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HOMOGENEOUS APPROXIMATION PROPERTY FOR VARIOUS DIRECTIONS

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Abstract. In the theory of signal reconstruction, the homogeneous approximation property (HAP) for wavelets is useful. In this paper, we consider the HAP for the continuous wavelet transform with matrix dilation. When the dilation of wavelet is different in various directions, we show that the HAP holds in $L^2(\mathbb{R}^d)$. The HAP also holds in $L^\infty(\mathbb{R}^d)$ when we add some conditions.

Keywords: continuous wavelet transform; finite interval; HAP; homogeneous approximation property.

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1. INTRODUCTION

In 1952, Duffin and Schaeffer proposed the frame of Hilbert space in [5]. Frames have become an important tool in many other disciplines, because they can provide many different expression of vectors. Frame theory plays an important role in signal processing and many other fields. In recent years, more and more scholars are interested in frame theory, especially Gabor frames and wavelet frames.

Wavelet frames are a class of important frames and have many useful properties [1, 2, 4, 6, 7, 15] during the development. Through limits a, b to discrete values, we can makes wavelet

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transform $\tau(a,b)\psi$ forms a frame. There have been plenty of results on various properties of wavelet frames, including necessary and sufficient conditions for wavelet systems to be frames.

The wavelet transform of $f \in L^2(\mathbb{R}^d)$ with respect to $\psi \in L^2(\mathbb{R}^d)$ is defined by

$$\langle f, \tau(a,b)\psi \rangle = a^{-d/2} \int_{\mathbb{R}^d} f(x) \overline{\psi\Big(\frac{x-b}{a}\Big)} dx,$$

where

$$(\tau(a,b)\psi)(x) = a^{-d/2}\psi(a^{-1}(x-b)), \qquad (a,b) \in \mathscr{G}.$$

Here $\mathscr{G} := \{(a,b) : a > 0, b \in \mathbb{R}^d\}$ is a group and the action on it is defined by

$$(a,b)(s,t) = (as, b+at).$$

The homogeneous approximation property (HAP) is an interesting properties of wavelet frames. If the wavelet frame has good generators, then it has the HAP. The HAP has been studied in [1, 8, 9, 10, 11, 14, 16, 17]. In addition, the HAP for wavelet frame with nice wavelet function and arbitrary expansive dilation matrix also was studied in [16, 17] recently.

The following results in recent years is the HAP for the continuous wavelet transform. In [12, 13], they show that every pair of admissible wavelets possesses the HAP in L^2-sense, while it is not true in general whenever pointwise convergence is considered. But if we add some conditions on the wavelets and function to be reconstructed, then the HAP holds in $L^\infty(\mathbb{R})$. In the case of multivariate, this result is still true in $L^\infty(\mathbb{R}^d)$, but the condition that wavelets and the function to be reconstructed are all compactly supported on $\mathbb{R}^d \backslash 0$ is just a sufficient condition, not the necessary condition [13].

In this paper, we consider the HAP for the continuous wavelet transform with matrix dilation. When the dilation of wavelet is different in various directions, we show that the HAP holds in $L^2(\mathbb{R}^d)$. The HAP also holds in $L^\infty(\mathbb{R}^d)$ when we add some conditions.

2. Notations and Preliminary Results

In this section, we will introduce some notations and definitions.

Define

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-ix\omega} dx, \quad \omega \in \mathbb{R}^d$$

then

$$f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{f}(\omega)e^{ix\omega}d\omega,\quad x\in\mathbb{R}^d,$$

and

$$\langle f,g\rangle=\frac{1}{(2\pi)^d}\langle\hat{f},\hat{g}\rangle.$$

For $d\times d$ invertible matrix M_1,M_2 and $b_1,b_2\in\mathbb{R}^d$, define

$$(M_1,b_1)(M_2,b_2)=(M_1M_2,b_1+M_1b_2).$$

For invertible matrix M, define

$$(\tau(M,b)\psi)(x)=|M|^{-1/2}\psi\left(M^{-1}(x-b)\right),$$

where $|M|=|detM|$. It is easy to see that

$$\|\tau(M,b)\psi\|_2=\|\psi\|_2 \quad\text{and}\quad (\tau(M,b)\psi)\hat{}(\omega)=|M|^{1/2}e^{-ib\omega}\hat{\psi}(M^T\omega).$$

where M^T denotes the transpose of M.

For $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)\in\mathbb{R}^d, u\in\mathbb{R}^+$, define

$$M_\alpha(u)=diag[u^{\alpha_1},\cdots,u^{\alpha_d}],\qquad |\alpha|=\alpha_1+\cdots+\alpha_d,$$

then $M_\alpha(u)$ is invertible and $M_\alpha^{-1}(u)=M_{-\alpha}(u), M_\alpha(u)M_{\alpha'}(u)=M_{\alpha+\alpha'}(u)$.

Let $\mathscr{G}=\{(a,b):a>0,b\in\mathbb{R}^d\}$. For every $(s,t)\in\mathscr{G},\alpha\in\mathbb{R}^d$, its $(A_1,A_2;B,\alpha)$-neighborhood is defined by

$$(s,t)Q_{A_1,A_2;B,\alpha}=\{(sa,t+M_\alpha(s)b):a\in[A_1,A_2],b\in[-B,B]^d\}.$$

A function $\psi\in L^2(\mathbb{R}^d)$ is called admissible about the matrix function $M_\alpha(a)$, if there exists $C>0$, such that

$$\int_0^{+\infty}|\hat{\psi}(M_\alpha(a)\omega)|^2\frac{1}{a}da\le C,\qquad a.e\qquad \omega\in\mathbb{R}^d$$

and

$$C_\psi^{M_\alpha}=\int_0^{+\infty}|\hat{\psi}(M_\alpha(a)\omega)|^2\frac{1}{a}da$$

is independent of ω.

Similar, we call a pair of function (ψ_1,ψ_2) admissible about the matrix functions $(M_\alpha(u),M_{\alpha'}(u))$, if both ψ_1,ψ_2 are admissible about the matrix functions $(M_\alpha(u),M_{\alpha'}(u))$ separately and

$$C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}=\int_0^{+\infty}\overline{\hat{\psi}_1(M_\alpha(a)\omega)}\hat{\psi}_2(M_{\alpha'}(a)\omega)\frac{1}{a}da$$

is independent of ω. when $\alpha=\alpha'$, we write simply $C_{\psi_1,\psi_2}^{M_\alpha}$ instead of $C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}$ and we call (ψ_1,ψ_2) admissible about matrix function $M_\alpha(u)$ at this time.

Now for (ψ_1,ψ_2), we give the definitions of homogeneous approximation property (HAP) for various directions in $L^2(\mathbb{R}^d)$ and in $L^\infty(\mathbb{R}^d)$ for continuous matrix wavelet transforms.

Definition 2.1. *We call that (ψ_1,ψ_2) possess the homogeneous approximation property about matrix functions $(M_\alpha(u),M_{\alpha'}(u))$ in $L^2(\mathbb{R}^d)$ if for any $f\in L^2(\mathbb{R}^d)$ and $\varepsilon>0$, there exist $A_2>A_1>0$ and $B>0$ such that*

$$\Bigg\|\tau(M_\alpha(s),t)f-\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\times$$

$$\cdot\iint_{(a,b)\in(s,t)Q_{A_1',A_2';B',\alpha}}\langle\tau(M_\alpha(s),t)f,\tau(M_\alpha(a),b)\psi_1\rangle\tau(M_{\alpha'}(a),b)\psi_2\frac{1}{a^{(|\alpha|+|\alpha'|+2)/2}}dadb\Bigg\|_2$$

$$\leq\varepsilon,\qquad\forall(s,t)\in\mathscr{G},A_2'\geq A_2,0<A_1'\leq A_1,B'\geq B.\tag{1}$$

Definition 2.2. *A pair of admissible wavelets (ψ_1,ψ_2) is said to possess the homogeneous approximation property in $L^\infty(\mathbb{R}^d)$ if for any $f\in L^2(\mathbb{R}^d)$, $x\in\mathbb{R}^d$ and $\varepsilon>0$, there exist some $A_2>A_1>0$ such that*

$$\Bigg\|(\tau(s,t)f)(x)-\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\times$$

$$\cdot\int_{A_1's}^{A_2's}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\int_{\mathbb{R}^d}\langle\tau(M_\alpha(s),t)f,\tau(M_\alpha(a),b)\psi_1\rangle\tau(M_{\alpha'}(a),b)\psi_2(x)db\Bigg\|_\infty$$

$$\leq\varepsilon,$$

$$\forall A_2'\geq A_2,\ 0<A_1'\leq A_1,(s,t)\in\mathscr{G}.\tag{2}$$

Now we give an important proposition.

Proposition 2.3. [3, Proposition 2.4.1] *For all $f,g\in L^2(\mathbb{R}^d)$, if (ψ_1,ψ_2) is admissible about matrix functions $(M_\alpha(u),M_{\alpha'}(u))$, then*

$$\iint\limits_{\mathscr{G}} \langle f,\tau(M_\alpha(a),b)\psi_1\rangle\overline{\langle g,\tau(M_{\alpha'}(a),b)\psi_2\rangle}\frac{1}{a^{(|\alpha|+|\alpha'|+2)/2}}dadb=C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\langle f,g\rangle. \tag{3}$$

From this proposition, we can see that the continuous wavelet transform can reconstruct a function f as following:

$$f(x)=C^{-1}_{\psi_1,\psi_2}\int_0^{+\infty}da\int_{\mathbb{R}^d}\frac{1}{a^{d+1}}\langle f,\tau(a,b)\psi_1\rangle\left(\tau(a,b)\psi_2\right)(x)db. \tag{4}$$

where the convergence is in the weak sense. Here (ψ_1,ψ_2) is a pair of admissible function. The following is a useful lemma in this paper.

Lemma 2.4. *Suppose that $\psi_1,\psi_2\in L^2(\mathbb{R}^d)$ and (ψ_1,ψ_2) is admissible about matrix functions $(M_\alpha(u),M_{\alpha'}(u))$ and $C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\neq 0$. For any $f\in L^2(\mathbb{R}^d)$ and $A_2>A_1>0$, define*

$$\begin{aligned}&f_{A_1,A_2}(x)\\ &=\left(C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right)^{-1}\int_{A_1}^{A_2}da\int_{\mathbb{R}^d}\langle f,\tau(M_\alpha(a),b)\psi_1\rangle\left(\tau(M_{\alpha'}(a),b)\psi_2\right)(x)\frac{1}{a^{(|\alpha|+|\alpha'|+2)/2}}db,\end{aligned} \tag{5}$$

then we have

$$\left(f_{A_1,A_2}\right)\hat{}(\omega)=\left(C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right)^{-1}\hat{f}(\omega)\int_{A_1}^{A_2}\overline{\hat{\psi}_1(M_\alpha(a)\omega)}\hat{\psi}_2(M_{\alpha'}(a)\omega)\frac{da}{a}. \tag{6}$$

Proof. For any $x\in\mathbb{R}^d$, we have

$$\begin{aligned}&\int_{A_1}^{A_2}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\int_{\mathbb{R}^d}|\langle f,\tau(M_\alpha(a),b)\psi_1\rangle\left(\tau(M_{\alpha'}(a),b)\psi_2\right)(x)|\,db\\ \leq\ &\int_{A_1}^{A_2}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\left(\int_{\mathbb{R}^d}|\langle f,\tau(M_\alpha(a),b)\psi_1\rangle|^2db\right)^{1/2}\left(\int_{\mathbb{R}^d}|(\tau(M_{\alpha'}(a),b)\psi_2)(x)|^2db\right)^{1/2}\\ =\ &\|\psi_2\|_2\int_{A_1}^{A_2}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}a^{|\alpha|}|\hat{f}(\omega)|^2|\hat{\psi}_1(M_\alpha(a)\omega)|^2d\omega\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{(2\pi)^{d/2}}\|\psi_2\|_2\left(\int_{A_1}^{A_2}\frac{da}{a^{|\alpha|+1}}\int_{\mathbb{R}^d}a^{|\alpha|}|\hat{f}(\omega)|^2|\hat{\psi}_1(M_\alpha(a)\omega)|^2d\omega\right)^{1/2}\left(\int_{A_1}^{A_2}\frac{da}{a^{|\alpha'|+1}}\right)^{1/2}\\
&\le \frac{1}{(2\pi)^{d/2}}\|\psi_2\|_2\left(\int_{0}^{+\infty}\frac{da}{a}\int_{\mathbb{R}^d}|\hat{f}(\omega)|^2|\hat{\psi}_1(M_\alpha(a)\omega)|^2d\omega\right)^{1/2}\left(\int_{A_1}^{A_2}\frac{da}{a^{|\alpha'|+1}}\right)^{1/2}\\
&= \frac{1}{(2\pi)^{d/2}}\|\psi_2\|_2\|\hat{f}\|_2(C_\psi^{M_\alpha})^{1/2}\left(\int_{A_1}^{A_2}\frac{da}{a^{|\alpha'|+1}}\right)^{1/2}\\
&< \infty.
\end{aligned}
\tag{7}
$$

Hence, f_{A_1,A_2} is well defined on $\mathbb{R}^d$.

For any $x,x'\in\mathbb{R}^d$, similar arguments show that

$$
\begin{aligned}
&|f_{A_1,A_2}(x)-f_{A_1,A_2}(x')|\\
=&\ \Bigg|\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\int_{A_1}^{A_2}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\times\\
&\cdot\int_{\mathbb{R}^d}\langle f,\tau(M_\alpha(a),b)\psi_1\rangle\Big((\tau(M_{\alpha'}(a),b)\psi_2)(x)-(\tau(M_{\alpha'}(a),b)\psi_2)(x')\Big)db\Bigg|\\
\le&\ \left|\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\right|\left(\frac{1}{(2\pi)^d}\int_{A_1}^{A_2}\int_{\mathbb{R}^d}|\hat{f}(\omega)|^2|\hat{\psi}_1(M_\alpha(a)\omega)|^2\frac{dad\omega}{a}\right)^{1/2}\times\\
&\cdot\left(\int_{A_1}^{A_2}\left\|\psi_2(M_{\alpha'}^{-1}(a)x-\cdot)-\psi_2(M_{\alpha'}^{-1}(a)x'-\cdot)\right\|_2^2\frac{da}{a^{|\alpha'|+1}}\right)^{1/2}\\
\le&\ \frac{1}{(2\pi)^{d/2}}\left|(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}})^{-1}\right|(C_\psi^{M_\alpha})^{1/2}\|\hat{f}\|_2\times\\
&\cdot\left(\int_{A_1}^{A_2}\left\|\psi_2(M_{\alpha'}^{-1}(a)x-\cdot)-\psi_2(M_{\alpha'}^{-1}(a)x'-\cdot)\right\|_2^2\frac{da}{a^{|\alpha'|+1}}\right)^{1/2}.
\end{aligned}
$$

Hence $\lim_{\|x-x'\|_2\to 0}|f_{A_1,A_2}(x)-f_{A_1,A_2}(x')|=0$. That is, f_{A_1,A_2} is uniformly continuous on $\mathbb{R}^d$.

Next we prove (6). By (7), for any $g\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$,

$$
\int_{\mathbb{R}^d}|g(x)|dx\int_{A_1}^{A_2}\frac{da}{a^{(|\alpha|+|\alpha'|+2)/2}}\int_{\mathbb{R}^d}|\langle f,\tau(M_\alpha(a),b)\psi_1\rangle\left(\tau(M_{\alpha'}(a),b)\psi_2\right)(x)|\,db<\infty.
$$

By Fubini's Theorem, we have

$$
\begin{aligned}
&\langle f_{A_1,A_2},g\rangle\\
=&\ \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\int_{A_1}^{A_2}da\int_{\mathbb{R}^d}\langle f,\tau(M_\alpha(a),b)\psi_1\rangle\langle\tau(M_{\alpha'}(a),b)\psi_2,g\rangle\frac{1}{a^{(|\alpha|+|\alpha'|+2)/2}}db.
\end{aligned}
$$

A similar argument as that done in [3, Proposition 2.4.1] shows that

$$\langle f_{A_1,A_2},g\rangle = \frac{1}{(2\pi)^d}\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\int_{\mathbb{R}^d}\hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega\int_{A_1}^{A_2}\overline{\hat{\psi}_1(M_\alpha(a)\omega)}\hat{\psi}_2(M_{\alpha'}(a)\omega)\frac{da}{a}.$$

Since $g\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ is arbitrary, we have

$$\left(f_{A_1,A_2}\right)\hat{}(\omega) = \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\hat{f}(\omega)\int_{A_1}^{A_2}\overline{\hat{\psi}_1(M_\alpha(a)\omega)}\hat{\psi}_2(M_{\alpha'}(a)\omega)\frac{da}{a}.$$

□

3. HAP for Various Directions in $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$

In this section, we first consider HAP for various directions in $L^2(\mathbb{R}^d)$.

Theorem 3.1. *Suppose that $\psi_1,\psi_2\in L^2(\mathbb{R}^d)$ is admissible about $M_\alpha(u),M_{\alpha'}(u)$ and that $C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\neq 0$, then (ψ_1,ψ_2) possess the homogeneous approximation property about matrix function $(M_\alpha(u),M_{\alpha'}(u))$ in $L^2(\mathbb{R}^d)$.*

Proof. Let $A_2>A_1>0$ and $B>0$ be constants to be determined later. Suppose that $A_2'>A_2$ and $0<A_1'\le A_1$. Then for any $f\in L^2(\mathbb{R})$ and $(s,t)\in\mathscr{G}$, we have

$$\begin{aligned}
&\Bigg\|\tau(M_\alpha(s),t)f-\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\times \\
&\cdot\iint_{(a,b)\in(s,t)Q_{A_1',A_2';B',\alpha}}\langle\tau(M_\alpha(s),t)f,\tau(M_\alpha(a),b)\psi_1\rangle\tau(M_{\alpha'}(a),b)\psi_2\frac{dadb}{a^{(|\alpha|+|\alpha'|+2)/2}}\Bigg\|_2 \\
=&\sup_{\|g\|_2=1}\Bigg|\Big\langle\tau(M_\alpha(s),t)f-\left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right)^{-1}\times \\
&\cdot\iint_{(a,b)\in(s,t)Q_{A_1',A_2';B',\alpha}}\langle\tau(M_\alpha(s),t)f,\tau(M_\alpha(a),b)\psi_1\rangle\tau(M_{\alpha'}(a),b)\psi_2\frac{dadb}{a^{(|\alpha|+|\alpha'|+2)/2}},g\Big\rangle\Bigg|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|g\|_2=1} \left| \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}} \right)^{-1} \times \right. \\
&\quad \left. \cdot \iint_{(a,b)\notin(s,t)Q_{A_1',A_2';B',\alpha}} \langle \tau(M_\alpha(s),t)f, \tau(M_\alpha(a),b)\psi_1 \rangle \langle \tau(M_{\alpha'}(a),b)\psi_2 \frac{dadb}{a^{(|\alpha|+|\alpha'|+2)/2}}, g \rangle \right|^2 \\
&\le \sup_{\|g\|_2=1} \left| \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}} \right)^{-2} \right| \iint_{(a,b)\notin(s,t)Q_{A_1',A_2';B',\alpha}} |\langle \tau(M_\alpha(s),t)f, \tau(M_\alpha(a),b)\psi_1 \rangle|^2 \frac{dadb}{a^{|\alpha|+1}} \\
&\qquad \cdot \iint_{\mathscr{G}} |\langle g, \tau(M_{\alpha'}(a),b)\psi_2 \rangle|^2 \frac{dadb}{a^{|\alpha'|+1}} \\
&\qquad \text{(using Proposition 2.3)} \\
&= C_{\psi_2}^{M_{\alpha'}} \left| \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}} \right)^{-2} \right| \iint_{(a,b)\notin(s,t)Q_{A_1',A_2';B',\alpha}} |\langle \tau(M_\alpha(s),t)f, \tau(M_\alpha(a),b)\psi_1 \rangle|^2 \frac{dadb}{a^{|\alpha|+1}} \\
&= C_{\psi_2}^{M_{\alpha'}} \left| (C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}})^{-2} \right| \iint_{(a,b)\notin(s,t)Q_{A_1',A_2';B',\alpha}} \left| \langle f, \tau(M_\alpha(\frac{a}{s}), M_\alpha^{-1}(s)(b-t))\psi_1 \rangle \right|^2 \frac{dadb}{a^{|\alpha|+1}} \\
&= C_{\psi_2}^{M_{\alpha'}} \left| \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}} \right)^{-2} \right| \iint_{(a,b)\notin Q_{A_1',A_2';B',\alpha}} |\langle f, \tau(M_\alpha(a),b)\psi_1 \rangle|^2 \frac{dadb}{a^{|\alpha|+1}} \\
&\qquad ((a,b) \to (as, t+M_\alpha(s)b)) \\
&\le C_{\psi_2}^{M_{\alpha'}} \left| \left(C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}} \right)^{-2} \right| \iint_{(a,b)\notin Q_{A_1,A_2;B}} |\langle f, \tau(M_\alpha(a),b)\psi_1 \rangle|^2 \frac{dadb}{a^{|\alpha|+1}} \\
&:= E_{A_1,A_2;B}.
\end{aligned}
$$

By Proposition 2.3, we can make $E_{A_1,A_2;B}$ arbitrary small by choosing A_2 and B large enough and A_1 small enough. This completes the proof. □

Now we show that the HAP for various directions also holds in $L^\infty(\mathbb{R}^d)$.

Theorem 3.2. *Suppose that $f \in L^2(\mathbb{R}^d)$ satisfies $\hat{f} \in L^1(\mathbb{R}^d)$, that (ψ_1, ψ_2) is admissible about matrix functions $(M_\alpha(u), M_{\alpha'}(u))$, then for $\forall \varepsilon, s_0 > 0$, there exist $A_2 > A_1 > 0$, for any $(s,t) \in \mathscr{G}$,* *which satisfy* $s \geq s_0 > 0$, *and any* $A_2' \geq A_2, 0 < A_1' \leq A_1$, *we have*

$$\begin{aligned} &\left\| \left(\tau(M_\alpha(s),t)f - \left(C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right)^{-1} \times \right.\right. \\ &\left.\int_{A_1's}^{A_2's} da \int_{\mathbb{R}^d} \langle \tau(M_\alpha(s),t)f, \tau(M_\alpha(a),b)\psi_1\rangle \left(\tau(M_{\alpha'}(a),b)\psi_2\right) \frac{dadb}{a^{(|\alpha|+|\alpha'|+2)/2}}\right\|_\infty \\ (8) \qquad &\leq \varepsilon. \end{aligned}$$

Proof. By Lemma 2.4, it is easy to see that for any $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} &\left\| \left(\tau(M_\alpha(s),t)f - \left(C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right)^{-1} \times \right.\right. \\ &\left.\cdot \int_{A_1's}^{A_2's} da \int_{\mathbb{R}^d} \langle \tau(M_\alpha(s),t)f, \tau(M_\alpha(a),b)\psi_1\rangle \left(\tau(M_{\alpha'}(a),b)\psi_2\right) \frac{dadb}{a^{(|\alpha|+|\alpha'|+2)/2}}\right\|_\infty \\ \leq\ & \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| |M_\alpha(s)|^{1/2} e^{-it\omega} \hat{f}(M_\alpha(s)\omega) \right. \\ &\left. - \left(C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right)^{-1} |M_\alpha(s)|^{1/2} e^{-it\omega} \hat{f}(M_\alpha(s)\omega) \int_{A_1's}^{A_2's} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right| d\omega \\ =\ & \frac{1}{(2\pi)^d \left|C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right|} \times \\ & \int_{\mathbb{R}^d} |M_\alpha(s)|^{1/2} \left|\hat{f}(M_\alpha(s)\omega)\right| \left| C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2} - \int_{A_1's}^{A_2's} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right| d\omega \\ \leq\ & \frac{1}{(2\pi)^d \left|C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right|} \int_{\mathbb{R}^d} |M_\alpha(s)|^{1/2} \left|\hat{f}(M_\alpha(s)\omega)\right| \times \\ &\qquad\qquad \cdot \left| \int_0^{A_1's} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right| d\omega \\ & + \frac{1}{(2\pi)^d \left|C^{M_\alpha,M_{\alpha'}}_{\psi_1,\psi_2}\right|} \int_{\mathbb{R}^d} |M_\alpha(s)|^{1/2} \left|\hat{f}(M_\alpha(s)\omega)\right| \times \\ &\qquad\qquad \cdot \left| \int_{A_2's}^{+\infty} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right| d\omega \\ (9) =\ & I + II. \end{aligned}$$

First, we estimate I. By substituting $M_\alpha^{-1}(s)\omega$ for ω, we have

$$\begin{aligned} I &= \frac{|M_\alpha(s)|^{-\frac{1}{2}}}{(2\pi)^d\left|C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right|}\int_{\mathbb{R}^d}|\hat{f}(\omega)|\left|\int_0^{A_1's}\overline{\hat{\psi}_1\left(M_\alpha\left(\frac{a}{s}\right)\omega\right)}\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\frac{1}{a}da\right|d\omega \\ &\leq \frac{s^{-\frac{|\alpha|}{2}}}{(2\pi)^d\left|C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right|}\int_{\mathbb{R}^d}|\hat{f}(\omega)|\int_0^{A_1's}\left|\overline{\hat{\psi}_1\left(M_\alpha\left(\frac{a}{s}\right)\omega\right)}\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\right|\frac{1}{a}dad\omega \\ &\leq \frac{s_0^{-\frac{|\alpha|}{2}}}{(2\pi)^d\left|C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right|}\int_{\mathbb{R}^d}|\hat{f}(\omega)|\int_0^{A_1's}\left|\overline{\hat{\psi}_1\left(M_\alpha\left(\frac{a}{s}\right)\omega\right)}\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\right|\frac{1}{a}dad\omega. \end{aligned}$$

Since both ψ_1 and ψ_2 are admissible about the matrix function $M_\alpha(a), M_{\alpha'}(a)$ separately, we have

$$\begin{aligned} &\int_0^{A_1's}\left|\overline{\hat{\psi}_1\left(M_\alpha\left(\frac{a}{s}\right)\omega\right)}\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\right|\frac{1}{a}da \\ \leq& \left(\int_0^{A_1's}\left|\hat{\psi}_1\left(M_\alpha\left(\frac{a}{s}\right)\omega\right)\right|^2\frac{1}{a}da\right)^{\frac{1}{2}}\left(\int_0^{A_1's}\left|\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\right|^2\frac{1}{a}da\right)^{\frac{1}{2}} \\ \leq& \left(\int_0^{A_1'}|\hat{\psi}_1(M_\alpha(a)\omega)|^2\frac{1}{a}da\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\left|\hat{\psi}_2(M_{\alpha'}(a)M_\alpha^{-1}(s)\omega)\right|^2\frac{1}{a}da\right)^{\frac{1}{2}} \\ \leq& C_{\psi_1}^{M_\alpha}C_{\psi_2}^{M_{\alpha'}}, \qquad a.e. \end{aligned}$$

and

$$\lim_{A_1'\to 0}\int_0^{A_1'}|\hat{\psi}_1(M_\alpha(a)\omega)|^2\frac{1}{a}da = 0, \qquad a.e.$$

Since

$$I \leq \frac{s_0^{-\frac{|\alpha|}{2}}C_{\psi_2}^{M_{\alpha'}}}{(2\pi)^d\left|C_{\psi_1,\psi_2}^{M_\alpha,M_{\alpha'}}\right|}\int_{\mathbb{R}^d}|\hat{f}(\omega)|\left(\int_0^{A_1'}|\hat{\psi}_1(M_\alpha(a)\omega)|^2\frac{1}{a}da\right)^{\frac{1}{2}}d\omega.$$

By the dominated convergence theorem, we have

$$\lim_{A_1'\to 0}\int_{\mathbb{R}^d}|\hat{f}(\omega)|\left(\int_0^{A_1'}|\hat{\psi}_1(M_\alpha(a)\omega)|^2\frac{1}{a}da\right)^{\frac{1}{2}}d\omega = 0.$$

Hence, we can choose some $A_1 > 0$ such that for any $s \geq s_0$ and $0 < A_1' \leq A_1$,

$$I \leq \frac{\varepsilon}{2}. \tag{10}$$

Next we estimate II. Using Hölder's inequality twice, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |M_\alpha(s)|^{1/2} \left|\hat{f}(M_\alpha(s)\omega)\right| \left| \int_{A_2' s}^{+\infty} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right| d\omega \\
\leq\ & \left(\int_{\mathbb{R}^d} |M_\alpha(s)| \left|\hat{f}(M_\alpha(s)\omega)\right|^2 d\omega \right)^{1/2} \times \\
& \cdot \left(\int_{\mathbb{R}^d} \left| \int_{A_2' s}^{+\infty} \overline{\hat{\psi}_1(M_\alpha(a)\omega)} \hat{\psi}_2(M_{\alpha'}(a)\omega) \frac{1}{a} da \right|^2 d\omega \right)^{1/2} \\
\leq\ & (2\pi)^{d/2} \|f\|_2 \times \\
& \cdot \left(\int_{\mathbb{R}^d} \left(\int_{A_2' s}^{+\infty} |\hat{\psi}_1(M_\alpha(a)\omega)|^2 \frac{1}{a} da \int_{A_2' s}^{+\infty} |\hat{\psi}_2(M_{\alpha'}(a)\omega)|^2 \frac{1}{a} da \right) d\omega \right)^{1/2} \\
\leq\ & \left(C_{\psi_2}^{M_{\alpha'}} \right)^{1/2} (2\pi)^{d/2} \|f\|_2 \left(\int_{A_2' s}^{+\infty} \frac{1}{a} da \int_{\mathbb{R}^d} |\hat{\psi}_1(M_\alpha(a)\omega)|^2 d\omega \right)^{1/2} \\
=\ & \left(C_{\psi_2}^{M_{\alpha'}} \right)^{1/2} (2\pi)^{d} \|f\|_2 \|\psi_1\|_2 \left(\int_{A_2' s}^{+\infty} \frac{1}{a^{|\alpha|+1}} da \right)^{1/2} \\
=\ & \frac{\left(C_{\psi_2}^{M_{\alpha'}} \right)^{1/2} (2\pi)^{d} \|f\|_2 \|\psi_1\|_2}{|\alpha|^{1/2} (A_2' s)^{|\alpha|/2}} \\
\leq\ & \frac{\left(C_{\psi_2}^{M_{\alpha'}} \right)^{1/2} (2\pi)^{d} \|f\|_2 \|\psi_1\|_2}{|\alpha|^{1/2} (A_2 s_0)^{|\alpha|/2}}.
\end{aligned}$$

Hence

$$II \leq \frac{\left(C_{\psi_2}^{M_{\alpha'}} \right)^{1/2} \|f\|_2 \|\psi_1\|_2}{\left| C_{\psi_1,\psi_2}^{M_\alpha, M_{\alpha'}} \right| |\alpha|^{1/2} (A_2 s_0)^{|\alpha|/2}}.$$

Take $A_2 = \left(4 C_{\psi_2}^{M_{\alpha'}} \|f\|_2^2 \|\psi_1\|_2^2 \Big/ \varepsilon^2 \left| C_{\psi_1,\psi_2}^{M_\alpha, M_{\alpha'}} \right|^2 |\alpha| d s_0^{|\alpha|} \right)^{1/|\alpha|}$. Then for any $A_2' \geq A_2$, we have

$$II \leq \varepsilon/2. \tag{11}$$

Putting (10) and (11) together, we get the conclusion. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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