2,k+1} \delta x_k \\ \psi_{2,k+1} = \psi_{2,k} - e^{-\theta k} p_1 q_1 e_{1_k} + \psi_{1,k+1} \gamma v_k - \psi_{2,k+1} \left(\delta v_k - \alpha y_k + 1 - m_1 - q_1 e_{1_k}\right) - \psi_{3,k+1} \beta y_k \\ \psi_{3,k+1} = \psi_{3,k} - e^{-\theta k} p_2 q_2 e_{2k} + \psi_{2,k+1} \alpha x_k - \psi_{3,k+1} \left(\beta x_k + 1 - m_2 - q_2 e_{2_k}\right) \\ \psi_{1,N} = \psi_{2,N} = \psi_{3,N} = 0\ \textit{the transversality conditions} \end{cases}$$

using the Forward-Backward Sweep Method (FBSM) [35] with an incorporation of a discrete progressive iterative scheme to stock at each iteration k, the values of the state variables corresponding to the above forward system with initial conditions, for using them in a discrete regressive iterative scheme incorporated for stocking at each time i, the values of the adjoint state variables corresponding to the two backward discrete-time adjoint systems with transversality conditions. Furthermore, at each time k, the values stocked of both state and adjoint state variables, are utilized in the characterization of optimal controls e_1^* and e_2^*.

Figure 1. depicts dynamics of the vegetation, grazers and predators populations v, x and y in the two cases when there is yet no control strategy and when we introduce the two optimal fishing effort functions e_1^* and e_2^* associated respectively to x and y variables. We also note that in this figure, we consider that the vegetation growth rate r is equal to 0.2. In fact, we choose

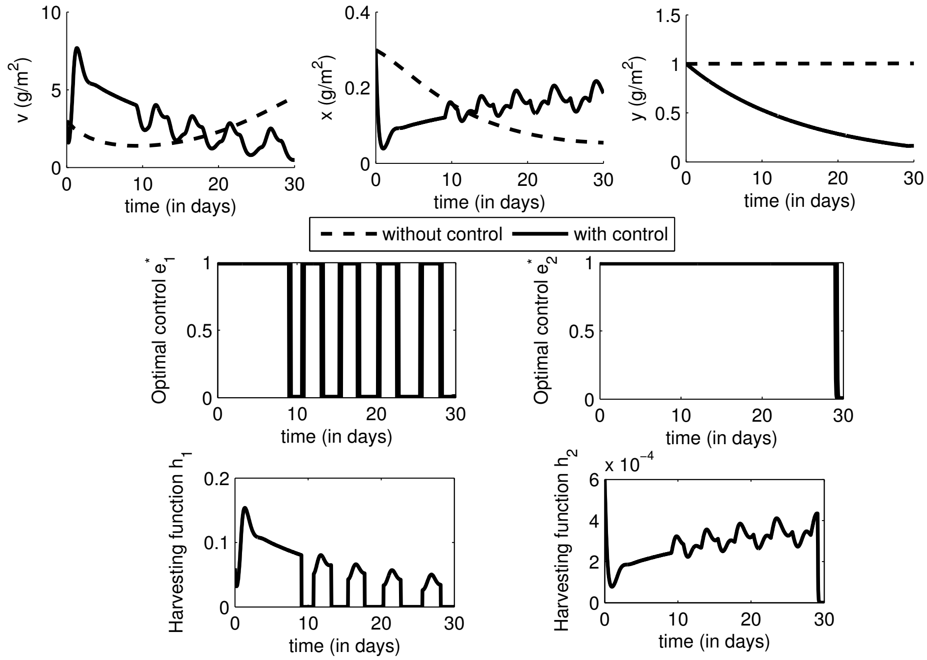

FIGURE 1. Harvesting case: Trophic-halieutic dynamics in the absence and presence of fishing effort. $q_1 = 0.02$, $q_2 = 0.002$, $r = 0.2$, $K = 10$, $m = 0.0000001$, $m_1 = 0.0001$, $m_2 = 0.00001$, $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 1$, $\delta = 0.02$.

to investigate the behaviors of v, x and y in cases of three different values of the parameter r, seeking to illustrate simulations with nonperiodic solutions.

Additionally, we note that in Figures 1., 2. and 3., we study the case when the catchability coefficient q_1 is strictly superior to q_2, which could represent a case of a fishing zone with important grazers biomass, while in Figure 4., we present numerical results when $q_1 < q_2$, which could represent a case of fishing nets with a big mesh size that target predators more.

When e_1^* and e_2^* are zero, we observe that the vegetation population v does not exceed values comprised between 2 and $4g/m^2$, while the grazers population x decreases from its initial condition towards the value $0.75g/m^2$ due to stabilization of the predators population y in important values very close to $1g/m^2$. On the other hand, when we suppose there are fishing fleets, we can observe the influence of both optimal controls e_1^* and e_2^* on the trophic-halieutic dynamics, by increasing the maximal value associated to the v population towards $7.5g/m^2$. In

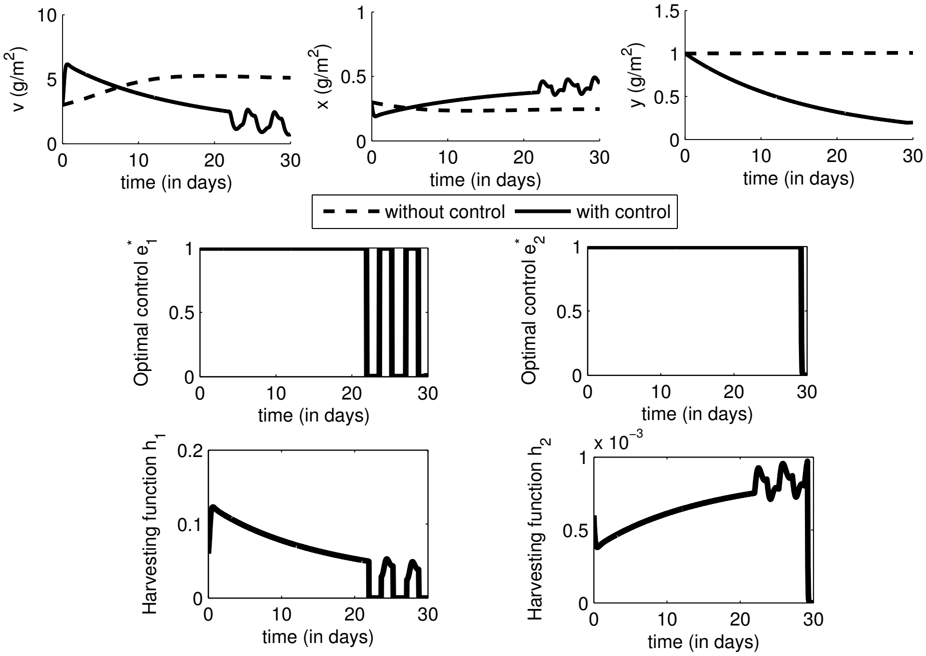

FIGURE 2. Harvesting case: Trophic-halieutic dynamics in the absence and presence of fishing effort. $q_1 = 0.02$, $q_2 = 0.002$, $r = 0.5$, $K = 10$, $m = 0.0000001$, $m_1 = 0.0001$, $m_2 = 0.00001$, $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 1$, $\delta = 0.02$.

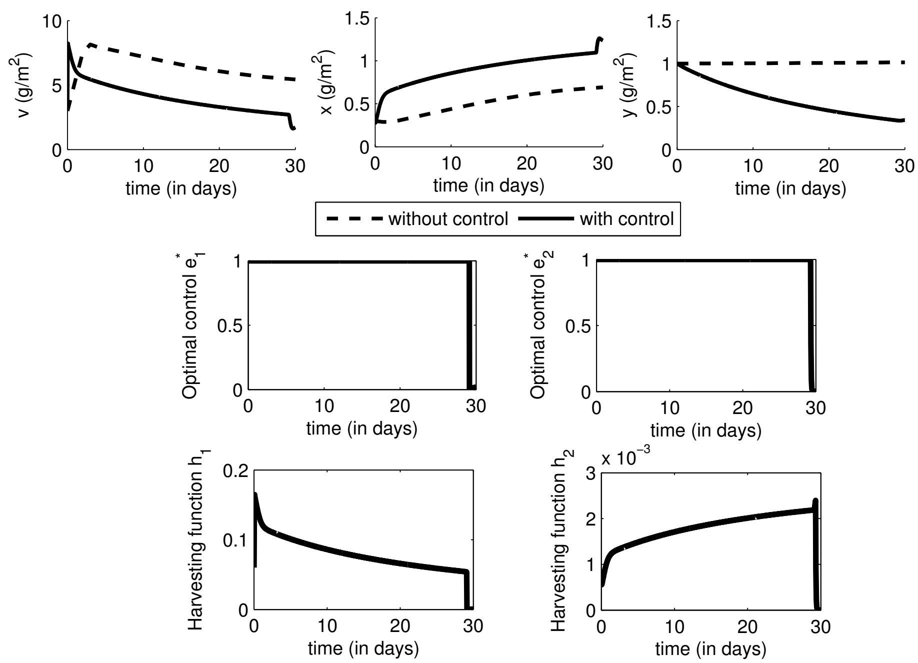

FIGURE 3. Harvesting case: Trophic-halieutic dynamics in the absence and presence of fishing effort. $q_1 = 0.02$, $q_2 = 0.002$, $r = 1$, $K = 10$, $m = 0.0000001$, $m_1 = 0.0001$, $m_2 = 0.00001$, $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 1$, $\delta = 0.02$.

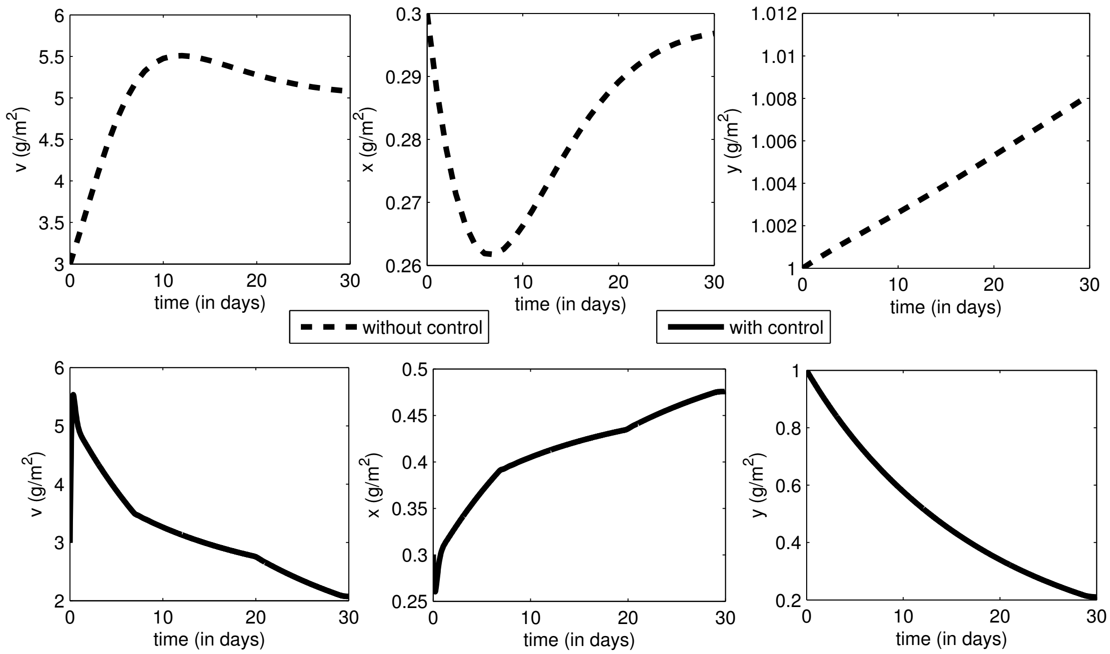

FIGURE 4. Bioeconomic case: Trophic-halieutic dynamics in the absence and presence of fishing effort. $q_1 = 0.002$, $q_2 = 0.02$, $r = 0.6$, $K = 10$, $m = 0.0000001$, $m_1 = 0.0001$, $m_2 = 0.00001$, $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 1$, $\delta = 0.02$, $c_1 = 5$, $c_2 = 6$, $p_1 = 100$, $p_2 = 600$.

addition, the optimal control e_1^* has shown its impact on the grazers by participating in a rapid decrease of the x variable towards $0.04g/m^2$ after the first day of fishing. Also, the x population amount increases after several ups and downs due to the periodic-like behavior of e_1^* towards a value equaling to $0.25g/m^2$. As regards to the y population, it decreases towards a smaller value equaling to $0.2g/m^2$ due to the important value taken by the optimal fishing effort e_2^* which stabilizes in the maximal value 1 along the fishing fleet period and decreases until a time strictly superior to the final time.

We mention that periodic solutions observed from the shapes of the v and x functions are also due to the nature of our system. In fact, periodic solutions are often met in the discrete-time case. This also shows more the analogy between the values taken by the optimal fishing effort e_1^* and other functions, since we can observe clearly that whenever it takes maximal and minimal values comprised between 0 and 1, v and x behave similarly towards their peaks. We

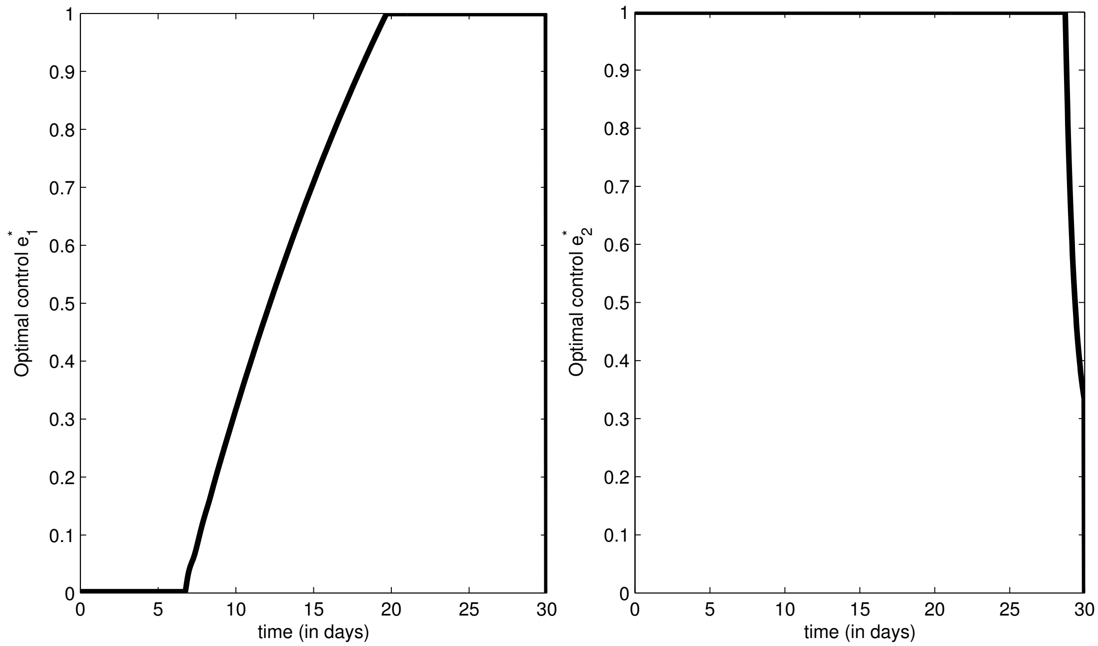

FIGURE 5. Bioeconomic case: Optimal fishing efforts e_1^* and e_2^*.

can also see there is an analogy between shapes of e_1^* and its associated harvesting function h_1, while the harvesting function h_2 increases periodically between 10 days and the final time due to the maximal value conserved by the optimal control e_2^*.

In Figure 2., we present simulations associated to the case when the vegetation growth rate r is equal to 0.5, and we can observe that in the case when there is yet no fishing fleets, the vegetation population amount does not exceed the value $5.1g/m^2$ while it takes a maximal peak equaling to $5.3g/m^2$ when we introduce the two optimal controls e_1^* and e_2^* due to the increase of the grazers population x towards the value $0.5g/m^2$ and which remains stabilizing in the value $0.275g/m^2$ when $e_1^* = 0$. The behavior of the predators population y does not change since the optimal control e_2^* takes similar shape as seen in the previous figure. We also deduce there are again some analogies between shapes of v, x and the harvesting function h_1 with the shape of the optimal control e_1^* with a reduction of the number of peaks observed in time of periodic solutions even in the shape of the harvesting function h_2.

As regards to Figure 3., we consider that the parameter r is equal to 1, and in this case,

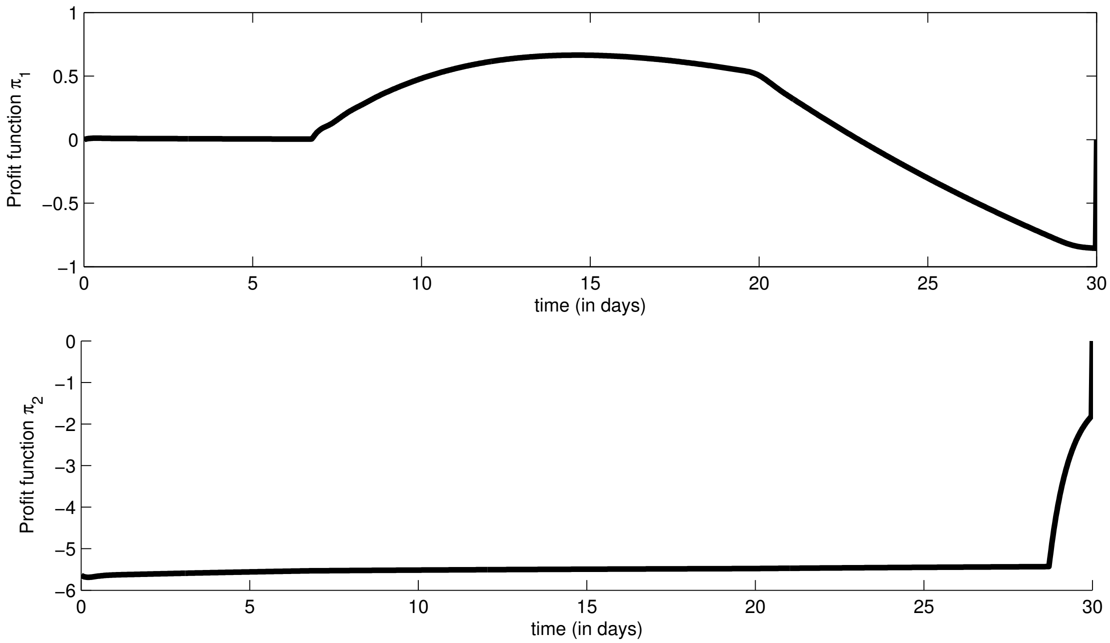

FIGURE 6. Bioeconomic case: Profit functions π_1 and π_2.

we can observe that in the case when $e_1^* = e_2^* = 0$, the v takes a maximal value which equals to $7.75g/m^2$ and decreases towards $6.5g/m^2$ due the important value of r, while it takes a maximal value which equals to $7.5g/m^2$ and decreases towards $3g/m^2$.

Figures 4., 5. and 6. depict numerical simulations associated to the bioeconomic case. In Figure 4. we can observe dynamics of v, x and y variables in the absence and presence of fishing fleets, and we can deduce that after the introduction of the two optimal controls e_1^* and e_2^* which stabilize in 1 value in , the vegetation decreases towards $2g/m^2$ because there is an important amount of predators that are harvested more than grazers ($q_1 < q_2$) and this leads to an important consumption of aquatic plants by grazers. In fact, when there are no controls, we can observe in the first 10 days that as more y function increases, as more x function decreases and then v function takes more important values.

We can also observe from Figure 5., that e_1^* increases gradually from 0 as the estimated initial condition towards 1 after only 7 days while e_2^* starts from 1 as its initial estimated condition and remains stabilized in this value for 28 days, which proves also the important decrease of y

function.

From Figure 6., we can understand there is an analogy between the two optimal controls e_1^* and e_2^* and the two profit functions π_1 and π_2. In fact, when the optimal fishing effort e_1^* has started to increase after 7 days of the fishing period, the profit function π_1 has increased simultaneously towards the positive peak 0.75. We note that in these first 7 days, there is no gain and no loss since π_1 remains zero. The result shows there is a gain in the profit associated to grazers fishes only between 7 and 21 days of the fishing fleet, and it changes to a loss after this period due to the fall of the function e_1^* at the final instant. As regards to the function π_2, we can see there is only loss in the profit regards the harvest of predators fishes, but π_2 stabilizes in a value around -5.5 in most periods of the fishing fleet due to the long stabilization of the optimal control e_2^* in the value 1, which also means that for larger values of the price p_2, there is a possibility that there will be a stable gain in profit.

In this paper, environmental and economic interests motivated the suggestion of two optimal control harvesting approaches applied to grazer-predator populations. There are studies which were based on other prey-predator models, showed that the harvesting effort has an impact on the evolution of fish populations, see for example [36], with possible or optimal control strategies as obtained also in [5]. Here, we show the effect of the optimal control or harvesting effort on the system (3) based on stability and control analysis with numerical simulations. Firstly, the stability of fixed points showed that in the absence of the two harvesting efforts e_1 and e_2 corresponding to grazers and predators populations respectively, system (3) tended to more stable states than the case when the two optimal controls have been introduced. In fact, the oscillatory fluctuations caused by the intrinsic growth of vegetation showed that when there is a maximal harvesting value which may correspond to overfishing, the values of x and y decreased and the life-cycle of grazer-predators population depended on periodicity of controls. This well-illustrates that an unreasonable harvest could lead to extinction of such species. Our study also illustrates that the two optimal controls introduced in the bioeconomic case, were proportional to the profit functions, which means there is a relationship between harvest and the market price of these fishes. Therefore, responsible entities in the fishing sector, should control the market price of species.

5. CONCLUSION

We devised a discrete-time mathematical model which describes dynamics of a food chain composed by three different marine species; the aquatic plants, grazers and predators. Firstly, we studied the stability of the fixed points in three different situations (trivial, axial and positive equilibria) based on the resolution of some cubic and quadratic equations. Secondly, we suggested two optimal harvesting policies which both aimed to minimize two discrete fishing effort functions related to grazer and predator variables, while maximizing their associated discrete harvesting and profit functions. The stability and control analysis developed in this paper, allow the investigation of stability of fixed points along with optimal harvesting policies when such approaches are applied to a hydraulic food chain model, framed in discrete time. By incorporating different values of the intrinsic growth rate of the vegetation, we observed how this tended to stabilize an instable or cyclic population process. This has especially been observed when the two optimal harvesting efforts were introduced, and then we can deduce that harvesting has a direct effect on the life-cycles of fishes. It would be also interesting to study the biological fluctuations from an other point, when for example, a set of such systems are described using stochastic differential equations since most ecological systems are inherently stochastic, due to the randomness of the environmental fluctuations.

Conflict of Interests

The authors declare that there is no conflict of interests.

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