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EKELAND'S VARIATIONAL PRINCIPLE AND FIXED POINT THEOREMS IN PARTIAL B-METRIC SPACES

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Abstract. In this paper, a new version of Ekeland's variational principle is presented in partial b-metric spaces via Cantor's intersection theorem. As a consequence, some Caristi type fixed point theorems are proved in partial b-metric spaces. Furthermore, the construction and number of fixed points are discussed.

Keywords: Ekeland's variational principle; partial b-metric space; fixed point; existence.

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1. INTRODUCTION

In 1974, Ekeland proposed a variational principle, which is the basis of modern variational calculus and has applications in many branches of mathematics, including optimization and fixed point theory etc.(see [1,2]). Since then, Bota [3] and Aydi [4] have applied Ekeland's variational principle to various generalized metric spaces, such as b-metric spaces, partial metric spaces(see [5,6]). Satish [7] compares the concepts of b-metric space and partial metric space and generates partial b-metric space. In his paper, an analog of the Banach contraction principle

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as well as the Kannan type fixed point theorem was proved, and some examples were put forward to illustrate this new space. (More papers concerning the fixed point theorems in b-metric spaces and partial metric spaces can be seen in [8-19] and references therein).

Motivated by above research, in this paper, we'll establish a new version of Ekeland's variational principle in partial b-metric space. Then, the principle is applied to derive some fixed point theorems in partial b-metric space.

This paper is organized as follows. In section 2, some definitions and notations are given. In section 3, Cantor's intersection theorem in partial b-metric spaces is proved, and a new version of Ekeland's variational principle is established. In section 4, some Caristi type fixed point theorems are prove in partial b-metric spaces. Furthermore, the construction and numbers of fixed points are also discussed.

2. PRELIMINARIES

In this section, some definitions and notations are presented.

Definition 2.1 [3] Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(bM1) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$

(bM2) $d(x,y) = d(y,x)$ for all $x, y \in X$

(bM3) there exists a real number $s \geq 1$, such that $d(x,y) \leq s[d(x,z) + d(z,y)]$

for all $x, y, z \in X$.

The pair (X,d) is called a b-metric space.

Definition 2.2 [4] Let $\mathbb{R}^+$ denote the set of all non-negative real numbers. A partial metric space is a pair(X, p) where X is a non-empty set and $p : X \times X \to \mathbb{R}^+$ is such that

(P1) $p(x,y) = p(y,x)$ (symmetry);

(P2) if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$ (equality);

(P3) $p(x,x) \leq p(x,y)$ (small self-distances);

(P4) $p(x,y) + p(z,z) \leq p(x,z) + p(y,z)$ (triangle inequality)

for all $x, y, z \in X$. We will use the abbreviation PMS for the partial metric space(X, p).

Remark 2.3 In a partial metric space (X,p), if $x,y \in X$ and $p(x,y) = 0$, then $x = y$, but the converse may not be true.

Example 2.4 Let $X = \mathbb{R}, 0 < r < 1$ a constant and $b : X \times X \to \mathbb{R}^+$ be defined by

$$p(x,y) = \max\{|x|^r, |y|^r\} \quad for\ all\ x,y \in X$$

Then (X,p) is a partial metric space.

Definition 2.5 [7] A partial $b-$metric on a nonempty set X is a function $b : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$,

(1) $b(x,y) = b(y,x)$;

(2) if $b(x,x) = b(x,y) = b(y,y)$, then $x = y$;

(3) $b(x,x) \leq b(x,y)$;

(4) there exists a real number $s \geq 1$, such that $b(x,y) \leq s[b(x,z) + b(y,z)] - b(z,z)$.

The partial b-metric space is a pair (X,b) such that X is a nonempty set and b is a b-metric on X. The number s is called the coefficient of (X,b). We will use the abbreviation $b - PMS$ for the partial b-metric space(X,b).

Remark 2.6 Let (X,b) be a b-PMS,

(i) if $b(x,y) = 0$, then $x = y$;

(ii)if $x \neq y$, then $b(x,y) > 0$.

Remark 2.7 It is obvious that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Example 2.8 [7] Let $X = \mathbb{R}^+, p > 1$ a constant and $b : X \times X \to \mathbb{R}^+$ be defined by

$$b(x,y) = [\max\{x,y\}]^p + |x-y|^p \quad for\ all\ x,y \in X$$

Then (X,b) is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space. Indeed, for any $x > 0$, we have $b(x,x) = x^p \neq 0$; therefore, b is not a b-metric on X. Also, for $x = 5, y = 1, z = 4$ we have

$$b(x,y) = 5^p + 4^p$$

and

$$b(x,z)+b(z,y)-b(z,z)=5^p+1+4^p+3^p-4^p=5^p+1+3^p$$

$$b(x,y)>b(x,z)+b(z,y)-b(z,z) \quad for\ all\ p>1,$$

therefore, b is not a partial metric on X.

We have the following diagram where arrows stand for inclusions. The inverse inclusions do not hold.

$$\begin{array}{ccc} metric\ space & \longrightarrow & \text{b}-metric\ space \\ \downarrow & & \downarrow \\ partial\ metric\ space & \longrightarrow & partial\ \text{b}-metric\ space \end{array}$$

Every partial b-metric "b" on a nonempty set X generates a topology τ_b on X whose base is the family of open b-balls $B_b(x,\varepsilon)$ where $\tau_b=\{B_b(x,\varepsilon):x\in X,\varepsilon>0\}$ and $B_p(x,\varepsilon)=\{y\in X:b(x,y)<\varepsilon+b(x,x)\}$. It is clear that the topological space (X,τ_b) is T_0, but need not be T_1.

Now, we recall the definitions of convergent sequence and Cauchy sequence in b-PMS.

Definition 2.9 [7] Let (X,b) be a b-PMS, let $\{x_n\}$ be any sequence in X and $x\in X$. Then

(1) a sequence $\{x_n\}$ is said to be convergent with respect to τ_b and converges to $x\in X$ if and only if $\lim_{n\to\infty} b(x_n,x)=b(x,x)$;

(2) a sequence $\{x_n\}$ in X is called Cauchy if and only if $\lim_{n,m\to\infty} b(x_n,x_m)$ exists and is finite;

(3) (X,b) is said to be complete b-PMS if every Cauchy sequence $\{x_n\}$ in X there exists $x\in X$, such that

$$\lim_{n,m\to\infty} b(x_n,x_m)=\lim_{n\to\infty} b(x_n,x)=b(x,x).$$

Definition 2.10 [20] Let(X,b) be a b-PMS. For a subset $A\subset X$, if $diam_r(A)=\sup\{p(x,y)-p(x,x):x,y\in A\}$ is finite, then we call A a radii-bounded set and $diam_r(A)$ the radii-diameter of A.

As an extension of strong b-metric space [21] , we give the concept of partial strong b-metric space as follows.

Definition 2.11 Let X be a nonempty set and let $s\geq 1$ be a given real number. A functional $b:X\times X\to\mathbb{R}^+$ is said to be a strong partial b-metric if and only if for all $x,y,z\in X$, the following conditions are satisfied:

(1) $b(x,y) = b(y,x)$;

(2) if $b(x,x) = b(x,y) = b(y,y)$, then $x = y$;

(3) $b(x,x) \leq b(x,y)$;

(4) there exists a real number $s \geq 1$, such that $b(x,y) \leq sb(x,z) + b(y,z) - b(z,z)$.

The pair (X,b) is called a strong partial b-metric space. The number s is called the coefficient of (X,b). We will use the abbreviation sb-PMS for the strong partial b-metric (X,b).

3. Main results

In this section, we'll present a new version of Ekeland's variational principle in b-PMS.

To prove the main results, we need the following Lemma.

Lemma 3.1 (Cantor's intersection theorem in b-PMS) Let (X,b) be a complete b-PMS, then for every descending nested sequence $\{A_n\}_{n\in N^+}$ of nonempty bounded closed subsets of X such that $\lim\limits_{n\to\infty} diam_r(A_n) = 0$, there exists $x \in X$ such that $\bigcap\limits_{n\in\mathbb{N}^+} A_n = \{x\}$.

Proof. Firstly, let us show $\bigcap\limits_{n\in\mathbb{N}^+} A_n \neq \emptyset$.

For each $n \in \mathbb{N}^+$, take $x_n \in A_n$. Since $\{A_n\}_{n\in N^+}$ is a descending nested sequence, we get $\{x_n\} \subset A_1$, then $\forall n \in \mathbb{N}^+, b(x_n,x_n) - b(x_1,x_1) \leq b(x_n,x_1) - b(x_1,x_1) \leq diam_r(A_1)$. Thus, $\{b(x_n,x_n)\}$ is bounded, and then there exists a convergent subsequence. Without loss of generality, we may assume that $\{b(x_n,x_n)\}$ is convergent.

Since $\lim\limits_{n\to\infty} diam_r(A_n) = 0$, we get $\lim\limits_{m,n\to\infty}\{b(x_m,x_n) - b(x_m,x_m)\} = 0$, then

$$\lim_{m,n\to\infty} b(x_m,x_n) = \lim_{n\to\infty} b(x_n,x_n)$$

i.e., $\{x_n\}_{n\in\mathbb{N}^+}$ is Cauchy. Thus, there is some $x \in X$ such that

$$\lim_{m,n\to\infty} b(x_m,x_n) = \lim_{n\to\infty} b(x,x_n) = b(x,x)$$

Since $\forall n \in \mathbb{N}^+, \forall i \in \mathbb{N}^+, x_{n+i} \in A_n$ and $\lim\limits_{i\to\infty} b(x,x_{n+i}) = b(x,x)$, we get $x \in \bar{A_n} = A_n$. Thus, $x \in \bigcap\limits_{n\in\mathbb{N}^+} A_n$, and hence, $\bigcap\limits_{n\in\mathbb{N}^+} A_n \neq \emptyset$.

Secondly, let us prove $\{A_n\}_{n\in N^+}$ is a singleton.

Conversely, suppose there is $y \in \bigcap_{n\in\mathbb{N}^+} A_n$ and $y \neq x$. From (Pb1), we have that $b(x,y), b(x,x), b(y,y)$ are not equal. We draw contradictions in two cases.

Case 1. If $b(x,y), b(x,x), b(y,y)$ are not equal to each other, then $b(x,y) - b(x,x) = \alpha > 0$.

Since $y \in \bigcap_{n\in\mathbb{N}^+} A_n$, we have $\lim_{n\to\infty} diam_r(A_n) = \sup\{b(x,y) - b(x,x) | x, y \in A_n\} = 0$. i.e., $\lim_{n\to\infty} diam_r(A_n) < \alpha = b(x,y) - b(x,x)$, which ensures that $y \notin A_n$. Hence, y cannot be in $\bigcap_{n\in N^+} A_n$, a contradiction.

Case 2. If $b(x,x) = b(y,y) \neq b(x,y)$, then $b(x,y) - b(x,x) = \alpha > 0$. Similar to case 1, a contradiction can be deduced. As for $b(x,x) = b(x,y) \neq b(y,y)$ or $b(x,y) = b(y,y) \neq b(x,x)$, a similar method can be used to draw contradictions

Finally, we have $\bigcap_{n\in\mathbb{N}^+} A_n = \{x\}$. □

Ekeland's variational principle in b-PMS is presented as follows.

Theorem 3.2 Let (X,b) be a complete b-PMS (with $s > 1$), such that the partial b-metric b is continuous and let $f : X \to \mathbb{R}^+$ be a lower semicontinuous, proper and lower bounded function. Let $\varepsilon > 0$ and $x_0 \in X$ be such that

$$f(x_0) \leq \inf_{x\in X} f(x) + \varepsilon,$$

then there exists a sequence $\{x_n\}_{n\in\mathbb{N}^+} \subset X$ and $x_\varepsilon \in X$ such that

(i) $\lim_{n\to\infty} x_n = x_\varepsilon$;

(ii) $b(x_\varepsilon, x_n) - b(x_n, x_n) \leq \frac{\varepsilon}{2^n} \quad n \in \mathbb{N}^+$;

(iii) $f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} b(x_\varepsilon, x_n) \leq f(x_0) + b(x_0, x_0)$;

(iv) $f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} b(x_\varepsilon, x_n) < f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} b(x, x_n),\ for\ every\ x \neq x_\varepsilon$.

Proof. Consider the set

$$T(x_0) = \{x \in X | f(x) + b(x, x_0) \leq f(x_0) + b(x_0, x_0)\}. \tag{1}$$

Noting that f is lower semicontinuous and $x_0 \in T(x_0)$, we have $T(x_0)$ is nonempty and closed in (X,b) and for every $y \in T(x_0)$

$$b(y,x_0) - b(x_0,x_0) \leq f(x_0) - f(y) \leq f(x_0) - \inf_{x\in X} f(x) \leq \varepsilon$$

Choose $x_1 \in T(x_0)$ such that

$$f(x_1) + b(x_1,x_0) \leq \inf_{x\in T(x_0)} \{f(x) + b(x,x_0)\} + \frac{\varepsilon}{2s}. \tag{2}$$

and let

$$T(x_1) = \{x \in T(x_0) | f(x) + \sum_{i=0}^{1} \frac{1}{s^i} b(x,x_i) \leq f(x_1) + b(x_0,x_1) + b(x_1,x_1)\}$$

Inductively, we can suppose that $x_{n-1} \in T(x_{n-2})$ was already chosen and we consider

$$\begin{aligned} T(x_{n-1}) &= \{x \in T(x_{n-2}) | f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} b(x,x_i) \\ &\leq f(x_{n-1}) + \sum_{i=0}^{n-1} \frac{1}{s^i} b(x_i,x_{n-1})\} \end{aligned} \tag{3}$$

Let us choose $x_n \in T(x_{n-1})$ such that

$$f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} b(x_n,x_i) \leq \inf_{x\in T(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} b(x,x_i)\} + \frac{\varepsilon}{2^n s^n} \tag{4}$$

and define the set

$$\begin{aligned} T(x_n) &= \{x \in T(x_{n-1}) | f(x) + \sum_{i=0}^{n} \frac{1}{s^i} b(x,x_i) \\ &\leq f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} b(x_i,x_n)\} \end{aligned} \tag{5}$$

$T(x_n)$ is nonempty and closed. From (4) and (5), it follows that for each $y \in T(x_n)$

$$\begin{aligned}
\frac{1}{s^n}b(y,x_n) - \frac{1}{s^n}b(x_n,x_n) &\leq [f(x_n) + \sum_{i=0}^{n-1}\frac{1}{s^i}b(x_i,x_n)] - [f(y) + \sum_{i=0}^{n-1}\frac{1}{s^i}b(y,x_i)] \\
&\leq [f(x_n) + \sum_{i=0}^{n-1}\frac{1}{s^i}b(x_i,x_n)] \\
&- \inf_{x\in T(x_{n-1})}\{f(x) + \sum_{i=0}^{n-1}\frac{1}{s^i}b(x,x_i)\} \\
&\leq \frac{\varepsilon}{2^n s^n}. \qquad (6)
\end{aligned}$$

Therefore, for all $y \in T(x_n)$

$$\frac{1}{s^n}b(y,x_n) - \frac{1}{s^n}b(x_n,x_n) \leq \frac{\varepsilon}{2^n s^n}. \qquad (7)$$

Noting that

$$\frac{1}{s^n}b(y,x_n) - \frac{1}{s^n}b(x_n,x_n) \to 0 \ (n \to \infty),$$

we have $diamT(x_n) \to 0$. Since (X,b) is a complete partial b-metric space, by Lemma 3.1, there exists $x_\varepsilon \in X$ such that $\bigcap_{n=0}^{\infty} T(x_n) = \{x_\varepsilon\}$. By (2) and (6), we know that $x_\varepsilon \in X$ satisfies (ii). Thus $x_n \to x_\varepsilon$ as $n \to \infty$.

Moreover, for all $x \neq x_\varepsilon$, we have $x \neq \bigcap_{n=0}^{\infty} T(x_n)$, so there exists $m \in N^+$ such that

$$f(x_m) + \sum_{i=0}^{m-1}\frac{1}{s^i}b(x_m,x_i) < f(x) + \sum_{i=0}^{m}\frac{1}{s^i}b(x,x_i)$$

By (1), (3) and (4), for every $q \geq m$, we obtain

$$\begin{aligned}
f(x_\varepsilon) + \sum_{i=0}^{q}\frac{1}{s^i}b(x_\varepsilon,x_i) &\leq f(x_q) + \sum_{i=0}^{q}\frac{1}{s^i}b(x_q,x_i) \\
&\leq f(x_m) + \sum_{i=0}^{m}\frac{1}{s^i}b(x_m,x_i) \\
&\leq f(x_0) + b(x_0,x_0).
\end{aligned}$$

Thus (iii) and (iv) hold.

This ends the proof. □

Corollary 3.3 Let (X,b) be a complete partial b-metric spaces (with $s>1$), such that the partial $b-$metric b is continuous and let $f:X\to\mathbb{R}^+$ be a lower semicontinuous, proper and lower bounded mapping. Then, for every $\varepsilon>0$, there exists a sequence $\{x_n\}_{n=0}^{\infty}\subset X$ and $x_\varepsilon\in X$ such that

(i) $f(x_0)\leq\inf\limits_{x\in X}f(x)+\varepsilon$

(ii) $\lim\limits_{n\to\infty}x_n=x_\varepsilon$

(iii) $f(x_\varepsilon)+\sum_{n=1}^{\infty}\frac{1}{s^n}b(x_\varepsilon,x_n)\leq f(x_0)$

(iv) $f(x_\varepsilon)+\sum_{n=0}^{\infty}\frac{1}{s^n}b(x_\varepsilon,x_n)<f(x)+\sum_{n=0}^{\infty}\frac{1}{s^n}b(x,x_n)$ for any $x\in X$ and $x\neq x_\varepsilon$.

Now, we'll present some generalizations of the Caristi type fixed point theorem.

Theorem 3.4 Let (X,b) be a complete partial b-metric spaces (with $s>1$), such that the partial b-metric b is continuous and let $T:X\to X$ be an operator for which there exists a lower semicontinuous mapping $f:X\to\mathbb{R}^+$, such that

$$b(T(u),v)\leq sb(u,T(u))+b(u,v)-b(u,u) \tag{8}$$

$$\frac{s^2}{s-1}b(u,T(u))\leq f(u)-f(T(u)) \tag{9}$$

for any $u,v\in X$. Then T has at least one fixed point.

Proof. By Corollary 3.3, for each $\varepsilon>0$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$, such that $x_n\to x_\varepsilon$ as $n\to\infty$, $x_\varepsilon\in X$ and

$$f(x_\varepsilon)+\sum_{n=0}^{\infty}\frac{1}{s^n}b(x_\varepsilon,x_n)\leq f(x)+\sum_{n=0}^{\infty}\frac{1}{s^n}b(x,x_n)\ \ \forall x\in X$$

In what follows, we'll prove that x_ε is a fixed point of T.

Conversely, suppose that $x_\varepsilon\neq T(x_\varepsilon)$. Let $x=T(x_\varepsilon)$, we get that

$$f(x_\varepsilon)-f(T(x_\varepsilon))<\sum_{n=0}^{\infty}\frac{1}{s^n}b(T(x_\varepsilon),x_n)-\sum_{n=0}^{\infty}\frac{1}{s^n}b(x_\varepsilon,x_n)$$

Using (8), for $u = x_\varepsilon, v = x_n$ we have

$$\begin{aligned} f(x_\varepsilon) - f(T(x_\varepsilon)) &< \sum_{n=0}^{\infty} \frac{1}{s^n} b(T(x_\varepsilon), x_n) - \sum_{n=0}^{\infty} \frac{1}{s^n} b(x_\varepsilon, x_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{s^n} [b(T(x_\varepsilon), x_n) - b(x_\varepsilon, x_n)] \\ &< \sum_{n=0}^{\infty} \frac{s}{s^n} b(x_\varepsilon, T(x_\varepsilon)) \quad (by\ condition\ (8)) \\ &= \frac{s^2}{s-1} b(x_\varepsilon, T(x_\varepsilon)). \end{aligned}$$

i.e.,

$$f(x_\varepsilon) - f(T(x_\varepsilon)) < \frac{s^2}{s-1} b(x_\varepsilon, T(x_\varepsilon)). \tag{10}$$

In (9),let $u = x_\varepsilon$, then

$$\frac{s^2}{s-1} b(x_\varepsilon, T(x_\varepsilon)) \leq f(x_\varepsilon) - f(T(x_\varepsilon)). \tag{11}$$

Combing the inequalities (10) with (11), we obtain that

$$\frac{s^2}{s-1} b(x_\varepsilon, T(x_\varepsilon)) \leq f(x_\varepsilon) - f(T(x_\varepsilon)) < \frac{s^2}{s-1} b(x_\varepsilon, T(x_\varepsilon)).$$

which is a contradiction. Thus $x_\varepsilon = T(x_\varepsilon)$, i.e., x_ε is a fixed point of T. □

Corollary 3.5 Assume the conditions of Theorem 4.1 are all satisfied.

(1) If there exists $\omega_0 \in X$, $f(\omega_0) = \inf_{x \in X} f(x)$, then ω_0 is fixed point of T;

(2) If for arbitrary $\omega \in X$, $f(\omega) > \inf_{x \in X} f(x)$, i.e., the infinimum of f can not be attained, then T has infinite fixed points in X;

(3) If T is continuous on X, then for any $u_0 \in X$, the iterative sequence $\{u_n\}$, (where $u_{n+1} = T(u_n),\ \ n = 0, 1, 2, \cdots$) converges to a fixed point of T in X.

Proof. (1) Let $u = \omega_0$. By condition (9), we have

$$\tfrac{s^2}{s-1} b(\omega_0, T(\omega_0)) \leq f(\omega_0) - f(T(\omega_0))$$

Thus

$$f(T(\omega_0)) \leq f(\omega_0) - \tfrac{s^2}{s-1} b(\omega_0, T(\omega_0))$$

Therefore, we must have

$$f(T(\omega_0)) = \inf_{x\in X} f(x) = f(\omega_0)$$

This implies that $b(\omega_0, T(\omega_0)) = 0$, that is, $\omega_0 = T(\omega_0)$. Hence, ω_0 is a fixed point of T.

(2) Let $\varepsilon_1 = \frac{1}{2}$. By Corollary 3.3, there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ and $x_{\varepsilon_1} \in X$ such that

$$f(x_{\varepsilon_1}) + \sum_{n=1}^{\infty} \frac{1}{s^n} b(x_{\varepsilon_1}, x_n) \le f(x_0) \le \inf_{x\in X} f(x) + \varepsilon_1$$

By Theorem 4.1, we have

$$x_{\varepsilon_1} = T(x_{\varepsilon_1}) \ \ and \ \ f(x_{\varepsilon_1}) \le \inf_{x\in X} f(x) + \varepsilon_1$$

Let $\varepsilon_2 = \min\{\frac{1}{2^2}, \frac{f(x_{\varepsilon_1}) - \inf_{x\in X} f(x)}{2^2}\}$. In a similar manner, we have a fixed point x_{ε_2}, which satisfies

$$x_{\varepsilon_2} = T(x_{\varepsilon_2}) \ \ and \ \ f(x_{\varepsilon_2}) \le \inf_{x\in X} f(x) + \varepsilon_2$$

Then, we have

$$f(x_{\varepsilon_2}) \le \inf_{x\in X} f(x) + \frac{f(x_{\varepsilon_1}) - \inf_{x\in X} f(x)}{2^2} = \frac{1}{4} f(x_{\varepsilon_1}) + \frac{3}{4} \inf_{x\in X} f(x) < f(x_{\varepsilon_1})$$

Let $\varepsilon_3 = \min\{\frac{1}{2^3}, \frac{f(x_{\varepsilon_2}) - \inf_{x\in X} f(x)}{2^3}\}$, we have $x_{\varepsilon_3} \in X$ such that $x_{\varepsilon_3} = T(x_{\varepsilon_3})$ and $f(x_{\varepsilon_3}) < f(x_{\varepsilon_2})$.

Generally, when x_{ε_k} exists, let $\varepsilon_{k+1} = \min\{\frac{1}{2^{k+1}}, \frac{f(x_{\varepsilon_k}) - \inf_{x\in X} f(x)}{2^{k+1}}\}$, there is $x_{\varepsilon_{k+1}} \in X$ such that $x_{\varepsilon_{k+1}} = T(x_{\varepsilon_{k+1}})$ and $f(x_{\varepsilon_{k+1}}) < f(x_{\varepsilon_k})$.

Continuing this process, we obtain a sequence $\{x_{\varepsilon_k}\} \subset X$ such that

$$x_{\varepsilon_{k+1}} = T(x_{\varepsilon_{k+1}})$$

$$f(x_{\varepsilon_{k+1}}) < f(x_{\varepsilon_k})$$

That is, T has infinite fixed points in X.

(3) For every $u_0 \in X, u_{n+1} = T(u_n) (n = 0, 1, 2, \cdots)$. By condition (9), we have

$$\frac{s^2}{s-1} b(u_n, u_{n+1}) \le f(u_n) - f(u_{n+1})$$

Then $\{f(u_n)\}$ is decreasing and lower bounded, $\{f(u_n)\}$ is convergent. Hence, for $\forall \varepsilon > 0$ and $\forall p \in \mathbf{N}$, there exists N, when $n > N$, we have $0 \le f(u_n) - f(u_{n+p}) < \varepsilon$. At this time, let $v = u_n$, by assumptions (8) and (9), we have

$$\begin{aligned}
b(u_n,u_{n+p}) &= b(T(u_{n+p-1}),u_n) \\
&\leq sb(u_{n+p-1},u_{n+p})+b(u_{n+p-1},u_n) \\
&\leq sb(u_{n+p-1},u_{n+p})+sb(u_{n+p-2},u_{n+p-1})+\cdots \\
&+ sb(u_{n+2},u_{n+1})+b(u_{n+1},u_n) \\
&\leq \frac{s-1}{s}[f(u_{n+p-1})-f(u_{n+p})]+\frac{s-1}{s}[f(u_{n+p-2})-f(u_{n+p-1})]+\cdots \\
&+ \frac{s-1}{s}[f(u_{n+1})-f(u_{n+2})]+\frac{s-1}{s^2}[f(u_n)-f(u_{n+1})] \\
&\leq f(u_{n+p-1})-f(u_{n+p})+f(u_{n+p-2})-f(u_{n+p-1})+\cdots \\
&+ f(u_{n+1})-f(u_{n+2})+f(u_n)-f(u_{n+1}) \\
&= f(u_n)-f(u_{n+p}). \qquad (12)
\end{aligned}$$

That is

$$b(u_n,u_{n+p})=f(u_n)-f(u_{n+p})<\varepsilon$$

i.e.,

$$\lim_{n\to\infty} b(u_n,u_{n+p})=0$$

Therefore, $\{u_n\}$ is Cauchy, there exists $\omega\in X$ such that $\lim_{n\to\infty} u_n=\omega$.

Since $u_{n+1}=T(u_n)$ and T is continuous, we'll obtain that

$$\omega=\lim_{n\to\infty} u_{n+1}=\lim_{n\to\infty} T(u_n)=T(\lim_{n\to\infty} u_n)=T(\omega)$$

Then, sequence $\{u_n\}$ converges to fixed point of T. □

Corollary 3.6 Let (X,b) be a complete partial sb-metric spaces (with $s>1$). Suppose $T:X\to X$ be a mapping such that

$$b(T(x),T(y))\leq \lambda b(x,y) \qquad for\ \ all\ \ x,y\in X \qquad (13)$$

where $\lambda\in(0,1)$. Then T has a unique fixed point $u\in X$ and $b(u,u)=0$

Proof. Let us first show the existence of fixed point. (X,b) is a complete partial sb-metric spaces implies

$$b(x,y) \leq sb(x,z) + b(y,z) - b(z,z) \qquad for\ all\ x,y,z \in X$$

Taking $x = T(u), y = v$ and $z = u$, we have

$$b(T(u),v) \leq sb(u,T(u)) + b(u,v) - b(u,u) \qquad for\ all\ u,v \in X$$

so (8) holds.

Define $f : X \to \mathbb{R}^+$ as follows

$$f(x) = \frac{s^2}{s-1}\frac{1}{1-\lambda}b(T(x),x)$$

Then $f : X \to \mathbb{R}^+$ is lower semi-continuous, bounded below and

$$\begin{aligned} f(x) - f(T(x)) &= \frac{s^2}{s-1}\frac{1}{1-\lambda}b(T(x),x) - \frac{s^2}{s-1}\frac{1}{1-\lambda}b(T^2(x),T(x)) \\ &\geq \frac{s^2}{s-1}\frac{1}{1-\lambda}[b(T(x),x) - \lambda b(T(x),x)] \\ &= \frac{s^2}{s-1}b(T(x),x) \end{aligned} \tag{14}$$

i.e.,

$$f(x) - f(T(x)) \geq \frac{s^2}{s-1}b(T(x),x)$$

Hence (9) holds. By Theorem 4.1, there exists $u \in X$ such that $u = T(u)$.

Let $v \in X$ is a fixed point of T, then

$$b(u,v) = b(Tu,Tv) \leq \lambda b(u,v)$$

Hence $b(u,v) = 0$, which means $u = v$. i.e., the fixed point of T is unique.

Finally, if u is a fixed point of T and $b(u,u) \neq 0$, then from (13), we have $b(u,u) = b(Tu,Tu) \leq \lambda b(u,u) < b(u,u)$, a contradiction. Therefore $b(u,u) = 0$. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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