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COMMON FIXED POINT THEOREMS FOR A NEW TYPE OF CONTRACTIVE MAPPINGS IN S-METRIC SPACES

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Abstract. We prove the existence and the uniqueness of common fixed point for theorems for a new type of contractive mappings in S-metric spaces. Our results generalize, extend and enrich recently fixed point results in existing literature.

Keywords: fixed point; *S*-metric space; point of coincidence; continuous.

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1. Introduction and Preliminaries

Fixed point theory plays a major role in many applications, including variational and linear inequalities, optimization and applications in the field of approximation theory and minimum norm problem. In 1922, S. Banach proved the famous and well known Banach contraction principle concerning the fixed of contraction mappings defined on a complete metric space. In recent years, Gahler [1, 2] introduced the notion of 2-metric spaces, while Dhage [3] introduced the concept of D-metric spaces. Later on, Mustafa and Sims [12] introduced a new notion of generalized metric space, called G-metric spaces. After then many authors studied fixed and common fixed points in generalized metric spaces see [11, 12, 13, 4, 14, 9, 15, 16, 17]. In [10],

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S. Sedghi, N. shobe and A. Aliouche have introduced the notion of an S-metric space. Moreover in [13, 4] we find some properties of $S-$metric spaces were represented. In the present paper, we going to prove the existence and the uniqueness of some common fixed point theorems by using a new contractive mappings on $S-$metric space.

Definition 1. [5] *Let X be a nonempty set. An S-metric on X is a function* $S : X \times X \times X \to [0,\infty)$ *that satisfies the following conditions, for each* $x,y,z,a \in X$

(S1): $S(x,y,z) \geq 0$

(S2): $S(x,y,z) = 0$ if and only if $x = y = z$

(S3): $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$ for all $x,y,z,a \in X$.

The pair (X,S) is called an $S-$metric space.

Some examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\|+\|y-z\|$ is an $S-$metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x,y,z) = \|x-z\|+\|y-z\|$ is an $S-$metric on X.

(3) Let X be a nonempty set, d is ordinary metric space on X, then $S(x,y,z) = d(x,y)+d(y,z)$ is an $S-$metric on X.

Lemma 2. [5], [6] *Let* (X,S) *be an* $S-$*metric space. Then* $S(x,x,y) \leq 2S(x,x,y)+S(y,y,z)$ *and* $S(x,x,z) \leq 2S(x,x,y)+S(z,z,y)$ *for all* $x,y,z \in X$.

Definition 3. [5] *Let* (X,S) *be an* $S-$*metric space. For* $r > 0$ *and* $x \in X$ *we define the open ball* $B_S(x,r)$ *and closed ball* $B_S[x,r]$ *with center x and radius r as follows respectively*

$$
\begin{aligned}
B_S(x,r) &= \{y \in X : S(y,y,x) < r\} \\
B_S[x,r] &= \{y \in X : S(y,y,x) \leq r\}
\end{aligned}
$$

Example 4. [5] *Let* $X = \mathbb{R}$. *Denote* $S(x,y,z) = |y+z-2x|+|y-z|$ *for all* $x,y,z \in \mathbb{R}$. *Thus* $B_S(1,2) = \{y \in X : S(y,y,1) < 2\} = (0,2)$.

Definition 5. [5] *Let (X,S) be an $S-$metric space and $A \subset X$.*

(1) *If for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$ then the subset A is called open subset of X.*

(2) *Subset A of X is said to be $S-$bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.*

(3) *A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is or each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x) < \varepsilon$ whenever $n \geq n_0$ and we denote this $\lim_{n\to\infty} x_n = x$.*

(4) *A sequence $\{x_n\}$ is called Cauchy sequence if for each $\varepsilon > 0$, then the sequence is convergent.*

(6) Let τ be the set of all $A \subset X$. with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X (induced by the $S-$metric space).

Example 6. *Any open ball $B_S(x,r)$, $x \in X$, $r > 0$ is an open set. Indeed, using lemma 2, $S(z,z,x) \leq 2S(z,z,a) + S(a,a,x) < r$. then we have $S(z,z,x) < r$, so $z \in B_S(x,r)$.*

Example 7. *Let x_0, $y_0 \in X$, considering the sets $B_1 = \{x \in X : S(x,x,x_0) < S(x,x,y_0)\}$ and $B_2 = \{x \in X : S(x,x,x_0) > S(x,x,y_0)\}$. $B_1$and B_2 are two open disjoint sets. Indeed, Let $z \in B_1$ then $S(z,z,x_0) < S(z,z,y_0)$ which implies $S(z,z,y_0) - S(z,z,x_0) > 0$. Setting $\rho = \frac{S(z,z,y_0)-S(z,z,x_0)}{4}$. We show that $B_S(z,\rho) \subset B_1$. Let $a \in B_S(z,\rho)$ then*

$$S(a,a,z) < \rho = \frac{S(z,z,y_0) - S(z,z,x_0)}{4}$$

therefore $2S(a,a,z) + S(z,z,x_0) < S(z,z,y_0) - 2S(a,a,z)$ by lemma 2 we have $S(a,a,x_0) \leq 2S(a,a,z) + S(z,z,x_0) < S(z,z,y_0) - 2S(a,a,z) \leq S(a,a,y_0)$ this means that $S(a,a,x_0) < S(a,a,y_0)$; the desired result follows. With the same way , we prove that B_2 is also an open set. Now, we prove that $B_1 \cap B_2 = \phi$. Assume that $B_1 \cap B_2 \neq \phi$, there exists $y \in B_1 \cap B_2$ then $S(y,y,x_0) < S(y,y,y_0)$ and $S(y,y,x_0) > S(y,y,y_0)$ which implies that $S(y,y,y_0) < S(y,y,y_0)$ which is a contradiction.

Theorem 8. *The $S-$metric space is a T_2 space.*

Proof. It is enaugh to use example7. □

Definition 9. [9] *Let f and g be singled-valued self mappings on a set X. If $\omega = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g .*

Definition 10. [9] *Let f and g be singled-valued self mappings on a set X. Mappings f and g are said to be weakly compatible if $fx = gx$ implies $fgx = gfx$, $x \in X$.*

Proposition 11. [9] *Let f and g be weakly compatible self mappings on a set X. If f and g have a unique point of coincidence $\omega = fx = gx$, then ω is the unique common fixed point of f and g.*

2. Main Results

Let Ψ denotes the class of the functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ which satisfies the following conditions:

(1) ψ is nondecreasing
(2) ψ is continuous
(3) $\psi(t) = 0 \iff t = 0$

The elements of Ψ are called altering distance functions.

Remark 12. *If $\psi \in \Psi$ and if $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then $\varphi(0) = 0$.*

Lemma 13. [7], [8] *Let (X,S) be a $S-$metric space and let $\{x_n\}$ be a sequence in it such that*

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(x_{m_k}, x_{m_k}, x_{n_k}), S(x_{m_k}, x_{m_k}, x_{n_k+1}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}), \ldots$$

Theorem 14. *Let (X,S) be an $S-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\psi(S(fx,fy,fz)) \leq \varphi(\max\{S(gx,gx,fx), S(gy,gy,fy), S(gz,gz,fz)\}) \tag{1}$$

for all $x, y, z \in X$. Where the function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function which satisfies the condition $\psi(t) > \varphi(t)$ for all $t > 0$ and $\psi \in \Psi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. Assume that f and g satisfy the condition (1). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is x_1 such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n-1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_{n-1} = gx_n$, then f and g have a point of coincidence. Thus we can suppose that $gx_{n+1} \neq gx_n$ for all $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, we obtain that

$$
\begin{aligned}
\psi(S(gx_n, gx_n, gx_{n+1})) &= \psi(S(fx_{n-1}, fx_{n-1}, fx_n)) \\
&\leq \varphi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\
&\quad S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}) \\
&\leq \varphi(\max\{S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}) \\
&= \varphi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\})
\end{aligned}
$$

If $\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_n, gx_n, gx_{n+1})\} = S(gx_n, gx_n, gx_{n+1})$,
then

$$\psi(S(gx_n, gx_n, gx_{n+1})) \leq \varphi(S(gx_n, gx_n, gx_{n+1}))$$

Since $gx_n \neq, gx_{n+1}$, then $S(gx_n, gx_n, gx_{n+1}) > 0$, so by the condition of the theorem, we have

$$\psi(S(gx_n, gx_n, gx_{n+1})) > \varphi(S(gx_n, gx_n, gx_{n+1}))$$

which leads to a contraduction. Therefore

$$\psi(S(gx_n, gx_n, gx_{n+1})) \leq \varphi(S(gx_{n-1}, gx_{n-1}, gx_n)) \text{ for all } n \geq 1 \tag{2}$$

and from (2) , it follows that

$$S(gx_n, gx_n, gx_{n+1}) \leq S(gx_{n-1}, gx_{n-1}, gx_n) \text{ for all } n \geq 1. \tag{3}$$

Indeed, if there exists $n_0 \in \mathbb{N}$, such that $S(gx_{n_0}, gx_{n_0}, gx_{n_0+1}) > S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0})$, since ψ is nondecreasing, we have $\psi(S(gx_{n_0}, gx_{n_0}, gx_{n_0+1})) \geq \psi(S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0}))$ and since from (2) we have $\psi(S(gx_{n_0}, gx_{n_0}, gx_{n_0+1})) \leq \varphi(S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0}))$, then $\psi(S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0})) \leq \varphi(S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0}))$, since $S(gx_{n_0-1}, gx_{n_0-1}, gx_{n_0}) > 0$, then we have a contradiction by the condition of the theorem. Now, setting $r_n = S(gx_n, gx_n, gx_{n+1})$, the sequence $\{r_n\}$ is nonincreasing $r_n \geq 0$, then, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} r_n = r$. We assume that $r > 0$, by going to the limit in (2), we get $\psi(r) \leq \varphi(r)$, by using the condition of the theorem, we obtain $r = 0$. Now we prove that $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in the $S-$metric space (X,S), there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) \text{ and } S(gx_{m_k}, gx_{m_k}, gx_{n_k}) \tag{4}$$

Putting now in (1) $x = y = x_{m_k}, z = x_{n_k}$ we obtain

$$\begin{aligned}
\psi(S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1})) &= \psi(S(fx_{m_k}, fx_{m_k}, fx_{n_k})) \\
&\leq \varphi(\max S(gx_{m_k}, gx_{m_k}, fx_{m_k}), \\
&\quad S(gx_{m_k}, gx_{m_k}, fx_{m_k}), S(gx_{n_k}, gx_{n_k}, fx_{n_k})\}) \\
&\leq \varphi(\max \left\{ \begin{array}{c} S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\ S(gx_n, gx_n, fx_n) \end{array} \right\}) \\
&= \varphi(\max \left\{ \begin{array}{c} S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), \\ S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}) \end{array} \right\})
\end{aligned}$$

If

$$\max\left\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\right\} = S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})$$

and since $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ we have

$$\psi(S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1})) \leq \varphi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}))$$

Letting $k \to \infty$, we obtain

$$\psi(\varepsilon) \leq \lim_{k\to\infty} \varphi(S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})) = \varphi(0) = 0$$

which implies that $\psi(\varepsilon) = 0$, so $\varepsilon = 0$ which is a contradiction.

Analogous, if

$$\max\left\{S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}), S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})\right\} = S(gx_{n_k}, gx_{n_k}, gx_{n_k+1})$$

we got a contraduction.

So, it follows that $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence in the $S-$metric space (X,S). By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (1), we have

$$\begin{aligned}
\psi(S(gx_n, gx_n, fp)) &= \psi(S(fx_{n-1}, fx_{n-1}, fp)) \\
&\leq \varphi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\
&\quad S(gp, gp, fp)\}) \\
&= \varphi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\})
\end{aligned}$$

Now we study the following cases:

• $\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp)\} = S(gx_{n-1}, gx_{n-1}, gx_n)$

we obtain that

$$\psi(S(gx_n, gx_n, fp)) \leq \varphi(S(gx_{n-1}, gx_{n-1}, gx_n))$$

By taking $n \to \infty$, we have $\psi(S(gp, gp, fp)) \leq \varphi(S(gp, gp, gp)) = \varphi(0) = 0$ which implies that $\psi(S(gp, gp, fp)) = 0$, so $S(gp, gp, fp) = 0$ and we have $gp = fp$.

• $\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gp, gp, fp))\} = S(gp, gp, fp)$

we obtain

$$\psi(S(gx_n, gx_n, fp)) \leq \varphi(S(gp, gp, fp))$$

By taking $n \to \infty$, we have $\psi(S(gp,gp,fp)) \le \varphi(S(gp,gp,fp)) = \varphi(0) = 0$ which implies that $\psi(S(gp,gp,fp)) = 0$, so $S(gp,gp,fp) = 0$ and we have $gp = fp$. Indeed, if $S(gp,gp,fp) > 0$ the condition of the theorem gives a contraduction. Therefore $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fl = gl$ for some $l \in X$. By applying the condition (1), it follows that,

$$\begin{aligned}\psi(S(gp,gp,gl)) &= \psi(S(fp,fp,fl)) \\ &\le \varphi(\max\{S(gp,gp,fp),S(gp,gp,fp), \\ &\quad S(gl,gl,fl)\}) \\ &= \varphi(0) = 0\end{aligned}$$

Therefore $gp = gl$. This implies that f and g have a unique point of coincidence. By proposition10, we can conclude that f and g have a unique common fixed point. □

Example 15. *Let* $X = [0,2]$, $S(x,y,z) = \max\{|x-y|,\ |y-z|,\ |x-z|\}$*and* $\psi \in \Psi$. *Define* $fx = 1$ *and* $gx = 2-x$, *we obtain that* f *and* g *satisfy* (1) *in theorem* 14 . *Indeed, we have* $S(fx,fy,fz) = 0$, *and* $\psi(\ S(fx,fy,fz)) = \psi(0) = 0$

$$\varphi(\max\{S(gx,gx,fx),S(gy,gy,fy),(gz,gz,fz)\}) = \varphi(\max\{|x-y|\,|y-z|\,,|x-z|\})$$

Hence

$$0 \le \varphi(\max\{|x-y|,\ |y-z|,\ |x-z|\})\ \textit{for all}\quad x,y,z \in X$$

It is obvious that the range of g *contains the range of* f *and* $g(X)$ *is a complete subspace of* (X,S). *Furthermore,* f *and* g *are weakly compatible. Thus all assumptions in Theorem*14 *are satisfied. This implies that* f *and* g *have a unique common fixed point which is* $x = 1$.

Corollary 16. *Let* (X,S) *be a* $S-$*metric space. Suppose that the mapping* $f,g : X \to X$ *satisfy*

$$\begin{aligned}(5)\qquad \psi(S(fx,fy,fz)) \;&\le\; \beta(\psi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\})) \\ &\quad \psi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\})\end{aligned}$$

for all $x,y,z \in X$. *Where the function* $\beta : [0,\infty) \longrightarrow [0,1)$ *is a continuous function which satisfies the condition* $\beta(t_n) \to 1$ *implies* $t_n \to 0$ *and* $\psi \in \Psi$. *If the range of* g *contains the range* *of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.*

Proof. It follows from theorem 14, by choosing $\varphi(x) = \beta(\psi(x))\psi(x)$. $\square$

Corollary 17. *Let (X,S) be a $S-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\begin{aligned} S(fx,fy,fz) &\leq \max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\} \\ &\quad -\psi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\}) \end{aligned}$$

for all $x,y,z \in X$. Where $\psi \in \Psi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. is a particular case of Theorem 14, for ψ, the identity function and $\varphi(x) = x - \psi(x)$ $\square$

Corollary 18. *Let (X,S) be a $S-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\begin{aligned} \psi(S(fx,fy,fz)) &\leq \psi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\}) \\ &\quad -\varphi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\}) \end{aligned}$$

for all $x,y,z \in X$. Where $\psi \in \Psi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. Is a particular case for $\varphi(x) = \psi(x) - \varphi_{1,2}(x)$, $\varphi_{1,2}$ is an altering function in Theorem 14. $\square$

Corollary 19. *Let (X,S) be a $S-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\begin{aligned} S(fx,fy,fz) &\leq \beta(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\}) \\ &\quad \max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\} \end{aligned} \tag{6}$$

for all $x,y,z \in X$. Where the function $\beta : [0,\infty) \longrightarrow [0,1)$ is a continuous function which satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$ and . If the range of g contains the range of

f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. It follows from theorem 14, by choosing $\varphi(x) = \beta(x)x$ and $\psi(x) = x$. □

Theorem 20. *Let (X,S) be a complete $S-$metric space. Suppose that the mapping $f : X \to X$ satisfy*

$$\psi(S(fx, fy, fz) \leq \max\{\varphi(S(x,x,fx)), \varphi(S(y,y,fy)), \varphi(S(z,z,fz))\}$$

for all $x,y,z \in X$. Where the function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$ and $\psi \in \Psi$. then f has a unique fixed point

Proof. The result follows by setting g the identity function on X. □

Theorem 21. *Let (X,S) be a $S-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\psi(S(fx, fy, fz)) \leq \varphi(k_1 S(gx, gx, fx) + k_2 S(gy, gy, fy)) \tag{7}$$

for all $x,y,z \in X$, $k_i \geq 0$, $i = 1,2$ and $k_1 + k_2 < 1$. Where the function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$ and $\psi \in \Psi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. Assume that f and g satisfy the condition (1). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is x_1 such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n-1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_{n-1} = gx_n$, then f and g have a point of coincidence. Thus we can suppose that $gx_{n+1} \neq gx_n$ for all $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
\psi(S(gx_n, gx_n, gx_{n+1})) &= \psi(S(fx_{n-1}, fx_{n-1}, fx_n)) \\
&\leq \varphi(k_1 S(gx_{n-1}, gx_{n-1}, fx_{n-1}) + \\
&\quad k_2 S(gx_{n-1}, gx_{n-1}, gx_n)) \\
&\leq \varphi((k_1+k_2) S(gx_{n-1}, gx_{n-1}, gx_n))
\end{aligned}$$

which implies that $S(gx_n, gx_n, gx_{n+1}) \leq (k_1+k_2) S(gx_{n-1}, gx_{n-1}, gx_n)$. Let $r = k_1+k_2 < 1$. Then

$$S(gx_n, gx_n, gx_{n+1}) \leq r^n S(gx_0, gx_0, gx_1)$$

This implies that $\lim\limits_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in the $S-$metric space (X,S), there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1}) \text{ and } S(gx_{m_k}, gx_{m_k}, gx_{n_k}) \tag{8}$$

Putting now in (7) $x = y = x_{m_k}$ and $z = x_{n_k}$, and using the fact that $S(gx_{m_k}, gx_{m_k}, gx_{m_k+1}) > 0$ and $S(gx_{n_k}, gx_{n_k}, gx_{n_k+1}) > 0$ we obtain

$$\begin{aligned}
\psi(S(gx_{m_k+1}, gx_{m_k+1}, gx_{n_k+1})) &= \psi(S(fx_{m_k}, fx_{m_k}, fx_{n_k})) \\
&\leq \varphi(k_1 S(gx_{m_k}, gx_{m_k}, fx_{m_k}) \\
&\quad +k_2 S(gx_{m_k}, gx_{m_k}, fx_{m_k})) \\
&\leq \varphi((k_1+k_2) S(gx_{m_k}, gx_{m_k}, gx_{m_k+1})
\end{aligned}$$

Letting $k \to \infty$ we obtain $\psi(\varepsilon) \leq \varphi(0) = 0$ which implies $\varepsilon = 0$. Contradiction. So the sequence $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence in the $S-$metric space (X,S), By the completeness of $g(X)$ or $f(X)$, we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. we will show that $gp = fp$. By (7), we have

$$\begin{aligned}
\psi(S(gx_n, gx_n, fp)) &= \psi(S(fx_{n-1}, fx_{n-1}, fp)) \\
&\leq \varphi(k_1 S(gx_{n-1}, gx_{n-1}, gx_n) \\
&\quad + k_2 S(gx_{n-1}, gx_{n-1}, gx_n)) \\
&\leq \varphi((k_1 + k_2) S(gx_{n-1}, gx_{n-1}, gx_n)
\end{aligned}$$

Letting $n \to \infty$ we have $\psi(S(gp, gp, fp)) \leq \varphi(0) = 0$. So $S(gp, gp, fp) = 0$ and $gp, = fp$. The proof of f and g have a unique point of coincidence is as in Theorem 14.So we omitted it. □

Later, from the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application of Theorem 14 above. For this purpose, let

$$Y = \left\{ \begin{array}{c} \chi,\, \chi : \mathbb{R}^+ \to \mathbb{R}^+, \text{ satisfies that } \chi \text{ is a Lebesgue integrable,} \\ \text{summable on each compact of subset of } \mathbb{R}^+ \\ \text{and } \int_0^\varepsilon \chi(t)dt > 0 \text{ for each } \varepsilon > 0 \end{array} \right\}$$

Theorem 22. *Let (X, S) be an $S-$metric space. Suppose that the mapping $f, g : X \to X$ satisfy*

$$\int_0^{\psi(S(fx,fy,fz))} \chi(t)dt \leq \int_0^{\varphi(\max\{S(gx,gx,fx),S(gy,gy,fy),S(gz,gz,fz)\})} \chi(t)dt \tag{9}$$

for all $x, y, z \in X$ and for all $\chi \in Y$. Where the function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function which satisfies the condition $\psi(t) > \varphi(t)$ for all $t > 0$ and $\psi \in \Psi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed common point.

Proof. For $\chi \in Y$, We consider the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$defined by $\Lambda(x) = \int_0^x \chi(t)dt$. we note that $\Lambda \in \Psi$. Thus the inequality (9)becomes

$$\Lambda(\psi(S(fx, fy, fz))) \leq \Lambda(\varphi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\})) \tag{10}$$

Setting in (10), $\Lambda \circ \psi = \psi_1$ and $\Lambda \circ \varphi = \varphi_1$, we obtain

$$\Lambda(\psi(S(fx, fy, fz))) \leq \Lambda(\varphi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}))$$

therefore from Theorem 14, the desired result follows. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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