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HYBRID METHOD FOR GENERALIZED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS INVOLVING MULTIVALUED NONLINEAR MAPPINGS

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Abstract. We introduce and study in this article, a new hybrid projection algorithm for approximating a common element of the set of fixed points of finite family of multivalued k_i- strictly pseudocontractive mapping and the set of solution of generalized mixed equilibrium problem in real Hilbert space. Under mild conditions, strong convergence of the sequence was proved. The result obtained improves and extends the important result of Bunyawat and Suantai [Fixed point theory and Applications 2013, 2013:236] and many other recent results.

Keywords: strictly pseudo contractive mapping; nonexpansive mapping; equilibrium problem; optimization problem; Hilbert spaces.

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1. Introduction

For several years, the study of fixed point theory for multi-valued nonlinear mappings has attracted the interest of several well known mathematicians (see, for example, Brouwer [3], Chang [1, 4], Chidume et al. [5] and the references therein). Interest in such studies stems, perhaps,

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mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations (see e.g.,[6]). Game theory is perhaps the most successful area of application of fixed point theory for multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium problems in game theory are mostly static in the sense that while they enhance the understanding of conditions under which equilibrium may be achieved, they do not indicate how to construct a process starting from a non-equilibrium point that will converge to an equilibrium solution. Iterative methods for fixed points of multivalued mappings are designed to address this problem. For more details, one may consult [4, 5].

For a nonempty subset of a real normed linear space, E, let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets, and of nonempty, proximinal and bounded subsets of K, respectively . The Hausdorff metric on $CB(K)$ is defined by

$$D(A,B) = \ max\{sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} \text{ for all } A,B \in CB(K).$$

Let $T : D(T) \subset E \to CB(E)$ be a multi-valued mapping on E. A point $x \in D(T)$ is called a fixed point of T if $x \in Tx$. The fixed point set of T is denoted by $F(T)$. A multi-valued mapping $T : D(T) \subset E \to CB(E)$ is called $L-$ Lipschitzian if there exists $L > 0$ such that

$$D(Tx,Ty) \leq L||x-y|| \ \forall x,y \in D(T). \tag{1}$$

When $L \in (0,1)$ in (1), we say that T is a contraction, and T is called nonexpansive if $L = 1$. Several papers dealwith the problem of approximating fixed points of multi-valued nonexpansive mappings (see, for example [1, 7] and the references therein)

Recently, Chidume et al. [5], introduced the class of multi-valued $k-$strictly pseudocontractive maps defined on a real Hilbert space H as follows.

Definition 1.1 A multi-valued map $T : D(T) \subset H \to CB(H)$ is called $k-$ strictly pseudocontractive if there exists $k \in [0,1)$ such that for all $x,y \in D(T)$,

$$\big(D(Tx,Ty)\big)^2 \leq ||x-y||^2 + k||x-y-(u-v)||^2 \ \forall u \in Tx, v \in Ty \tag{2}$$

In the case that T is single-valued, definition (1) reduces to the definition introduced and studied by Browder and Petryshn [2] as an important generalization of the class of nonexpansive mappings. Chidume et al.[4], proved strong convergence theorems for approximating fixed points of this class of mappings using a Krasnoselskii-type algorithm, [5] which is well known to be superior to the recursion formula of Mann [14] or Ishikawa [15].

The generalized mixed equilibrium problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, mixed equilibrium problems and equilibrium problems as special cases (see e.g., [8, 12, 18]). Numerous problems in Physics, optimization and economics reduce to find a solution of generalized mixed equilibrium problem. Several methods have been proposed to solve the fixed point problems, variational inequality problems , mixed equilibrium problems and generalized mixed equilibrium problems in the literature. (See e.g., [8, 9, 11, 12, 13, 20] and the refernces therein).

It is known that Manns iterations have only weak convergence even in the Hilbert spaces. To overcome this problem, Takahash [19] introduced a new method, known as shrinking projection method, which is a hybrid method of Manns iteration, and the projection method, and obtained strong convergence results of such method. Using Takahashi's method, in [7], Bunyawat Suantai used the shrinking projection method to define a new hybrid method for mixed equilibrium problem (MEP) and fixed point problem for a family of nonexpansive multivalued mappings.

Motivated by the results of Bunyawat and Suantai [7], we introduce and study a new hybrid method for finding a common element of the set of solutions of generalized mixed equilibrium problem and the set of common fixed points of a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1,2,...,m$ in real Hilbert spaces, and proved strong convergence theorem for the sequence generated by our proposed method. The result obtained extends the result of Bunyawat and Suantai [7] to the more general class of multivalued $k-$ strictly pseudocontractive mappings and to generalized mixed equilibrium problem, and many other interesting results in the literature.

2. Preliminaries

Let $\phi : C \subset H \to \mathbb{R}$ be a real-valued function and $A : C \to H$ be a nonlinear mapping. Suppose that $F : C \times C \to \mathbb{R}$ is an equilibrium bi-function, that is, $F(u,u) = 0, \forall u \in C$. The generalized mixed equilibrium problem is to find $x \in C$ such that

$$F(x,y)+\varphi(y)-\varphi(x)+\langle Ax, y-x\rangle \geq 0,\ \forall y \in C. \tag{3}$$

We shall denote the set of solutions of the generalized mixed equilibrium problem by Ω. Thus

$$\Omega := \{x \in C : F(x,y)+\varphi(y)-\varphi(x)+\langle Ax, y-x\rangle \geq 0\ \forall y \in C\}$$

Lemma 2.1 Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x,y,z \in H$ and also given $a \in \mathbb{R}$, the set $\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z,v\rangle + a\}$ is convex and closed .

For solving the generalized mixed equilibrium problem, we assume that F,φ and the set C satisfy the following conditions:

(A1) $F(x,x) = 0\ for\ all\ x \in C$,

(A2) F is monotone, that is $F(x,y)+F(y,x) \leq 0\ for\ all\ x,y \in C$,

(A3) for each $x,y,z \in C$, $\limsup_{t\uparrow 0} F(tz+(1-t)x,y) \leq F(x,y)$.

(A4) $F(x,.)$ is convex and lower semi-continuous for each $x \in C$,

(B1) for each $x \in H$ and $r > 0$. there exists a bounded subset $C_x \subseteq C$ and $y_x \in C\bigcap dom\varphi$ such that for any $z \in C\backslash C_x$,

$$F(z,y_x)+\varphi(y_x)+\frac{1}{r}\langle y_x - z, z-x\rangle < \varphi(z), \tag{4}$$

(B2) C is a bounded set.

Then, we have the following lemma.

Lemma 2.2(Wangkeeree and Wangkeeree [22]) Let C be a nonempty closed and convex subset of a real Hilbert spaces H. Let $F: C\times C \to \Re$ be a bifunction satisfying conditions (A1)- (A4) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper semicontinuous and convex function such that $C\bigcap dom\varphi \neq \emptyset$. For $r > 0$ and $x \in C$ define a mapping $T_r^{(F,\varphi)} : H \to C$ as follows:

$T_r^{(F,\varphi)}(x) = \{z \in C : F(z,y)+\varphi(y)-\varphi(x)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ y \in C\}$

for all $x \in H$. Assume that either (B1) or (B2) holds. Then the following conclusions hold:

(1) for each $x \in H, T_r^{(F,\varphi)}(x) \neq \emptyset$,

(2) $T_r^{(F,\varphi)}$ is single- valued,

(3) $T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e. $||T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y|| \leq \langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x-y\rangle$

(4) $F(T_r^{(F,\varphi)}) = GMEP(F,\varphi)$,

(5) $GMEP(F,\varphi)$ is closed and convex.

Lemma 2.3([17]) Let D be a nonempty closed convex subset of a real Hilbert space H and $P_D : H \to D$ be the metric projection form. Then the following inequality holds:
$\|y - P_D x\|^2 + \|x - P_D x\|^2 \leq \|x - y\|^2 \ \forall\, x \in H, y \in D.$

Lemma 2.4 Let H be a normed space and $T : C \to P(C)$ be a multivalued mapping. Let $P_T(x) = \{y \in Tx : \|x - y\| = d(x,Tx)\}$. Then the following are equivalent:

(1) $x \in Tx$,

(2) $P_T x = \{x\}$,

(3) $x \in F(P_T)$.

Moreover, $F(T) = F(P_T)$.

Lemma 2.5([10]) Let H be a real Hilbert space and $\{x_i, i = 1,2,...,m\} \subset H$. For $\alpha_i \in (0,1), i = 1,2,\cdots,m$, such that $\sum_{i=1}^m \alpha_i = 1$,the following identity holds:
$\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^m \alpha_i \alpha_j \|x_i - x_j\|^2.$

3. Main results

Theorem 3.1 Let H be a real Hilbert space and C be a nonempty, closed and subset of H. Let $T_i : C \to P(C)$ be finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1,2,...,m$. Let $F : C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous and convex function such that $C \bigcap dom\varphi \neq \emptyset$, and $\psi : C \to H$ be a β inverse strongly monotone mapping . Assume that $\Gamma := \bigcap_{i=1}^m F(T_i) \bigcap \Omega \neq \emptyset$ and the sequence $\{x_n\}$ be generated by

$$(5) \qquad \begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = \alpha_{n,0} u_n + \sum_{i=1}^m \alpha_{n,i} w_n^i, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, n \geq 0, \end{cases}$$

where $r_n \in (0, 2\beta]$ with $\liminf_{n\to\infty} r_n > 0$ and $w_n^i \in P_{T_i}u_n, P_{T_i}u_n := \{w_n^i \in T_iu_n : \|w_n^i - u_n\| = d(u_n, T_iu_n)\}$,

with conditions

(i) $k \in (0,1)$, where $k := max\{k_i, i = 1, 2, \cdots, m\}$

(ii) $\alpha_{n,0} \in (k,1), \alpha_{n,i} \in (0,1), i = 0, 1, 2, ..., m,$ *such that* $\sum_{i=0}^{m} \alpha_{n,i} = 1.$

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$.

Proof. We divide the proof into 7 steps.

Step 1: We show that $P_{C_{n+1}}x_0$ is well defined for $x_0 \in C$.

It follows from Lemma 2.1 and Lemma 2.2 that Γ is a closed and convex subset of C. Also, by Lemma 2.1 , we obtain that C_{n+1} is closed and convex for $n \geq 0$.

Let $x^* \in \Gamma$, then by Lemma 2.4, we have that $P_{T_i}(x^*) = \{x^*\}, i = 1, 2, ..., m$. Since $u_n = T_{r_n}^{(F,\varphi)}(x_n - r_n\psi x_n) \in dom\varphi$, we have

$$\|u_n - x^*\| = \|T_{r_n}^{(F,\varphi)}(x_n - r_n\psi x_n) - T_{r_n}^{(F,\varphi)}(x^* - r_n\psi x^*)\| \leq \|x_n - x^*\|. \tag{6}$$

From Lemma 2.5 and the fact that T_i is k_i-strictly pseudocontractive mapping, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|\alpha_0 u_n + \sum_{i=1}^{m} \alpha_{n,i} w_n^i - x^*\|^2 \\
&= \|\alpha_{n,0}(u_n - x^*) + \sum_{i=1}^{m} \alpha_{n,i}(w_n^i - x^*)\|^2 \\
&= \alpha_{n,0}\|u_n - x^*\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\|w_n^i - x^*\|^2 - \sum_{i=1}^{m} \alpha_0\alpha_{n,i}\|u_n - w_n^i\|^2 - \sum_{i,j=1, i\neq j} \alpha_{n,i}\, \alpha_{n,j}\|w_n^i - w_n^j\|^2 \\
&\leq \alpha_{n,0}\|u_n - x^*\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\Pi^2(T_iu_n, T_ix^*) - \sum_{i=1}^{m} \alpha_0\alpha_{n,i}\|u_n - w_n^i\|^2 \\
&\leq \alpha_{n,0}\|u_n - x^*\|^2 + \sum_{i=1}^{m} \alpha_{n,i}[\|u_n - x^*\|^2 + k_i\|u_n - w_n^i\|^2] - \sum_{i=1}^{m} \alpha_{n,0}\alpha_{n,i}\|u_n - w_n^i\|^2 \\
&= \|u_n - x^*\|^2 + (k - \alpha_{n,0})\sum_{i=1}^{m} \alpha_{n,i}\|u_n - w_n^i\|^2 \qquad (3.3) \\
&\leq \|u_n - x^*\|^2
\end{aligned}$$

which implies that $x^* \in C_{n+1}$. Hence, $\Gamma \subset C_{n+1}$, thus $P_{C_{n+1}}x_0$ is well defined.

Step 2: We show that $\underset{n\to\infty}{lim}\|x_n - x_0\|$ exist.

Since $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n \ \forall\ n \geq 0$. We have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \tag{3.4}$$

For $x^* \in \Gamma \subset C_{n+1} \subset C_n$, we have

$$\|x_n - x_0\| \leq \|x^* - x_0\|. \tag{3.5}$$

From (3.4) and (3.5), we have that $\{x_n\}$ is a non-decreasing and bounded sequence.

Therefore, $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 3: We show that the sequence $\{x_n\}$ converges strongly to $\bar{x} \in C$. Since $x_n = P_{C_m}x_0 \in C_m \subset C_n$ for $m > n$ we obtain from Lemma 2.3, we have that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.6}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exist, we have from (3.6)that

$$\lim_{n\to\infty}\|x_m - x_n\| = 0$$

Thus $\{x_n\}$ is a Cauchy sequence. By the completeness of H and the closeness of C there exists $\bar{x} \in C$ such that $\{x_n\}$ converges to $\bar{x}$.

Step 4: We show that $\lim_{n\to\infty}\|w_n^i - x_n\| = 0, i = 1,2,...,N$. From (3.1), we have

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \qquad (3.7) \\ &\leq 2\|x_n - x_{n+1}\| \to 0,\ as\ n \to \infty. \end{aligned}$$

From (3.3), we have

$$\begin{aligned} (\alpha_{n,0} - k)\sum_{i=1}^{m}\alpha_{n,i}\|u_n - w_n^i\|^2 &\leq \|u_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \qquad (3.8) \end{aligned}$$

and for each $i = 1,2,...,m$, we have

$$\begin{aligned}(\alpha_{n,0}-k)\alpha_{n,i}\|u_n-w_n^i\|^2 &\leq \|x_n-x^*\|^2-\|y_n-x^*\|^2 \\ \leq & \quad M\|x_n-y_n\| \to 0, \ as\ n\to\infty, \qquad (3.9)\end{aligned}$$

Where $M = \sup_{n\geq 0}\{\|x_n-x^*\|+\|y_n-x^*\|\}$.
Using condition (i) and (ii) in (3.9), we have

$$\lim_{n\to\infty}\|u_n-w_n^i\|^2 = 0, i = 1,2,...,m. \qquad (3.10)$$

From Lemma 2.2 (3), we have

$$\begin{aligned}\|u_n-x^*\| &= \|T_{r_n}^{(F,\varphi)}(x_n-r_n\psi x_n)-T_{r_n}^{(F,\varphi)}(x^*-r_n\psi x^*)\|^2 \\ &\leq \langle u_n-x^*, x_n-x^*\rangle \\ &= \frac{1}{2}[\|u_n-x^*\|^2+\|x_n-x^*\|-\|x_n-u_n\|^2], \qquad (3.11)\end{aligned}$$

which implies

$$\begin{aligned}\|u_n-x^*\|^2 &\leq \|x_n-x^*\|^2-\|x_n-u_n\|^2 .i.e., \qquad (3.12) \\ \|y_n-x^*\|^2 &\leq \|x_n-x^*\|^2-\|x_n-u_n\|^2. \qquad (3.13) \\ &\leq M\|x_n-y_n\|\to 0, \ as\ n\to\infty\end{aligned}$$

From (3.10) and (3.13), we have

$$\lim_{n\to\infty}\|w_n^i-x_n\| = 0, i = 1,2,...,m. \qquad (3.14)$$

Step 5 : We show that $\bar{x}\in GMEP(F,\varphi)$.

From (3.13) and by condition $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\|\tfrac{x_n-u_n}{r_n}\| = 0 \qquad (3.15)$$

Since $\{x_n\}$ is bounded, we have that there exists a subsequence, $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\bar{x}$. From (3.15), we get that $\{u_{n_j}\}$ also converges weakly to $\bar{x}$. Since $u_n = T_{r_n}^{(F,\varphi)}(x_n-r_n\psi x_n)\in dom\varphi$, we have $F(u_n,y)+\varphi(y)-\varphi(u_n)+\langle\psi x_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq 0, \ \forall\ y\in C$.

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain:

$$\varphi(y)-\varphi(u_{n_j})+\left\langle \psi x_{n_j},y-u_{n_j}\right\rangle+\frac{1}{r_{n_j}}\left\langle y-u_{n_j},u_{n_j}x_{n_j}\right\rangle \geq F(y,u_{n_j}). \tag{7}$$

Let $z_t:=ty+(1-t)\bar{x}$ for all $t\in(0,1]$ and $y\in C$. This implies that $z_t\in C$. Then, by (7), we have

$$\begin{aligned}
\left\langle z_t-u_{n_j},\psi z_t\right\rangle &\geq \varphi(u_{n_j})-\varphi(z_t)+\left\langle z_t-u_{n_j},\psi z_t\right\rangle\\
&- \left\langle z_t-u_{n_j},\psi x_{n_j}\right\rangle-\left\langle z_t-u_{n_j},\frac{u_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle+F(z_t,u_{n_j})\\
&= \varphi(u_{n_j})-\varphi(z_t)+\left\langle z_t-u_{n_j},\psi z_t-\psi u_{n_j}\right\rangle\\
&+ \left\langle z_t-u_{n_j},\psi u_{n_j}-\psi x_{n_j}\right\rangle-\left\langle z_t-u_{n_j},\frac{u_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle\\
&\quad +F(z_t,u_{n_j}).
\end{aligned}$$

Since $||x_{n_j}-u_{n_j}||\to 0,\ j\to\infty$, we obtain $||\psi x_{n_j}-\psi u_{n_j}||\to 0,\ j\to\infty.$ Furthermore, by the monotonicity of ψ, we obtain $\langle z_t-u_{n_j},\psi z_t-\psi u_{n_j}\rangle\geq 0$. Then, by $(A4)$ we obtain as $j\to\infty$,

$$\langle z_t-\bar{x},\psi z_t\rangle\geq\varphi(\bar{x})-\varphi(z_t)+F(z_t,\bar{x}). \tag{8}$$

Using $(A1)$, $(A4)$ and (8) we also obtain

$$\begin{aligned}
0 &= F(z_t,z_t)+\varphi(z_t)-\varphi(z_t)\leq tF(z_t,y)+(1-t)F(z_t,\bar{x})+t\varphi(y)\\
&+ (1-t)\varphi(\bar{x})-\varphi(z_t)+t\varphi z_t-t\varphi z_t\\
&\leq t\big[F(z_t,y)+\varphi(y)-\varphi(z_t)\big]+(1-t)\langle z_t-\bar{x},\psi z_t\rangle
\end{aligned}$$

and hence

$$0\leq F(z_t,y)+\varphi(y)-\varphi(z_t)+(1-t)\langle y-\bar{x},\psi z_t\rangle.$$

Letting $t\to 0$, we obtain, for each $y\in C$,

$$0\leq F(u,y)+\varphi(y)-\varphi(\bar{x})+\langle y-u,\psi\bar{x}\rangle. \tag{9}$$

This implies that $\bar{x}\in GMEP$.

Step 6: We show that $\bar{x} \in \bigcap_{i=1}^{m} F(T_i)$.

From (3.14), we have

$$\lim_{n\to\infty} d(u_n, T_i u_n) = \lim_{n\to\infty} \|u_n - w_n^i\| = 0, i = 1, 2, ..., m. \tag{3.16}$$

Since $\{x_n\}$ converges to $\bar{x}$, then by (3.13), we have that $\{u_n\}$ converges to $\bar{x}$.

Hence, by (3.16), we have that

$$\bar{x} \in \bigcap_{n=1}^{m} F(T_i).$$

Step 7: We finally show that $\bar{x} = P_T x_0$.

Since $x_n = P_{C_n} x_0$, we have

$$\langle z - x_n, x_0 - x_n \rangle \leq 0, \ \forall \ z \in C_n.$$

Since $\Gamma \subset C_n$, we have

$$\langle x^* - x_n, x_0 - x_n \rangle \leq 0 \ \forall \ x^* \in \Gamma \tag{3.17}$$

Letting $n \to \infty$ in (3.17), we obtain

$$\langle x^* - \bar{x}, x_0 - \bar{x} \rangle \leq 0 \ \forall \ x^* \in \Gamma.$$

which implies that $\bar{x} = P_\Gamma x_0$.

Setting $\varphi \equiv 0, A \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.2

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T_i : C \to P(C)$ be finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1, 2, ..., m$. Let $F : C \times C \to \mathbb{R}$ satisfying $(A_1 - A_4)$. Assume that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap EP(F) \neq \emptyset$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C \\ y_n = \alpha_{n,0} u_n + \sum_{i=1}^{m} \alpha_{n,i} w_n^i, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, n \geq 0, \end{cases}$$

Where $r_n \in (0,\infty)$ with $\lim_{n\to\infty} r_n > 0$ and $w_n^i \in P_{T_i}u_n, P_{T_i}u_n = \{w_n^i \in T_iu_n : \|w_n^i - u_n\| = d(u_n, T_iu_n)\}$, with conditions

(i) $k \in (0,1)$, where $k = max\{k_i, i = 1,2,...,m\}$,

(ii) $\alpha_{n,0} \in (k,1), \alpha_{n,i} \in (0,1), i = 0,1,2,...,m$, such that $\sum_{i=0}^{m} \alpha_i = 1$

Then the sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

Remark 3.1 (a) Setting $F \equiv 0$ in Theorem 3.1, we obtain immediately corresponding result for $\Gamma = \bigcap_{i=1}^{m} F(T_i) \cap CMP(\varphi) \neq \emptyset$, where $CMP(\varphi)$ stands for convex minimization problem with respect to φ.

(b) If the mappings T_i are single valued for each i in Theorem 3.1, we easily get a corresponding result for such mappings which have been studied by many researchers.

Remark 3.2

(i) Let $\{\alpha_{n,i}\}$ be double sequence in $(0,1]$. Let (a) and (b) be the following conditions:

(a) $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0 \; \forall \; i \in \mathbb{N}$

(b) $\lim_{n\to\infty} \alpha_{n,i}$ *exist and lie in* $(0,1]$ *for all* $i = 0,1,2,...,m$

(c) So Theorem 3.1 and Corollary 3.2 hold true when the control double sequence $\alpha_{n,i}$ satisfies the condition (a).

(ii) The following double sequences are examples of the control sequences in Theorem 3.1 and Corollaries 3.2.

(1)

$$\alpha_{n,k} = \begin{cases} \frac{1}{3^k}\left(\frac{n}{n+2}\right), & n \geq k \\ 1 - \frac{n}{n+2}\left(\sum_{k=1}^{n} \frac{1}{3^k}\right), & n = k-1 \\ 0, & n < k-1 \end{cases}$$

That is,

$$(10) \quad \alpha_{n,k} = \left(\begin{array}{cccccccccc}
\frac{1}{9} & \frac{8}{9} & 0 & 0 & 0 & 0 & \dots & 0 & \dots & \\
\frac{1}{6} & \frac{1}{18} & \frac{7}{9} & 0 & 0 & 0 & \dots & 0 & & \\
\frac{1}{5} & \frac{1}{15} & \frac{1}{45} & \frac{32}{45} & 0 & 0 & \dots & 0 & \dots & \dots \\
\frac{2}{9} & \frac{2}{27} & \frac{2}{81} & \frac{2}{243} & \frac{163}{243} & 0 & \dots & 0 & \dots & \\
. & . & . & . & . & . & . & . & & \\
. & . & . & . & . & . & . & . & & \\
. & . & . & . & . & . & . & . & & \\
\frac{n}{3(n+2)} & \frac{n}{9(n+2)} & \frac{n}{27(n+2)} & \frac{n}{81(n+2)} & \frac{n}{243(n+2)} & \frac{n}{729(n+2)} & \dots & \frac{n}{3^k n+2} & \dots &
\end{array}\right)$$

We see that $\lim_{n\to\infty} \alpha_{n,k} = \frac{1}{3^k}$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,k} = \frac{1}{3^{k+1}},\ for\ some\ k = 1,2,3,\dots$

(2)

$$\alpha_{n,k} = \begin{cases} \frac{1}{3^k}\left(\frac{n}{n+2}\right), & n \geq k \text{ and } n \text{ is odd} \\ \frac{1}{3^{k+1}}\left(\frac{n}{n+2}\right), & n \geq k \text{ and } n \text{ is even} \\ 1 - \frac{n}{n+2}\left(\sum_{k=1}^{n} \frac{1}{3^k}\right), & n = k-1 \text{ and } n \text{ is odd} \\ 1 - \frac{n}{n+2}\left(\sum_{k=1}^{n} \frac{1}{3^{k+1}}\right), & n = k-1 \text{ and } n \text{ is even} \\ 0, & n < k-1 \end{cases}$$

That is,

$$(11) \quad \alpha_{n,k} = \left(\begin{array}{cccccccccc}
\frac{1}{9} & \frac{8}{9} & 0 & 0 & 0 & 0 & \dots & 0 & \dots & \\
\frac{1}{18} & \frac{1}{54} & \frac{25}{27} & 0 & 0 & 0 & \dots & 0 & & \\
\frac{1}{5} & \frac{1}{15} & \frac{1}{45} & \frac{32}{45} & 0 & 0 & \dots & 0 & \dots & \dots \\
\frac{2}{27} & \frac{2}{81} & \frac{2}{243} & \frac{2}{729} & \frac{649}{729} & 0 & \dots & 0 & \dots & \\
. & . & . & . & . & . & . & . & & \\
. & . & . & . & . & . & . & . & & \\
. & . & . & . & . & . & . & . & & \\
\frac{2n-1}{3(2n+1)} & \frac{2n-1}{9(2n+1)} & \frac{2n-1}{27(2n+1)} & \frac{2n-1}{81(2n+1)} & \frac{2n-1}{243(2n+1)} & \frac{2n-1}{729(2n+1)} & \dots & \frac{2n-1}{3^k(2n+1)} & \dots & \\
\frac{2n}{9(2n+2)} & \frac{2n}{27(2n+2)} & \frac{2n}{81(2n+2)} & \frac{2n}{243(2n+2)} & \frac{2n}{729(2n+2)} & \frac{2n}{2187(2n+2)} & \dots & \frac{2n}{3^{k+1}(2n+2)} & \dots &
\end{array}\right)$$

The $\lim_{n\to\infty} \alpha_{n,k}$ does not exist and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,k} = \frac{1}{3^{k+3}},\ for\ k = 1,2,3,\dots$

Remark 4.1 (1) The main result of our work Theorem 3.1 extends the main result of Bunyawat and Suantai [7] from multivalued nonexpansive mappings to multivalued strictly pseudo contractive mappings. Furthermore, while Bunyawat and Suantai solved the problem of approximating solution of mixed equilibrium problem, our result is applicable to the more general problem of approximating solution of generalized mixed equilibrium problem .
(2) Also, we gave an example which satisfies the conditions imposed on our itertative parameter.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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