Available online at http://scik.org
Adv. Fixed Point Theory, 2020, 10:3
https://doi.org/10.28919/afpt/4442
ISSN: 1927-6303

HYBRID-TYPE ITERATION SCHEME FOR APPROXIMATING FIXED POINTS OF LIPSCHITZ α-HEMICONTRACTIVE MAPPINGS

I. K. AGWU* AND D. I. IGBOKWE

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

Abstract. We propose a two-step iteration Scheme of hybrid-type for α-hemicontractive mappings and establish some weak and strong convergence theorems of the scheme to the fixed points of the mappings in real Hilbert spaces. Our results extend and generalize the results of Wang [18], compliments the results of Osilike and Onah [17] and other numerous results currently existing in literature.

Keywords: weak and strong convergence theorem; α-hemicontractive mapping; fixed point; real Hilbert space; Ishikawa-type iteration.

2010 AMS Subject Classification: 47H09, 47H10, 47J05, 65J15.

1.0 INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let K be a nonempty closed convex subset of H. Let $T: K \to K$ be a mapping. We use $F(T)$ to denote the set of fixed points of T and $x_n \to x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively weak) convergence of the sequence $\{x_n\}$ to x.

T is said to be *L-Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in K. \tag{1.1}$$

T is *nonexpansive* if

---

*Corresponding author
E-mail address: agwuimo@gmail.com
Received December 23, 2019

$$\|Tx - Ty\| \le \|x - y\|, \forall x, y \in K. \quad (1.2)$$

and *quasi-nonexpansive* if

$$\|Tx - p\| \le \|x - p\|, \forall x \in K \text{ and } p \in F(T). \quad (1.3)$$

From the definition, it easy to see that every nonexpansive mapping is Lipchistizian with $L = 1$. Recall that a nonexpansive mapping with a nonempty fixed point set is *quasi-nonexpansive*.

T is said to be k-strictly pseudocontraction (see, for example [9]) if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in K. \quad (1.4)$$

lf $k = 1$ in (1.4), then T is a pseudocontraction. It is well known that in real Hilbert spaces, the class of nonexpansive maps is a proper subclass of the class of k-strictly pseudocontractions. Furthermore, the class of k-strictly pseudocontractions is a proper subclass of the class of pseudocontractive maps.

T is said to be *demicontractive* (see, for example [13]) if $F(T) = \{x \in K: Tx = x\} \ne \emptyset$ and there exists $k \in [0,1)$ such that

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in K \text{ and } p \in F(T). \quad (1.5)$$

T is said to satisfy *Condition A* (see, for example [11]) if $F(T) = \{x \in K: Tx = x\} \ne \emptyset$ and there exists $\lambda > 0$ such that

$$\langle x - Tx, x - p\rangle \ge \lambda\|x - Tx\|^2, \forall x \in K \text{ and } p \in F(T). \quad (1.6)$$

The class of k-strictly pseudocontraction with a nonempty fixed point set is a proper subclass of the class of demicontractive maps.

T is said to be *hemicontraction* (see, for example [17]) if $k = 1$ in (1.5). The class of pseudocontractive maps with a nonempty fixed point set is a proper subclass of the class of hemicontractive maps.

The class of demicontractive maps is a proper subclass of the class of hemicontractive maps (see, for example [8]). These two classes of mappings have been studied extensively by many researches (see, for example [8], [13], [17]).

T is said to be α*-demicontraction* (see, for example [11]) if $F(T) = \{x \in K: Tx = x\} \ne \emptyset$ and there exists $\lambda > 0, \alpha \ge 1$ such that

$$\langle x - Tx, x - \alpha p\rangle \ge \lambda\|x - Tx\|^2, \forall x \in K \text{ and } p \in F(T). \quad (1.7)$$

Clearly, (1.7) is equivalent to

$$\|Tx - \alpha p\|^2 \le \|x - \alpha p\|^2 + k\|x - Tx\|^2, \forall x \in K \text{ and } p \in F(T), \quad (1.8)$$

where $k = 1 - 2\lambda \in [0,1)$. The following proposition shows that α-demicontractive map is a Lipschitizian map.

Proposition 1.1. Let K be a nonempty subset of a real Hilbert space H and let $T: K \to K$ be α-demicontractive mapping. Assume that $x \in K$ and $\alpha \geq 1$. Then, T is Lipschitizian.

Proof. Using the fact that T is α-demicontractive, we obtain the following estimates:

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + k\|x - Tx\|^2$$
$$\leq (\|x - \alpha p\| + \sqrt{k}\|x - Tx\|)^2,$$

so that

$$\|Tx - \alpha p\| \leq \|x - \alpha p\| + \sqrt{k}\|x - \alpha p + \alpha p - Tx\|$$
$$\leq \|x - \alpha p\| + \sqrt{k}\|x - \alpha p\| + \sqrt{k}\|\alpha p - Tx\|.$$

Hence,

$$\|Tx - \alpha p\| \leq \left(\frac{1+\sqrt{k}}{1-\sqrt{k}}\right)\|x - \alpha p\|$$

Therefore, T is L-Lipschitizian with $L = \frac{(1+\sqrt{k})}{(1-\sqrt{k})}$.

T is said to be α-*hemicontraction* (see, for example [8]) if $F(T) = \{x \in K: Tx = x\} \neq \emptyset$ and there exists $\alpha \geq 1$ such that

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + \|x - Tx\|^2, \forall x \in K \text{ and } p \in F(T). \tag{1.9}$$

Observe that (1.7) is equivalent to

$$\langle x - Tx, x - \alpha p\rangle \geq 0, \forall x \in K \text{ and } p \in F(T). \tag{1.10}$$

In [17], an example of an α-hemicontractive mapping with $\alpha > 1$ which is not hemicontrative (see ([17], Example 2.2) is given and observe that there are hemicontractive (1-hemicontractive) maps which are not α-hemicontraction for $\alpha > 1$ (see, for example ([17], Example 2.1)). Also, in [17] an example of a mapping which is hemicontractive (1-hemicontractive) and α-hemicontractive for $\alpha > 1$ but neither demicontractive (1-demicontraive) nor α-demicontractive for $\alpha > 1$ (see, for example ([17], Example 2.3) is given.

In addition, if $k = 1$ in (1.8), we get (1.9). Thus, the class of α-demicontractive mapping with $\alpha > 1$ is a subclass of the class α-hemicontractive mapping. For other properties of α-hemicontractive mappings, the reader may consult [17].

A mapping $T: H \to H$ is said to be η-strongly monotone if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \leq \eta\|x - y\|^2, \forall x, y \in H \tag{1.11}$$

Iterative techniques for approximating fixed points of nonexpansive mappings have been extensively studied based on the Mann's or Mann-type scheme by several authors (see, for example, [1], [5], [6], [16]). Recently, Wang [18] and Igbokwe and Jim [9] introduced the hybrid iteration method which has been used in solving variational inequalities.

Let H be a Hilbert space, $T: H \to H$ a nonexpansive mapping with $F(T) = \{x \in K: Tx = x\} \neq \emptyset$ and $F: H \to H$ an η-strongly monotone and Lipschiz mapping. Let $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty}$ be real sequences in $[0,1), \mu > 0$. The sequence $\{x_n\}_{n=1}^{\infty}$ is generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, n \geq 1, \quad (1.12)$$

where $T^{\lambda_{n+1}} x = Tx - \lambda_{n+1} \mu F(Tx), \mu > 0$.

Observe that if either $\lambda_n = 0, \forall n \geq 1$ or $F \equiv 0$, then (1.10) reduces to the well known Mann iteration method

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T \quad x_n, n \geq 1, \quad (1.13)$$

which has been used by several authors to approximate the fixed points of operators or operator equations.

In 2007, Wang [18] obtained weak and strong convergence of (1.12) to the fixed point of T as follows:

Theorem W [18]. Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping. If for any given $x_0 \in H, \{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, n \geq 1$$

where $\{\alpha_n\}$ and $\{\lambda_n\} \subset [0,1)$ satisfy the following conditions

(I) $\alpha \leq \alpha_n \leq \beta$ for some $\alpha, \beta \in (0,1)$

(II) $\sum_{n=1}^{\infty} \lambda_n < \infty$

(III) $0 < \mu < {}^{2\eta}/_{k^2}$,

then,

(II) $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in F(T)$;

(II) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Motivated by the work of Wang [18], Igbokwe and Jim [9] and the results of Osilike and Onah [17], it is our purpose in this paper to extend the results of Wang [18] and other related results currently in literature from nonexpansive mapping to the more general α-hemicontractive mapping. Our results are much more general and also more applicable than the results of Wang [18] because the strong monotonicity condition imposed on F is not required in our results.

2. 0 PRELIMINARIES.

For the sake of convenience, we restate the following concepts and results.

Let E be a real Banach space. A mapping T with domain $D(T)$ in E is said to be demiclosed at 0, if for any sequence $x_n \subset E,\ x_n \rightharpoonup p \in D(T)$ and $\|x_n - Tx_n\| \to 0$, then $Tp = p$.

A Banach space E is said to have Opial property, if for any sequence $\{x_n \subset E\}$ with $x_n \rightharpoonup p$, we have

$$\lim_{n\to\infty} \inf\|x_n - p\| < \lim_{n\to\infty} \inf\|x_n - q\|,$$

for all $q \in E$ with $q \neq p$. It is well known [19] that a Hilbert space H satisfies Opial's condition.

Let H and K maintain their usual meaning. For each point $x \in H$, there exists a unique nearest point of K, denoted by $P_K x$, such that $\|x - P_K x\| \le \|x - y\|, \forall y \in K$. Such a P_K is called metric projection from H onto K. It is well known that P_K is a firmly nonexpansive mapping from H onto K, i.e,

$$\|P_K x - P_K y\|^2 \le \langle P_K x - P_K y, x - y\rangle, \forall x, y \in H.$$

Furthermore, for any $x \in H$ and $z \in K, z = P_K x$ if and only if

$$\langle x - z, z - y\rangle \ge 0, \forall y \in K.$$

Lemma 2.1 (See [17]). In a real Hilbert space H, the following inequalities hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \forall t \in [0,1], \forall x, y \in H.$$

Lemma 2.2 (See [18]). Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \ge 0, \forall n \ge 1$. Let

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 (see [9]). Let E be an arbitrary normed space and let $\{t_n\}_{n=1}^{\infty}$ be a sequence satisfying the following conditions:

(i) $0 \le t_n \le t \le 1$, for some $t \in (0,1)$,

(ii) $\sum_{n=1}^{\infty} t_n = +\infty$.

Let $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be two sequences such that

(iii) $U_{n+1} = (1 - t_n)U_n + t_n V_n, n \ge 1$,

(iv) $\lim_{n\to\infty} |U_n| = d$, for some $d \in [0, \infty)$ and $n \ge 1$,

(v) $\lim_{n\to\infty} sup|V_n| \leq d, n \geq 1,$

(vi) $\sum_{n=1}^{\infty} t_j V_j$ is bounded.

Then, $d = 0$.

Theorem 2.1 [4]. A Banach space E is reflexive if and only if every (normed) bounded sequence in E has a subsequence which converges weakly to an element of E.

3.0 MAIN RESULTS.

Theorem 3.1 . Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T: K \to K$ be a Lipschitz α-hemicontractive self map with Lipschitz constant $L, L > 0$. Suppose $F: H \to H$ is an L_1-Lipschizian mapping. For any $x_1 \in H$, let $\{x_n\}$ be generated by

$$\left.\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) T y_n{}^{\lambda_{n+1}} \\ y_n &= P_K\left[\gamma_n x_n + (1-\gamma_n) T x_n{}^{\lambda_{n+1}}\right] \end{aligned}\right\} \tag{3.1}$$

where

$Tx^{\lambda_{n+1}} = Tx - \lambda_{n+1}\mu F(Tx),\ \mu > 0\ \forall x \in H,\ \{\gamma_n\}, \lambda_n \subset (0,1)$ and $\{\beta_n\} \subset \left]0, \frac{1}{2}\right]$ satisfy the following conditions:

(i) $0 < \beta < \beta_n < \gamma_n < 1$ for some $\beta \in \left(0, \frac{1}{2}\right)$

(ii) $\sum_{n=1}^{\infty}(1-\beta) = +\infty$

(iii) $\sum_{n=1}^{\infty}(1-\beta_n)^2 < +\infty$

(iv) $\sum_{n=1}^{\infty} \lambda_n < +\infty$

Then,

(a) $\lim_{n\to\infty} \|x_n - \alpha p\|$ exists for each $p \in F(T)$

(b) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$

(c) $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim_{n\to\infty} \inf d\big(x_n, F(T)\big) = 0$

Proof:

Using the well known identity

$$\|\beta x + (1-\beta)y\|^2 \leq \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2 \tag{3.2}$$

which holds $\forall x, y \in H, \beta \in (0,1)$, (3.1), and the fact that T and F are L-lipschizian, we estimate as follows: Since $F(T)$ is nonempty, let $\alpha p \in F(T)$. Let $x \in K$ be arbitrary, then we have:

$$\|x_{n+1}-\alpha p\|^2 = \left\|\beta_n(x_n-\alpha p)+(1-\beta_n)(T^{\lambda_{n+1}}y_n-\alpha p)\right\|^2$$

$$= \beta_n\|x_n-\alpha p\|^2 + (1-\beta_n)\left\|T^{\lambda_{n+1}}y_n-\alpha p\right\|^2 - \beta_n(1-\beta_n)\left\|x_n-T^{\lambda_{n+1}}y_n\right\|^2$$

Using $Tx^{\lambda_{n+1}} = Tx - \lambda_{n+1}\mu F(Tx)$, we obtain

$$\|x_{n+1}-\alpha p\|^2 \le \beta_n\|x_n-\alpha p\|^2 + (1-\beta_n)[\|Ty_n-\alpha p\|^2 + \lambda_{n+1}\mu(\|Ty_n-\alpha p\|^2$$
$$+\|F(Ty_n)\|^2) + \lambda_{n+1}^2\mu^2\|F(Ty_n)\|^2] - \beta_n(1-\beta_n)[\|x_n-Ty_n\|^2$$
$$+\lambda_{n+1}\mu(\|x_n-Ty_n\|^2 + \|F(Ty_n)\|^2) + \lambda_{n+1}^2\mu^2\|F(Ty_n)\|^2],$$

and which on simplification gives

$$\|x_{n+1}-\alpha p\|^2 \le \beta_n\|x_n-\alpha p\|^2 + (1-\beta_n)(1+\lambda_{n+1}\mu)\|Ty_n-\alpha p\|^2$$
$$+(1-\beta_n)\lambda_{n+1}\mu(1+\lambda_{n+1}\mu)\|F(Ty_n)\|^2$$
$$-\beta_n(1-\beta_n)(1+\lambda_{n+1}\mu)\|(x_n-\alpha p)+(\alpha p-Ty_n)\|^2$$
$$-\beta_n(1-\beta_n)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu\|F(Ty_n)\|^2$$
$$\le \beta_n\|x_n-\alpha p\|^2 + (1-\beta_n)(1+\lambda_{n+1}\mu)\|Ty_n-\alpha p\|^2$$
$$+(1-\beta_n)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu\|F(Ty_n)\|^2 - \beta_n(1-\beta_n)(1+\lambda_{n+1}\mu)$$
$$\times[\|x_n-\alpha p\|^2 + \|x_n-\alpha p\|^2 + \|Ty_n-\alpha p\|^2 + \|Ty_n-\alpha p\|^2]$$
$$-\beta_n(1-\beta_n)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu\|F(Ty_n)\|^2$$
$$\le \beta_n - 2\beta_n(1+\lambda_{n+1}\mu)\|x_n-\alpha p\|^2 + (1-\beta_n)^2(1+\lambda_{n+1}\mu)L^2\|y_n-\alpha p\|^2$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu\|F(Ty_n)\|^2, \tag{3.3}$$

since $-2\beta_n < -\beta_n$.

From

$$\|F(Ty_n)\| \le \|F(Ty_n)-F(T\alpha p)\| + \|F(T\alpha p)\|$$
$$\le L_1L\|y-\alpha p\| + \|F(T\alpha p)\|, \tag{3.4a}$$

it follows that

$$\|F(Ty_n)\|^2 \le L_1L(1+LL_1)\|y_n-\alpha p\|^2 + (1+L_1L)\|F(T\alpha p)\|^2 . \tag{3.4b}$$

Also, from (3.1), we get

$$\|y_n-\alpha p\|^2 \le \left\|\gamma_n x_n + (1-\gamma_n)Tx_n^{\lambda_{n+1}} - \alpha p\right\|^2,$$

$$= \left\|\gamma_n(x_n-\alpha p) + (1-\gamma_n)\left(Tx_n^{\lambda_{n+1}}-\alpha p\right)\right\|^2$$

$$= \gamma_n\|x_n-\alpha p\|^2 + (1-\gamma_n)\left\|Tx_n^{\lambda_{n+1}}-\alpha p\right\|^2 - \gamma_n(1-\gamma_n)\left\|x_n-Tx_n^{\lambda_{n+1}}\right\|^2$$

Using $Tx^{\lambda_{n+1}} = Tx - \lambda_{n+1}\mu F(Tx)$, we obtain

$$\|y_n-\alpha p\|^2 = \gamma_n\|x_n-\alpha p\|^2 + (1-\gamma_n)[\|\alpha p - Tx_n\|^2$$

$$+2\lambda_{n+1}\mu\|\alpha p - Tx_n\|\|F(Tx_n)\| + \lambda^2{}_{n+1}\mu^2\|F(Tx_n)\|^2]$$
$$-\gamma_n(1-\gamma_n)[\|x_n - Tx_n\|^2 + 2\lambda_{n+1}\mu\|x_n - Tx_n\|\|F(Tx_n)\|$$
$$+\lambda^2{}_{n+1}\mu^2\|F(Tx_n)\|^2$$
$$\leq \gamma_n\|x_n - \alpha p\|^2 + (1-\gamma_n)[\|\alpha p - Tx_n\|^2$$
$$+\lambda_{n+1}\mu(\|Tx_n - \alpha p\|^2 + \|F(Tx_n)\|^2) + \lambda^2{}_{n+1}\mu^2\|F(Tx_n)\|^2]$$
$$-\gamma_n(1-\gamma_n)[\|x_n - Tx_n\|^2 + \lambda_{n+1}\mu(\|x_n - Tx_n\|^2 + \|F(Tx_n)\|^2)$$
$$+\lambda^2{}_{n+1}\mu^2\|F(Tx_n)\|^2$$
$$= \gamma_n\|x_n - \alpha p\|^2 + (1-\gamma_n)[(1+\lambda_{n+1}\mu)\|Tx_n - \alpha p\|^2$$
$$+\lambda_{n+1}\mu(1+\lambda_{n+1}\mu)\|F(Tx_n)\|^2] - \gamma_n(1-\gamma_n)[(1+\lambda_{n+1}\mu)\|x_n - Tx_n\|^2$$
$$+\lambda_{n+1}\mu(1+\lambda_{n+1}\mu)\|F(Tx_n)\|^2] \qquad (3.5)$$

Since T is α -hemicontractive, we obtain (from (3.5)

$$\|y_n - \alpha p\|^2 \leq \gamma_n\|x_n - \alpha p\|^2 + (1-\gamma_n)[(\|x_n - \alpha p\|^2 + \|x_n - Tx_n\|^2)$$
$$+\lambda_{n+1}\mu(1+\lambda_{n+1}\mu)\|F(Tx_n)\|^2]$$
$$-\gamma_n(1-\gamma_n)[(1+\lambda_{n+1}\mu)\|x_n - Tx_n\|^2$$
$$+\lambda_{n+1}\mu(1+\lambda_{n+1}\mu)\|F(Tx_n)\|^2]$$
$$= [\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu)]\|x_n - \alpha p\|^2$$
$$+\left[(1-\gamma_n)\left(1-\gamma_n(1+\lambda_{n+1}\mu)\right)\right]\|x_n - Tx_n\|^2$$
$$+(1-\gamma_n)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu(1-\gamma_n)\|F(Tx_n)\|^2 \qquad (3.6)$$

Putting (3.4b) into (3.6), with y_n replaced by x_n, we have:

$$\|y_n - \alpha p\|^2 \leq [\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu)]\|x_n - \alpha p\|^2$$
$$+\left[(1-\gamma_n)\left(1-\gamma_n(1+\lambda_{n+1}\mu)\right)\right]\|x_n - Tx_n\|^2$$
$$+(1-\gamma_n)^2(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu[L_1L(1+L_1L)\|x_n - \alpha p\|^2$$
$$+(1+L_1L)\|F(\alpha p)\|^2]$$
$$= [\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu) + (1-\gamma_n)^2(1+\lambda_{n+1}\mu)$$
$$\times (1+L_1L)\lambda_{n+1}\mu L_1L]\|x_n - \alpha p\|^2 + [(1-\gamma_n)^2(1+\lambda_{n+1}\mu)]\|x_n - Tx_n\|^2$$
$$+(1-\gamma_n)^2(1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu\|F(\alpha p)\|^2. \qquad (3.7)$$

Putting (3.4b) into (3.3) we get:

$$\|x_{n+1} - \alpha p\|^2 \leq [\beta_n - 2\beta_n(1+\lambda_{n+1}\mu)]\|x_n - \alpha p\|^2 + (1-\beta_n)^2(1+\lambda_{n+1}\mu)L^2\|y_n - \alpha p\|^2$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu[LL_1(1+LL_1)\|y_n - \alpha p\|^2$$
$$+(1+LL_1)\|F(\alpha p)\|^2]$$

$$= [\beta_n - 2\beta_n(1+\lambda_{n+1}\mu)]\|x_n - \alpha p\|^2 + (1-\beta_n)^2(1+\lambda_{n+1}\mu)$$
$$\times [L^2 + \lambda_{n+1}\mu(1+LL_1)LL_1]\|y_n - \alpha p\|^2$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu(1+L_1L)\|F(\alpha p)\|^2 \quad (3.8)$$

Putting (3.7) into (3.8), we obtain

$$\|x_{n+1} - \alpha p\|^2 \leq [\beta_n - 2\beta_n(1+\lambda_{n+1}\mu)]\|x_n - \alpha p\|^2$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)[L^2 + \lambda_{n+1}\mu(1+L_1L)L_1L]\{[\gamma_n + (1-\gamma_n)$$
$$\times (1+\lambda_{n+1}\mu) + (1-\gamma_n)^2(1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu L_1L]\|x_n - \alpha p\|^2$$
$$+[(1-\gamma_n)^2(1+\lambda_{n+1}\mu)]\|x_n - Tx_n\|^2$$
$$+(1-\gamma_n)^2(1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu\|F(\alpha p)\|^2\}$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu\|F(\alpha p)\|^2$$
$$\leq \{1 + (1-\beta_n)^2(1+\lambda_{n+1}\mu)[L^2 + \lambda_{n+1}\mu(1+LL_1)LL_1$$
$$\times (\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu) + (1-\gamma_n)^2$$
$$\times (1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu L_1L)]\}\|x_n - \alpha p\|^2$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)[(1-\gamma_n)^2(1+\lambda_{n+1}\mu)]\|x_n - Tx_n\|^2$$
$$+(1-\beta_n)^2(1+L_1L)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu[1+(1-\gamma_n)^2]\|F(\alpha p)\|^2 \quad (3.9)$$

Since,

$$\|x_n - Tx_n\| \leq \|x_n - \alpha p\| + \|Tx_n - \alpha p\|,$$

it follows that

$$\|x_n - Tx_n\|^2 \leq (1+L)^2\|x_n - \alpha p\|^2 \quad (3.10)$$

Putting (3.10) into (3.9) imply that

$$\|x_{n+1} - \alpha p\|^2 \leq \{1 + (1-\beta_n)^2(1+\lambda_{n+1}\mu)[L^2 + \lambda_{n+1}\mu(1+\lambda_{n+1}\mu)(1+L_1L)L_1L$$
$$\times (\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu) + (1-\gamma_n)^2$$
$$\times (1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu L_1L)$$
$$+[(1-\gamma_n)^2(1+\lambda_{n+1}\mu)(1+L)^2]\}\|x_n - \alpha p\|^2$$
$$+(1-\beta_n)^2(1+L_1L)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu[1+(1-\gamma_n)^2]\|F(\alpha p)\|^2.$$

Let

$$\delta_n = (1-\beta_n)^2(1+\lambda_{n+1}\mu)[L^2 + \lambda_{n+1}\mu(1+\lambda_{n+1}\mu)(1+L_1L)L_1L$$
$$\times (\gamma_n + (1-\gamma_n)(1+\lambda_{n+1}\mu) + (1-\gamma_n)^2(1+\lambda_{n+1}\mu)(1+L_1L)\lambda_{n+1}\mu L_1L)]$$
$$+(1-\beta_n)^2(1+\lambda_{n+1}\mu)^2(1-\gamma_n)^2(1+L)^2$$

and

$\rho_n = (1-\beta_n)^2(1+L_1L)(1+\lambda_{n+1}\mu)\lambda_{n+1}\mu[1+(1-\gamma_n)^2]\|F(\alpha p)\|^2.$

Then,

$$\|x_{n+1}-\alpha p\|^2 \le (1+\delta_n)\|x_n-\alpha p\|^2+\rho_n\,. \qquad (3.11)$$

From conditions (iii) and (iv), $\sum_{n=1}^{\infty}\delta_n < \infty$ *and* $\sum_{n=1}^{\infty}\rho_n < \infty$.

Thus, using Lemma 2.2, it follows that $\lim_{n\to\infty}\|x_n-\alpha p\|$ exists. Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded and this completes the proof of (a).

Since $Tx^{\lambda_{n+1}} = Tx-\lambda_{n+1}\mu F(Tx)$, using (3.1), we obtain

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \beta_n\|x_n-Tx_n\|+\beta_n\|Tx_n-Tx_{n+1}\|+(1-\beta_n)\|Ty_n-Tx_{n+1}\| \\ &\quad +(1-\beta_n)\lambda_{n+1}\mu\|F(Tx_n)\| \\ &\le \beta_n\|x_n-Tx_n\|+\beta_nL\|x_n-x_{n+1}\|+(1-\beta_n)L\|y_n-x_{n+1}\| \\ &\quad +(1-\beta_n)\lambda_{n+1}\mu\|F(Tx_n)\| \qquad (3.12)\end{aligned}$$

Now, we estimate $\|x_n-x_{n+1}\|$:

Again, using (3.1) and the fact that $Tx^{\lambda_{n+1}} = Tx-\lambda_{n+1}\mu F(Tx)$, we obtain

$$\|x_n-x_{n+1}\| \le (1-\beta_n)\|x_n-Ty_n\|+(1-\beta_n)\lambda_{n+1}\mu\|F(Ty_n)\| \qquad (3.13)$$

Observe that

$$\|x_n-Ty_n\| \le \|x_n-\alpha p\|+L\|y_n-\alpha p\|,$$

which, using $Tx^{\lambda_{n+1}} = Tx-\lambda_{n+1}\mu F(Tx)$ and (3.1), gives

$$\begin{aligned}\|x_n-Ty_n\| &\le \|x_n-\alpha p\|+L\|\gamma_n(x_n-\alpha p)+(1-\gamma_n)(Tx_n-\alpha p)\| \\ &\quad +(1-\gamma_n)\lambda_{n+1}\mu L\|F(Tx_n)\| \\ &\le \|x_n-\alpha p\|+\gamma_nL\|x_n-\alpha p\|+(1-\gamma_n)L^2\|x_n-\alpha p\| \\ &\quad +(1-\gamma_n)\lambda_{n+1}\mu L\|F(Tx_n)\| \\ &= [1+\gamma_nL+(1-\gamma_n)L^2]\|x_n-\alpha p\|+(1-\gamma_n)\lambda_{n+1}\mu L\|F(Tx_n)\| \\ &\le (1+L)^2\|x_n-\alpha p\|+\lambda_{n+1}\mu L\|F(Tx_n)\| \qquad (3.14)\end{aligned}$$

Putting (3.14) into (3.13), we get

$$\begin{aligned}\|x_n-x_{n+1}\| &\le (1-\beta_n)[(1+L)^2\|x_n-\alpha p\|+\lambda_{n+1}\mu L\|F(Tx_n)\|] \\ &\quad +(1-\beta_n)\lambda_{n+1}\mu\|F(Ty_n)\|.\end{aligned}$$

Substituting for $\|F(Tx_n)\|$ and $\|F(Ty_n)\|$ using $(3.4a)$, we get

$$\begin{aligned}\|x_n-x_{n+1}\| &\le (1+L)^2(1-\beta_n)\|x_n-\alpha p\|+(1-\beta_n)\lambda_{n+1}\mu L(L_1L\|x_n-\alpha p\|+\|F(T\alpha p)\|) \\ &\quad +(1-\beta_n)\lambda_{n+1}\mu(L_1L\|y_n-\alpha p\|+\|F(T\alpha p)\|) \\ &= (1-\beta_n)[(1+L)^2+\lambda_{n+1}\mu L_1L]\|x_n-\alpha p\|+2(1-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\|\end{aligned}$$

$$+(1-\beta_n)\lambda_{n+1}\mu L_1 L\|y_n-\alpha p\|. \quad (3.15)$$

Furthermore, since $Tx^{\lambda_{n+1}} = Tx-\lambda_{n+1}\mu F(Tx)$, we get (from (3.1)) that

$$\|y_n-\alpha p\| \le (1-\gamma_n)\|x_n-Tx_n\|+\|x_n-\alpha p\|+(1-\gamma_n)\lambda_{n+1}\mu\|F(Tx_n)\|$$
$$\le (1-\gamma_n)\|x_n-\alpha p\|+(1-\gamma_n)\|Tx_n-\alpha p\|+\gamma_n\|x_n-\alpha p\|$$
$$+(1-\gamma_n)\lambda_{n+1}\mu\|F(Tx_n)\|,$$

which, using $(3.4a)$, gives

$$\|y_n-\alpha p\| \le [(1+L)+\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|+\lambda_{n+1}\mu\|F(T\alpha p)\|. \quad (3.16)$$

Putting (3.16) into (3.15) and simplifying, we get

$$\|x_n-x_{n+1}\| \le (1-\beta_n)[(1+L)^2+\lambda_{n+1}\mu L_1 L^2]\|x_n-\alpha p\|+2(1-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\|$$
$$+(1-\beta_n)\lambda_{n+1}\mu L_1 L\{[1+(1+\lambda_{n+1}\mu L_1)L]\|x_n-\alpha p\|+\lambda_{n+1}\mu\|F(T\alpha p)\|\}$$
$$\le 2(1-\beta_n)^2\{[(1+L)^2+\lambda_{n+1}\mu L_1 L^2(2+(1+\lambda_{n+1}\mu L_1)L]\|x_n-\alpha p\|$$
$$+\lambda_{n+1}\mu\times(2+\lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\|\}, \quad (3.17)$$

since $(1-\beta_n)^2=(1-\beta_n)-\beta_n(1-\beta_n). \Longrightarrow (1-\beta_n)=2(1-\beta_n)^2$.

Next, we estimate $\|y-x_{n+1}\|$:

From (3.1) and the fact that $Tx^{\lambda_{n+1}} = Tx-\lambda_{n+1}\mu F(Tx)$, we obtain

$$\|y-x_{n+1}\| \le \|y_n-Ty_n\|+\beta_n\|x_n-Ty_n\|+(1-\beta_n)\lambda_{n+1}\mu\|F(Ty_n)\| \quad (3.18)$$

But,

$$\|y_n-Ty_n\| \le \left\|\gamma_n x_n+(1-\gamma_n)\big(Tx_n-\lambda_{n+1}\mu F(Tx_n)\big)-Ty_n\right\|$$
$$\le (1-\gamma_n)\|x_n-Tx_n\|+\|x_n-Ty_n\|+(1-\gamma_n)\lambda_{n+1}\mu\|F(Tx_n)\| \quad (3.19)$$

Putting (3.14) into (3.19), we get

$$\|y_n-Ty_n\| \le (1-\gamma_n)\|x_n-Tx_n\|+(1+L)^2\|x_n-\alpha p\|+\lambda_{n+1}\mu L\|F(Tx_n)\|$$
$$+(1-\gamma_n)\lambda_{n+1}\mu\|F(Tx_n)\|$$
$$\le (1-\beta_n)\|x_n-Tx_n\|+(1+L)^2\|x_n-\alpha p\|+(1+L-\beta_n)\lambda_{n+1}\mu\|F(Tx_n)\|$$

Using $(3.4a)$, the last inequality becomes

$$\|y_n-Ty_n\| \le (1-\beta_n)\|x_n-Tx_n\|+(1+L)^2\|x_n-\alpha p\|$$
$$+(1+L-\beta_n)\lambda_{n+1}\mu(L_1 L\|x_n-p\|+\|F(T\alpha p)\|)$$
$$\le (1-\beta_n)\|x_n-Tx_n\|+[(1+L)^2+(1+L-\beta_n)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$
$$+(1+L-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\| \quad (3.20)$$

Putting (3.20) and (3.14) into (3.18), we get

$$\|y_n-x_{n+1}\| \le (1-\beta_n)\|x_n-Tx_n\|+[(1+L)^2+(1+L-\beta_n)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$

$$+(1+L-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\| + \beta_n[(1+L)^2\|x_n-\alpha p\|$$
$$+\lambda_{n+1}\mu L\|F(Tx_n)\|] + (1-\beta_n)\lambda_{n+1}\mu\|F(Ty_n)\|$$
$$\leq (1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2 + (1+L-\beta_n)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$
$$+(1+L-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\| + \lambda_{n+1}\mu L\|F(Tx_n)\|$$
$$+(1-\beta_n)\lambda_{n+1}\mu\|F(Ty_n)\|,$$

which, using (3.4a) and simplifying, gives

$$\|y_n - x_{n+1}\| \leq (1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2 + (1+L-\beta_n)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$
$$+(1+L-\beta_n)\lambda_{n+1}\mu\|F(T\alpha p)\| + \lambda_{n+1}\mu L(L_1 L\|x_n-\alpha p\| + \|F(T\alpha p)\|)$$
$$+(1-\beta_n)\lambda_{n+1}\mu(L_1 L\|y_n-\alpha p\| + \|F(T\alpha p)\|)$$
$$\leq (1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2 + (1-\beta_n+2L)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$
$$+ [2(1+L)-\beta_n]\lambda_{n+1}\mu\|F(T\alpha p)\| + \lambda_{n+1}\mu L_1 L\|y_n-\alpha p\|$$

Substituting for $\|y_n - \alpha p\|$ using (3.16), we have:

$$\|y_n - x_{n+1}\| \leq (1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2 + (1-\beta_n+2L)\lambda_{n+1}\mu L_1 L]\|x_n-\alpha p\|$$
$$+[2(1+L)-\beta_n]\lambda_{n+1}\mu\|F(T\alpha p)\|$$
$$+ \lambda_{n+1}\mu L_1 L\{[(1+L)+\lambda_{n+1}\mu L_1 L]\times\|x_n-\alpha p\| + \lambda_{n+1}\mu\|F(T\alpha p)\|\}$$
$$\leq (1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2$$
$$+ \lambda_{n+1}\mu L_1 L(3+3L+\lambda_{n+1}\mu L_1 L)]\|x_n-\alpha p\|$$
$$+ \lambda_{n+1}\mu[2(1+L)+\lambda_{n+1}\mu L_1 L]\|F(T\alpha p)\| \tag{3.21}$$

Putting (3.4a), (3.17) and (3.21) into (3.12), we get:

$$\|x_{n+1} - Tx_{n+1}\| \leq \beta_n\|x_n - Tx_n\| + 2\beta_n L(1-\beta_n)^2\{[(1+L)^2 + \lambda_{n+1}\mu L_1 L^2$$
$$\times(2+(1+\lambda_{n+1}\mu L_1)L]\|x_n-\alpha p\|$$
$$+\lambda_{n+1}\mu(2+\lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\|\} + (1-\beta_n)L\{(1-\beta_n)\|x_n - Tx_n\|$$
$$+[2(1+L)^2 + \lambda_{n+1}\mu L_1 L(2+3L+\lambda_{n+1}\mu L_1 L)]\|x_n-\alpha p\|$$
$$+\lambda_{n+1}\mu[2(1+L)+\lambda_{n+1}\mu L_1 L]\|F(T\alpha p)\|\} + (1-\beta_n)\lambda_{n+1}\mu\|F(Tx_n)\|.$$
$$\Rightarrow \|x_{n+1} - Tx_{n+1}\| \leq (\beta_n + (1-\beta_n)^2 L)\|x_n - Tx_n\| + 2\beta_n L(1-\beta_n)^2\{[(1+L)^2$$
$$+\lambda_{n+1}\mu\mu L_1 L^2(2+(1+\lambda_{n+1}\mu L_1)L]\|x_n-\alpha p\|] + \lambda_{n+1}\mu$$
$$\times(2+\lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\|\} + (1-\beta_n)L[2(1+L)^2 + \lambda_{n+1}\mu L_1 L$$
$$\times(2+3L+\lambda_{n+1}\mu L_1 L)]\|x_n-\alpha p\| + (1-\beta_n)\lambda_{n+1}\mu[2(1+L)$$
$$+\lambda_{n+1}\mu L_1 L]\|F(T\alpha p)\| + (1-\beta_n)\lambda_{n+1}\mu[L_1 L\|x_n-\alpha p\| + \|F(T\alpha p)\|]$$
$$\leq (1+(1-\beta_n)^2 L)\|x_n - Tx_n\| + 2(1-\beta_n)^2\{[L\beta_n(1+L)^2$$

$$+\lambda_{n+1}\mu L_1 L^2\beta_n(2+(1+\lambda_{n+1}\mu L_1)L]\big]+(1-\beta_n)^2L[2(1+L)^2$$
$$+\lambda_{n+1}\mu L_1 L(2+3L+\ \lambda_{n+1}\mu L_1 L)]+(1-\beta_n)^2\lambda_{n+1}\mu L_1 L\}M$$
$$+2\lambda_{n+1}\mu(1-\beta_n)^2\{L\beta_n(2+\lambda_{n+1}\mu L_1 L)+L[2(1+L)$$
$$+\lambda_{n+1}\mu L_1 L]+1\}\|F(T\alpha p)\|.$$

$$\therefore \|x_{n+1}-Tx_{n+1}\|\leq(1+\delta_n)\|x_n-Tx_n\|+\rho_{n,} \qquad (3.22)$$

where

$$\delta_n=(1-\beta_n)^2L$$

and

$$\rho_n=2(1-\beta_n)^2\{[L\beta_n(1+L)^2+\lambda_{n+1}\mu L_1 L^2\beta_n(2+(1+\lambda_{n+1}\mu L_1)L]\big]$$
$$+(1-\beta_n)^2L[2(1+L)^2+\lambda_{n+1}\mu L_1 L(2+3L+\ \lambda_{n+1}\mu L_1 L)]+(1-\beta_n)^2\lambda_{n+1}\mu L_1 L\}M$$
$$+2\lambda_{n+1}\mu(1-\beta_n)^2\{L\beta_n(2+\lambda_{n+1}\mu L_1 L)+L[2(1+L)+\lambda_{n+1}\mu L_1 L]+1\}\|F(T\alpha p)\|$$

From conditions (iii) and (iv), it follows that $\sum\delta_n<\infty$ and $\sum\rho_n<\infty$. Also, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n-Tx_n\|$ exists. Let $\lim_{n\to\infty}\|x_n-Tx_n\|=d$, and set

$U_n=(x_n-Tx_n)$. It follows (3.11*) that:

$$U_{n+1}=(1-t_n)U_n+t_nV_{n,} \qquad \ldots \qquad (3.23)$$

where $t_n=1-\beta_n$ and

$$t_nV_n=\beta_n(Tx_n-Tx_{n+1})+(1-\beta_n)(Ty_n-Tx_{n+1})+(1-\beta_n)\lambda_{n+1}\mu\, F(Ty_n)$$
$$\Longrightarrow t_nV_n\leq\beta_n(Tx_n-Tx_{n+1})+2(1-\beta_n)^2(Ty_n-Tx_{n+1})+2(1-\beta_n)^2\lambda_{n+1}\mu\, F(Ty_n)$$
$$\Longrightarrow V_n=(x_n-Tx_n)+\frac{\beta_n}{(1-\beta_n)}((Tx_n-Tx_{n+1})+2(1-\beta_n)(Ty_n-Tx_{n+1})$$
$$+2(1-\beta_n)\lambda_{n+1}\mu\, F(Ty_n))-(x_n-Tx_n)$$
$$\leq(x_n-Tx_n)+\frac{\beta_n}{(1-\beta_n)}((Tx_n-Tx_{n+1})+(1-\beta_n)\left(1+\frac{\beta_n}{(1-\beta_n)}\right)(Ty_n-Tx_{n+1})$$
$$+2(1-\beta_n)\lambda_{n+1}\mu\, F(Ty_n))$$
$$\Rightarrow\|V_n\|\leq\ \|x_n-Tx_n\|+\frac{\beta_n}{(1-\beta_n)}\|Tx_n-Tx_{n+1}\|+2(1-\beta_n)\|Ty_n-Tx_{n+1}\|$$
$$+2(1-\beta_n)\lambda_{n+1}\mu\,\|F(Ty_n)\|$$
$$\leq\ \|x_n-Tx_n\|+\frac{L\beta_n}{(1-\beta_n)}\|x_n-x_{n+1}\|+2(1-\beta_n)L\|y_n-x_{n+1}\|$$
$$+2(1-\beta_n)\lambda_{n+1}\mu\,\|F(Ty_n)\| \qquad (3.24)$$

Putting (3.17), (3.21) and (3.4a) into (3.24), we get:

$$\|V_n\|\leq\ \|x_n-Tx_n\|+\frac{L\beta_n}{(1-\beta_n)}\{2(1-\beta_n)^2\{[(1+L)^2+\lambda_{n+1}\mu L_1 L^2(2+(1+\lambda_{n+1}\mu L_1)L]$$

$$\times \|x_n - \alpha p\| + \lambda_{n+1}\mu(2 + \lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\|\} + 2L(1-\beta_n)$$
$$\times \{(1-\beta_n)\|x_n - Tx_n\| + [2(1+L)^2 + \lambda_{n+1}\mu L_1 L(2 + 3L + \lambda_{n+1}\mu L_1 L)]\|x_n - \alpha p\|$$
$$+ \lambda_{n+1}\mu[2(1+L) + \lambda_{n+1}\mu L_1 L]\|F(T\alpha p)\|\} + 2(1-\beta_n)$$
$$\times \lambda_{n+1}\mu\{L_1 L\|y_n - \alpha p\| + \|F(T\alpha p)\|\},$$

which using (3.16), gives

$$\|V_n\| \leq \|x_n - Tx_n\| + (1-\beta_n)L\{[(1+L)^2 + \lambda_{n+1}\mu L_1 L^2(2 + (1+\lambda_{n+1}\mu L_1)L]\|x_n - \alpha p\|$$
$$+\lambda_{n+1}\mu(2 + \lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\|\} + 2L(1-\beta_n)^2\|x_n - Tx_n\|$$
$$+2L(1-\beta_n)\big[2(1+L)^2 + \lambda_{n+1}\mu L_1 L\big(2 + 3L + (2 + (1+\lambda_{n+1}\mu L_1)L)\big)\big]\|x_n - \alpha p\|$$
$$+\lambda_{n+1}\mu(1-\beta_n)(4 + \lambda_{n+1}\mu L_1 L)\|F(T\alpha p)\| + \lambda_{n+1}\mu(1-\beta_n)$$
$$\times \{[2 + (1+\lambda_{n+1}\mu L_1)L]\|x_n - \alpha p\| + \lambda_{n+1}\mu\|F(T\alpha p)\|\}$$

$$\leq (1 + 2L(1-\beta_n)^2)\|x_n - Tx_n\| + 2(1-\beta_n)^2\{L[(1+L)^2 + \lambda_{n+1}\mu L_1 L^2$$
$$\times (2 + (1+\lambda_{n+1}\mu L_1)L] + L[2(1+L)^2 + \lambda_{n+1}\mu L_1 L(2 + 3L + \lambda_{n+1}\mu L_1)L]$$
$$+\lambda_{n+1}\mu L_1 L(1+L) + (1+\lambda_{n+1}\mu L_1)L\}M + 2\lambda_{n+1}\mu(1-\beta_n)^2$$
$$\times [3(1 + \lambda_{n+1}\mu L_1 L) + 2(1+L)]\|F(T\alpha p)\|.$$

It follows from the last inequality that:

$$\|V_n\| \leq (1 + 2L(1-\beta_n)^2)\|x_n - Tx_n\| + \psi_n, \tag{3.25}$$

where

$$\psi_n = 2(1-\beta_n)^2\{L[(1+L)^2 + \lambda_{n+1}\mu L_1 L^2(2 + (1+\lambda_{n+1}\mu L_1)L] + L[2(1+L)^2$$
$$+\lambda_{n+1}\mu L_1 L^2(2 + 3L + \lambda_{n+1}\mu L_1)L] + \lambda_{n+1}\mu L_1 L(1+L) + (1+\lambda_{n+1}\mu L_1)L\}M$$
$$+2\lambda_{n+1}\mu(1-\beta_n)^2[3(1 + \lambda_{n+1}\mu L_1 L) + 2(1+L)]\|F(T\alpha p)\|.$$

From condition (iii) and (iv), $\sum_{n=1}^{\infty}\psi_n < \infty$, hence $\lim_{n\to\infty}\psi_n$ exists.

Since $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists, $\|V_n\| \leq (1 + L(1-\beta_n)^2)\|x_n - Tx_n\| + Q, Q > 0$.

Therefore, $\lim_{n\to\infty} sup\|V_n\| \leq d$. Observe that:

$$\|\sum_{i=1}^{n} t_i V_i\| = \left\|\sum_{i=1}^{n}\big(L(x_i - x_{i+1}) + L(1-\beta_i)(y_i - x_{i+1}) + \lambda_{i+1}\mu F(Ty_i)\big)\right\|$$
$$\leq L\|\sum_{i=1}^{n}(x_i - x_{i+1})\| + L\|\sum_{i=1}^{n}(1-\beta_i)(y_i - x_{i+1})\| + \mu\|\sum_{i=1}^{n}(\lambda_{i+1})F(Ty_i)\|$$
$$\leq L\|x_1 - x_{n+1}\| + L\|y_1 - x_{n+1}\| + \mu D\sum_{i=1}^{n}\lambda_{i+1}$$
$$= L\|x_1 - \alpha p + \alpha p - x_{n+1}\| + L\|y_1 - \alpha p + \alpha p - x_{n+1}\| + \mu D\sum_{i=1}^{n}\lambda_{i+1}$$
$$\leq L\|x_1 - \alpha p\| + L\|x_{n+1} - \alpha p\| + L\|y_1 - \alpha p\| + L\|x_{n+1} - \alpha p\| + \mu D\sum_{i=1}^{n}\lambda_{i+1}$$
$$= L\|x_1 - \alpha p\| + 2L\|x_{n+1} - \alpha p\| + L\|y_1 - \alpha p\| + \mu D\sum_{i=1}^{n}\lambda_{i+1} \leq K^*,$$

for all $n \geq 1$ and for some $K^* > 0$. Hence, $\{\sum_{i=1}^{n} t_i V_i\}_{n=1}^{\infty}$ is bounded. It now follows that, from Lemma 2.3,

$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This completes the proof of (b).

Furthermore, from (3.11), we have:

$$\|x_{n+1} - \alpha p\|^2 \leq (1+\delta_n)\|x_n - \alpha p\|^2 + \rho_n$$
$$\leq \|x_n - \alpha p\|^2 + \delta_n M^2 + \rho_n.$$
$$\Rightarrow \|x_{n+1} - \alpha p\|^2 \leq \|x_n - \alpha p\|^2 + b_n, \tag{3.27}$$

where $b_n = \delta_n M^2 + \rho_n$. Hence,

$$[d(x_{n+1},\ F(T))]^2 \leq \left[d\left(x_n,\ F(T)\right)\right]^2 + b_n.$$

Since $\sum b_n < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, F(T))$ exists. If $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point αp of T, then $\lim_{n\to\infty} \|x_n - \alpha p\| = 0$.

Since $0 \leq d(x_n,\ F(T)) \leq \|x_n - \alpha p\|$, we have $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$.

Conversely, suppose $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$, then we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, for arbitrary $\varepsilon > 0, \exists N_1 > 0 \ni d(x_n, F(T)) < \frac{\varepsilon}{4},\ \forall_n \geq N_1$.

Also, $\sum b_n < \infty$ implies that there exists an $N_2 > 0 \ni \sum_{j=1}^{\infty} b_j < \frac{\varepsilon}{4}, \forall_n \geq N_2$.

Choose $N = max(N_{,1} N_2)$, then $d(x_N, F(T)) < \frac{\varepsilon}{4}$ and $\sum_{j=1}^{\infty} b_j < \frac{\varepsilon}{4}$. It follows, from (3.27), that $\forall n, m \geq N$ and $\forall p \in F(T)$, we have

$$\|x_n - x_m\| \leq \|x_n - \alpha p\| + \|x_m - \alpha p\|$$
$$\leq \|x_N - \alpha p\| + \sum_{j=1}^{n} b_j + \|x_N - \alpha p\| + \sum_{j=1}^{m} b_j$$
$$\leq 2\|x_N - \alpha p\| + 2\sum_{j=N}^{\infty} b_j.$$

Taking the infimum over all $\alpha p \in F(T)$, we obtain:

$$\|x_n - x_m\| \leq 2d\left(x_N, F(T)\right) + 2\sum_{j=1}^{\infty} b_j < 2\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n, m \geq N.$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim_{n\to\infty} x_n = u$, then since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we have $u \in F(T)$. This completes the proof of (c).

Theorem 3.2. Let $T: K \to K$ be a Lipschitz α-hemicontractive mapping, with Lipschiz constant L, $L > 0$, where K is a closed convex subset of a real Hilbert space H. Let the conditions of Theorem 3.1 be satisfied. If $(I - T)$ is demiclosed at zero, then for any arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to the fixed point of T.

Proof:

From Theorem 3.1, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The closedness of the subspace K guarantees its completeness, and K is a Hilbert space which is reflexive. Therefore, from the boundedness of $\{x_n\}_{n=1}^{\infty}$, and reflexivity of K, Theorem 2.1 guarantees the existence of a subsequence $\{x_{nj}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, such that $\{x_{nj}\}_{j=1}^{\infty}$converges weakly to $q \in K$. Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. and $I - T$ is demiclosed at zero, then $Tq = q$.

Hence, $q \in F(T)$. To prove uniqueness of q, we consider a subsequence $\{x_{nk}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, which converges weakly to $q' \in K$. As before, $q' \in F(T)$.

Since $q, q' \in F(T)$, for each $x, y \in K$, if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $K \subseteq H$ which converges weakly to z, by Marino and Xu [12],

$$\lim_{n\to\infty} sup\|x_n - \alpha p\|^2 = \lim_{n\to\infty} sup\|x_n - z\|^2 + \|z - \alpha p\|^2$$

Hence,

$$\lim_{n\to\infty}\|x_n - \alpha q\|^2 = \lim_{j\to\infty}\left\|x_{nj} - \alpha q\right\|^2 = \lim_{j\to\infty}\left\|x_{nj} - \alpha q'\right\|^2 + \|\alpha q' - \alpha q\|^2$$

$$= \lim_{j\to\infty}\left\|x_{mj} - \alpha q'\right\|^2 + \|\alpha q' - \alpha q\|^2$$

Therefore, $\alpha q = \alpha q'$ proving the uniqueness of αq which implies that $\{x_n\}_{n=1}^{\infty}$ converges to αq, completing the proof of Theorem 3.2.

Corollary

Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T: K \to K$ be α-demicontractive self map. Suppose $F: H \to H$ is an L_1-Lipschizian mapping. For any $x_1 \in H$, let $\{x_n\}$ be generated by

$$\left.\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n)T{y_n}^{\lambda_{n+1}} \\ y_n &= P_K\left[\gamma_n x_n + (1-\gamma_n)T{x_n}^{\lambda_{n+1}}\right] \end{aligned}\right\}$$

where

$Tx^{\lambda_{n+1}} = Tx - \lambda_{n+1}\mu F(Tx)$, $\mu > 0\ \forall x \in H$, $\{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\lambda_n \subset (0,1)$ satisfy the following conditions:

(v) $0 < \beta < \beta_n \leq \gamma_n < 1$ for some $\beta \in (0,1)$

(vi) $\sum_{n=1}^{\infty}(1-\beta) = +\infty$

(vii) $\sum_{n=1}^{\infty}(1-\beta_n)^2 < +\infty$

(viii) $\sum_{n=1}^{\infty} \lambda_n < +\infty$

Then,

(d) $\lim_{n\to\infty} \|x_n - \alpha p\|$ exists for each $p \in F(T)$

(e) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$

(f) $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim_{n\to\infty} \inf d\big(x_n, F(T)\big) = 0$.

Remark: The proof follows since an α-demicontractive map is Lipschitizian and α-hemicontractive.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] Browder, F. E. Nonlinear mappings of nonexpansive and accretive-type in Banach spaces. Bull. Amer. Math. Soc. 73(1967), 875 – 882.

[2] Browder, F. E. and Petryshyn, W. V., Cconstruction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20(1967), 197 -228.

[3] Chidume, C. E., Picards Iteration for nonlinear Lipschitz strong pseudocontractions in uniformly strong Banach spaces, ICTP Preprint, IC/951, (1995), 88.

[4] Chidume, C. E., Geometric properties of banach spaces and nonlinear iterations, Springer – Verlag London Limited. (2009).

[5] Chidume, C. E. and Moore, C., Fixed point iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127(1999), 1163 – 1170.

[6] Chidume, C. E. and Mutangadura, S. A., An example of the mann iterative method for lipschitz pseudocontractions, Proc. Amer. Math. Soc. 13(1974), 2359 – 2363.

[7] Hicks, T. L. and Kubicek, J. R., On the Mann iteration process in Hilbert space, J. Math. Anal. Appl. 59(1979), 498 – 504.

[8] Hussain, N., Rafiq, A. and Kang, M. S., Iteration schemes for two hemicontractive mappings in arbitrary Banach spaces, Int. J. Math. Anal. 7(18) (2013), 863 – 871.

[9] Igbokwe D. I., Construction of fixed points of strictly pseudocontractive mappings of Browder-Petryshn-type in arbitrary Banach spaces, J. Fixed Point Theory Appl. 4(2004), 137 - 147.

[10] Epuke, E. E., Approximation of fixed points and solutions of variational inequalities for certain classes of mappings using hybrid iteration scheme. M.Sc. Thesis, University of Nigeria, Nsukka, (2010).

[11] Liu, Q., On Naimpally Singh's open question, J. Math. Anal. Appl. 124 (1987), 157 – 164.

[12] Marino G. and Xu, H., Weak and Strong Convergence Theorems for Strictly Pseudocontrative Mappings in Hibert spaces, J. Math. Anal. Appl. 329(1)(2007), 336-346.

[13] Maruster, L. and Maruster, S., Strong convergence of α-demicontractive mappings, Math. Computer Model. 54 (2011), 2486 – 2492.

[14] Noor, M. A. and Noor, K. I., Some aspects of variational inequalities, J. Comput. Appl. Math. 47(1993), 285 – 312.

[15] Osilike, M. O., Strong and weak convergence of Ishikawa iteration methods for a class of nonlinear equations, Bull. Korean Math. Soc. 37(1) (2000), 11 – 127.

[16] Osilike, M. O. and Isiogugu, F. O. Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces, J. Math. Anal. 74 (2011), 1814 – 1822.

[17] Osilike, M. O. and Onah, A. C., Strong convergence of the Ishikawa iteration for Lipschitz α-hemicontractive mappings. Seria Mat. Inform. LIII(1) (2015), 151 – 161.

[18] Wang, L., An iteration method for nonexpansive mappings in Hilbert spaces, J. Nonlinear Anal. 74(2007), 80 - 38.

[19] Xu, H. K. and Tan K. K., Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. 43(3)(1993), 301 - 308.

[20] Yao, Y. H., Liou, Y. C. and Marino, G., A hybrid algorithm for pseudocontractive mappings, J. Nonlinear Anal. 17(2009), 4997 – 5002.

[21] Zhou, H., Convergence theorems of fixed points for lipschitz-pseudocontractions in Hilbert spaces, J. Math. Anal. Appl. 343(2008), 546 – 556.

[22] Zhou, H., Convergence theorems for λ-strict pseudo-contractions in smooth Banach spaces, J. Nonlinear Anal. (2009), 3140 – 3145.

[23] Akuchu B., Weak and Strong convergence of the Ishikawa iteractive sequence to fixed points of Lipschitz pseudocontractive maps in Hilbert spaces. Adv. Fixed Point Theory, 5(2015), 147-157.

[24] Veluhan, E. I., Weak and Strong Convergence Algorithm for Lipschitz Pseudocontractive Maps in Hilbert Spaces. Unpublished M.Sc. Thesis, Department of Mathematics, University of Nigeria, Nsukka, (2014).

[25] Owojori, O. O., Some convergence results for fixed point of hemicontractive operators in some Banach spaces, Kragujevac J. Math. 31(1974), 111 – 129.

[26] Morales, C. H. and Jung, J. S., Convergence of paths for pseudocontractive mappings in Banach spaces, Proc. Amer. Math. Soc. 128(2000), 3411 – 3419.