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GENERALIZATION OF THE FIXED POINT THEOREMS FOR DASS-GUPTA AND CHATTERJEE TYPE MAPPINGS

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Abstract. In this paper, we generalize the Chatterjee type Mappings and prove some fixed point theorems for such generalized type mappings.

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1. Introduction

In 1912, L. E. J. Brouwer [2] proved his famous fixed point theorem, if R^n is the n-Euclidian space with the usual metric and T is a continuous mapping from the unit ball of R^n into itself, then T has fixed point.

In 1922, S. Banach [1] introduced his Banach contraction principle. (The basic theorem of fixed point theory long time ago). If X is a complete metric space and T is r-contraction mapping from X into itself, then T has a unique fixed point $y \in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x \in X$.

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Mathematicians in the field of fixed point theory try to improve the results of this theorem in which reduce the contractivity assumption imposed on the given mapping or changing the completeness condition on the given topological space.

In 1965, [5] F.Edelstein theorem was introduced to prove that if X is a compact metric space and T is a contractive mapping from X into itself, then T has a unique fixed point $y \in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x \in X$.

In 1975, B. K. Dass and S. Gupta [4] generalized the Banach's contraction principle for mappings T defined from a complete metric space X into X that satisfied

$$d(T(x), T(y) \leq \alpha d(x, y) + \beta[\frac{1 + d(x, T(x))}{1 + d(x, y)}]d(y, T(y)) \;\forall x,\, y \in X$$

for some nonnegative real constants α and β with $\alpha + \beta < 1$ and proved the existence of a unique fixed point of such defined mappings.

In 1979, H. K. Chatterjee [3] generalized the Dass and Gupta fixed point theorem as follows:

Let T be a mapping defined from a complete metric space X into itself that satisfied

$$d(T^{n+1}(x), T^{n+2}(y) \leq \alpha d(T^n(x), T^{n+1}(y)) + \gamma[\frac{1 + d(T^n(x), T^{n+1}(x))}{1 + d(T^n(x), T^{n+1}(y))}]d(T^{n+1}(y), T^{n+2}(y))$$

for all $x,\, y \in X$, any natural number $n \in N$, and for some nonnegative real constants α and γ with $\alpha + \gamma < 1$ and proved the existence of a unique fixed point of such defined mappings.

In this paper we generalize the Chatterjee type mappings for mappings T defined from a complete metric space X into itself that satisfy

$$d(T^{n+1}(x),T^{n+2}(y) \leq \alpha d(T^n(x),T^{n+1}y)+[\frac{(\beta+\gamma)\{d(T^n(x),T^{n+1}(x))\}+\gamma}{1+d(T^n(x),T^{n+1}(y))}]d(T^{n+1}(y),T^{n+2}(y))$$

for all x, $y \in X$, any natural number $n \in N$, and for some nonnegative real constants α, β, and γ with $\alpha+\beta+\gamma<1$ and proved the existence of a unique fixed point of such defined mappings. These inequalities are symmetric with respect to β and γ. That is

$$d(T^{n+1}(x),T^{n+2}(y) \leq \alpha d(T^n(x),T^{n+1}y)+[\frac{(\beta+\gamma)\{d(T^n(x),T^{n+1}(x))\}+\beta}{1+d(T^n(x),T^{n+1}(y))}]d(T^{n+1}(y),T^{n+2}(y)).$$

2. Preliminaries

Throughout this paper, we assume that T is a mapping from a metric space X into itself.

3. Main results

Lemma 3.1. *Let T be a mapping of a space X into itself and F be an arbitrarily mapping from $X \times X$ into $[0,\infty)$ and*

$$F(T(x),T^2(x)) \leq \alpha F(x,T(x))+[\frac{(\beta+\gamma)F(x,T(x))+\gamma}{1+F(x,T(x))}]F(T(x),T^2(x)) \tag{1}$$

for every $x \in X$ and for some nonnegative real constants α, β, and γ with $\alpha+\beta+\gamma<1$. Then every element $x \in X$ fulfils the following:

$$\lim_{n\to\infty} F(T^n(x),T^{n+1}(x))=0 \tag{2}$$

If in addition the mapping F satisfies the triangle inequality

$$F(x,y) \leq F(x,z)+F(z,y)\ \forall x,y,z \in X,$$

then we have

$$\lim_{n,m\to\infty} F(T^n(x),T^m(x))=0 \tag{3}$$

Proof. we have

$$\begin{aligned}F(T(x),T^2(x)) &\leq \alpha F(x,T(x)) + [\frac{(\beta+\gamma)F(x,T(x))+\gamma}{1+F(x,T(x))}]F(T(x),T^2(x))\\ &\leq \alpha F(x,T(x)) + [\frac{\beta F(x,T(x))+\gamma(1+F(x,T(x)))}{1+F(x,T(x))}]F(T(x),T^2(x))\\ &\leq \alpha F(x,T(x)) + [\frac{\beta F(x,T(x))}{1+F(x,T(x))}+\frac{\gamma(1+F(x,T(x)))}{1+F(x,T(x))}]F(T(x),T^2(x))\\ &\leq \alpha F(x,T(x)) + (\beta+\gamma)F(T(x),T^2(x)),\end{aligned}$$

hence

$$F(T(x),T^2(x)) \leq [\frac{\alpha}{1-(\beta+\gamma)}]F(x,T(x)). \tag{4}$$

Repeating the last step with $T(x)$ instead of x proves that

$$F(T^2(x),T^3(x)) = F(T^2(x),T(T^2(x))) \leq [\frac{\alpha}{1-(\beta+\gamma)}]F(T(x),T^2(x)).$$

Using (4) shows that

$$F(T^2(x),T^3(x)) \leq [\frac{\alpha}{1-(\beta+\gamma)}]^2F(x,T(x)).$$

Repeating the last steps n-times proves that

$$F(T^n(x),T^{n+1}(x)) \leq [\frac{\alpha}{1-(\beta+\gamma)}]^nF(x,T(x)).$$

for every natural number $n \in N$. Taking the limit as $n \to \infty$ proves (2) because $r = \frac{\alpha}{1-(\beta+\gamma)} < 1$. Using the triangle inequality for $n, m \in N$ and $n < m$ we get

$$F(T^n(x),T^m(x)) \leq [\frac{r^n}{1-r}]F(x,T(x)).$$

Taking the limit as $n \to \infty$ proves (3).

Corollary 3.3. *Let T be a mapping of a metric space (X,d) into itself and*

$$d(T(x),T^2(x)) \leq \alpha d(x,T(x)) + [\frac{(\beta+\gamma)d(x,T(x))+\gamma}{1+d(x,T(x))}]d(T(x),T^2(x)) \tag{5}$$

for every $x \in X$ and for some nonnegative real constants α, β, and γ with $\alpha+\beta+\gamma < 1$. Then the sequence of iterates $\{T^n(x)\}_{n\in N}$ is cauchy sequence for every element $x \in X$.

Proof. Using Lemma (3.1) with $F = d$ proves that $\lim_{n,m\to\infty} d(T^n(x),T^m(x)) = 0$. Therefore $\{T^n(x)\}_{n\in N}$ is cauchy sequence.

Corollary 3.4. *If T is a mapping of a metric space (X,d) into itself and*

$$d(T(x),T(y)) \leq \alpha d(x,y) + [\frac{(\beta+\gamma)d(x,T(x))+\gamma}{1+d(x,y)}]d(y,T(y)) \tag{6}$$

for every $x,y \in X$ and for some nonnegative real constants α, β, and γ, then

$$d(T(x),T^2(x)) \leq \alpha d(x,T(x)) + [\frac{(\beta+\gamma)d(x,T(x))+\gamma}{1+d(x,T(x))}]d(T(x),T^2(x)) \tag{7}$$

for every $x \in X$.

Proof. Let x be an arbitrarily element in X. Then taking $y = T(x)$ is clearly giving (7).

we have the following main theorem:

Theorem 3.1. *Let T be an arbitrarily mapping of a complete metric space (X,d) into itself and*

$$d(T(x),T(y)) \leq \alpha d(x,y) + [\frac{(\beta+\gamma)d(x,T(x))+\gamma}{1+d(x,y)}]d(y,T(y)) \tag{8}$$

for every $x,y \in X$ and for some nonnegative real constants α, β, and γ with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point $y \in X$. Moreover; if x_0 is an arbitrarily element of X, then the sequence of iterates $\{T^n(x_0)\}_{n\in N}$ is converging strongly to such a unique fixed point.

Proof. Let x_0 be an arbitrarily element of X. Using Corollaries (3.3) and (3.4)

proves that the sequence of iterates $\{T^n(x_0)\}_{n\in N}$ is cauchy sequence in X, since X is complete there is $y \in X$ such that $\lim_{n\to\infty} T^n(x_0) = y$. Now

$$\begin{aligned}
d(y,T(y)) &\leq d(y,T^n(x_0)) + d(T^n(x_0),T(y)) \\
&\leq d(y,T^n(x_0)) + d(T(T^{n-1}(x_0)),T(y)) \\
&\leq d(y,T^n(x_0)) + \alpha d(T^{n-1}(x_0),y) + [\frac{(\beta+\gamma)d(T^{n-1}(x_0)),T^n(x_0))+\gamma}{1+d(T^{n-1}(x_0)),y)}]d(y,T(y)).
\end{aligned}$$

Taking the limit as n tends to infinity yields that

$$d(y,T(y)) \leq \gamma d(y,T(y))$$

hence $d(y,T(y)) = 0$ because $\gamma < 1$ and we proved that y is a fixed point of T, $T(y) = y$. Finally we prove the uniqueness of such a fixed point by contrarily assumption, suppose that there are two distinct fixed points y and z of T, we have the following contradiction:

$$d(y,z) = d(T(y),T(z)) \leq \alpha d(y,z) + [\frac{(\beta+\gamma)d(y,T(y))+\gamma}{1+d(y,z)}]d(z,T(z))$$
$$\leq \alpha d(y,z) < d(y,z).$$

Using Theorem (3.1) , we proved Dass-Gupta and Chatterjee fixed point theorem as given in the following corollary:

Corollary 3.5. *Let T be a mapping of a complete metric space (X,d) into itself and*

$$d(T(x),T(y)) \leq \alpha d(x,y) + [\frac{\gamma\{1+d(x,T(x))\}}{1+d(x,y)}]d(y,T(y)) \tag{9}$$

for every $x, y \in X$ and for some nonnegative real constants α and β with $\alpha+\beta < 1$. Then T has a unique fixed point $y \in X$. Moreover; if x_0 is an arbitrarily element of X, then the sequence of iterates $\{T^n(x_0)\}_{n\in N}$ is converging strongly to such a unique fixed point.

Proof.

Using Theorem (3.1) with $\beta = 0$ gives a direct proof.

3. Conclusion

In this paper, we generalized both Dass-Gupta and Chatterjee type mappings and proved the existence of a unique fixed points of such type of mappings.

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