Available online at http://scik.org
Advances in Fixed Point Theory, 2 (2012), No. 3, 258-272
ISSN: 1927-6303

STEPANOV-LIKE PSEUDO-ALMOST AUTOMORPHIC MILD SOLUTIONS FOR SOME ABSTRACT DIFFERENTIAL EQUATIONS

RONG-HUA HE*

Department of Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610100, P. R. China

Abstract. In this work, we give some theorems on pseudo-almost automorphic solutions for some abstract semilinear differential equations with uniform continuity. To facilitate this we give a new composition theorem of Stepanov-like pseudo-almost automorphic functions. Since S^p-pseudo-almost automorphic functions are more general and complicated than almost automorphic functions and S^p-pseudo-almost periodic functions, our work improves the known results by making use of a uniform continuity condition instead of the Lipschitz condition.

Keywords: S^p-pseudo-almost automorphic, uniform continuity, fixed point, C_0-semigroup.

2000 AMS Subject Classification: 43A60

1. Introduction

The qualitative theory of differential equations involving almost periodicity and almost automorphism has been an attractive topic for nearly a century because of their significance and applications in areas such as physics and control theory. Consequently, differential equations, partial differential equations, and functional differential equations with the properties such as almost periodicity and almost automorphism have been

*Corresponding author

Received June 17, 2012

of great interest to many authors and there is a vast literature on the subject(see, for example, [1]-[12] and the references therein).

Recently, Xiao, Liang and Zhang [9] introduced a new concept of a function called a pseudo-almost automorphic function. They established a general existence and uniqueness theorem for pseudo-almost automorphic mild solutions to some semilinear abstract differential equations as well as solving a basic problem on the Banach space $(PAA(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty)$.

In this paper, we introduce and study the notion of S^p-pseudo-almost automorphy (or Stepanov-like pseudo-almost automorphy), which generalizes the concepts such as pseudo-almost automorphy and S^p-pseudo-almost periodicity. As applications, some existence theorems for pseudo-almost automorphic solutions for abstract differential equations were obtained. We notice that a Lipschitz condition is needed in the composition theorem and its applications in abstract differential equations(see [3], Theorem 3.5). So it is interesting and worthwhile to consider the same problem under a uniform continuity condition instead of the Lipschitz condition. This seems reasonable and necessary since the uniform continuity condition is the main condition needed for the composition theorems of almost-automorphic functions and pseudo-almost automorphic functions(see [13]).

The aim of this paper is to give some theorems on stepanov-like pseudo-almost automorphic (mild) solutions of the abstract semilinear differential equations

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), t \in \mathbb{R}, \tag{1}$$

under a uniform continuity condition. For this purpose, we give a new composition theorems for stepanov-like pseudo-almost automorphic functions, which improves the one given in [13] because of S^p-pseudo-almost automorphic functions are more general and complicated than almost automorphic functions.

Throughout the rest of the paper, we set $q = 1 - \frac{1}{p}$. Note that $q \neq 0$, as $p > 1$, and we suppose that $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is densely defined closed linear operator, and the operator A is the infinitesimal generator of a compact C_0-semigroup $(T(t))_{t\geq 0}$, which is exponentially stable. Namely, there exist some constants $M, \delta > 0$ such that $\|T(t)\| \leq Me^{-\delta t}$ for every $t \geq 0$.

2. Preliminaries

Throughout this paper, we always assume that $(\mathbb{X}, \|\cdot\|)$ is a Banach space. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) be the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{X}$ (respectively, $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$), and $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by $\|f\| = \sup\limits_{t\in\mathbb{R}} \|f(t)\|$, is a Banach space.

Definition 2.1 ([13])**.** (i) A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for each sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that $g(t) = \lim_{n\to\infty} f(t + \tau_n)$ is well-defined in $t \in \mathbb{R}$, and $\lim_{n\to\infty} g(t - \tau_n) = f(t)$ for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in \mathbb{K}$, where $\mathbb{K}$ is any bounded subset of $\mathbb{X}$. That is to say, for each sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that $g(t, x) = \lim_{n\to\infty} f(t + \tau_n, x)$ is well-defined in $t \in \mathbb{R}$ for all $x \in \mathbb{K}$, and $\lim_{n\to\infty} g(t - \tau_n, x) = f(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{K}$. Denote by $AA(\mathbb{R} \times \mathbb{X})$ the set of all such functions.

Remark 2.1. The function g in definition 2.1 is measurable, but not necessarily continuous. If f is almost automorphic, then its range is relatively compact.

Define the classes of functions $PAP_0(\mathbb{X})$ and $PAP_0(\mathbb{R} \times \mathbb{Y})$ respectively as follows:

$$PAP_0(\mathbb{X}) := \{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(s)\| ds = 0\},$$

and $PAP_0(\mathbb{R} \times \mathbb{Y})$ is the collection of all functions $F \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(t, u)\| dt = 0$$

uniformly in $u \in \mathbb{Y}$.

Define $AA_0(\mathbb{R} \times \mathbb{Y})$ as the collection of all functions $F \in BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(t, u)\| dt = 0$$

uniformly $u \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

Obviously,

$$PAP_0(\mathbb{R}\times\mathbb{Y})\subset AA_0(\mathbb{R}\times\mathbb{Y}).$$

Now we are ready to introduce the set $PAA(\mathbb{X})$ (resp.,$PAA(\mathbb{R}\times\mathbb{X})$) of pseudo-almost automorphic functions.

Definition 2.2 ([13])**.** A continuous function $f:\mathbb{R}\mapsto\mathbb{X}$ (resp., $\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$) is said to be pseudo-almost automorphic if it can be decomposed as $f=g+\phi$, where $g\in AA(\mathbb{X})$ (resp., $AA(\mathbb{R}\times\mathbb{X})$) and $\phi\in AA_0(\mathbb{X})$ (resp., $AA_0(\mathbb{R}\times\mathbb{X})$). Denoted by $PAA(\mathbb{X})$ (resp., $PAA(\mathbb{R}\times\mathbb{X})$) the set of all such functions.

The functions g and ϕ in Definition 2.2 are called the almost automorphic and the ergodic perturbation components of f, respectively. Moreover, the decomposition $g+\phi$ of f is unique, and $AA_0(\mathbb{X})$ (resp., $AA_0(\mathbb{R}\times\mathbb{X})$) and $PAA(\mathbb{X})$ (resp., $PAA(\mathbb{R}\times\mathbb{X})$) are all Banach spaces with the norm inherited from $BC(\mathbb{R},\mathbb{X})$ (resp., $BC(\mathbb{R}\times\mathbb{X},\mathbb{X})$) (see [14]).

Definition 2.3 ([15])**.** The Bocher transform $f^b(t,s),t\in\mathbb{R},s\in[0,1]$ of a function $f:\mathbb{R}\mapsto\mathbb{X}$ is defined by $f^b(t,s):=f(t+s)$.

Definition 2.4 ([3])**.** Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R};L^p((0,1),\mathbb{X}))$. This is a Banach space with the norm $\|f\|_{S^p}:=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}(\int_t^{t+1}\|f(\tau)\|^p d\tau)^{\frac{1}{p}}$.

Definition 2.5 ([16])**.** The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions consists of all $f\in BS^p(\mathbb{X})$ such that $f^b\in AA(L^p(0,1),\mathbb{X})$. That is, a function $f\in L^p_{loc}(\mathbb{R};\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p((0,1),\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in N}$, there exist a subsequence $(s_n)_{n\in N}$ and a function $g\in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$[\int_t^{t+1}\|f(s_n+s)-g(s)\|^p ds]^{\frac{1}{p}}\to 0,\quad \text{and}$$

$$[\int_t^{t+1}\|g(s-s_n)-f(s)\|^p ds]^{\frac{1}{p}}\to 0,\quad \text{as}\quad n\to\infty\quad \text{pointwise on}\quad \mathbb{R}.$$

Definition 2.6 ([3]). A function $F:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$ with $F(\cdot,u)\in L^p_{loc}(\mathbb{R};\mathbb{X})$ for each $u\in\mathbb{Y}$, is said to be S^p-almost automorphic in $t\in\mathbb{R}$ uniformly in $u\in\mathbb{Y}$ if $t\mapsto F(t,u)$ is S^p-almost automorphic for each $u\in\mathbb{Y}$, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u)\in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$[\int_t^{t+1}\|F(s_n+s,u)-G(s,u)\|^p ds]^{\frac{1}{p}}\to 0, \quad \text{and}$$

$$[\int_t^{t+1}\|G(s-s_n,u)-F(s,u)\|^p ds]^{\frac{1}{p}}\to 0, \quad \text{as} \quad n\to\infty \quad \text{pointwise on} \quad \mathbb{R} \quad \text{for each} \quad u\in\mathbb{Y}.$$

The collection of those S^p-almost automorphic functions $F:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$ will be denoted by $AS^p(\mathbb{R}\times\mathbb{Y})$.

Definition 2.7 ([3]). (i) A function $f\in BS^p(\mathbb{X})$ is called S^p-pseudo-almost automorphic if it can be expressed as $f=h+\varphi$, where $h^b\in AA(L^p((0,1),\mathbb{X}))$ and $\varphi^b\in AA_0(L^p((0,1),\mathbb{X}))$. The collection of such functions will be denoted by $PAA^p(\mathbb{X})$.

(ii) A function $F:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$ is called S^p-pseudo-almost automorphic if there exist two functions $H,\Phi:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ such that $F=H+\Phi$, where $H^b\in AA(\mathbb{R}\times L^p((0,1),\mathbb{X}))$ and $\Phi^b\in AA_0(\mathbb{R}\times L^p((0,1),\mathbb{X}))$. The collection of such functions will be denoted by $PAA^p(\mathbb{R}\times\mathbb{Y})$.

Example 2.1. Let $l^\infty(\mathbb{X})$ be the space of all two-sided bounded sequences with values in the Banach space $\mathbb{X}$ equipped with the sup norm defined by

$$\|x\|_\infty=\sup_{n\in\mathbb{Z}}\|x_n\| \quad \textit{for all} \quad x=(x_n)_{n\in\mathbb{Z}}.$$

Define the space $pap_0(\mathbb{X})$ *by*

$$pap_0(\mathbb{X})=\{x=(x_n)_{n\in\mathbb{Z}}\in l^\infty(\mathbb{X}):\lim_{n\to+\infty}\frac{1}{2n}\sum_{k=-n}^{n}\|x_k\|=0\}.$$

It is well-known that a sequence $x=(x_n)_{n\in\mathbb{Z}}\in pap_0(\mathbb{X})$ *if there exists a function* $\varphi\in pap_0(\mathbb{X})$ *such that* $x_n=\varphi(n)$ *for all* $n\in\mathbb{Z}$, *see* [5]. *Fix* $\varepsilon_0\in(0,1)$. *Let* g *be any* $\mathbb{X}$*-valued almost automorphic function , but not almost periodic. Clearly,* $g\in AA(\mathbb{X})\subset AS^p(\mathbb{X})$ *for* $p>1$ *and* $g\notin AP(\mathbb{X})$. *Let* $b=(b_n)_{n\in\mathbb{Z}}\subset pap_0(\mathbb{X})$ *and suppose* b *is not the zero sequence.*

Let φ be the function defined by

$$\varphi(t) = \begin{cases} b_n, \ if \ t \in (n, n+\varepsilon_0), \\ \\ 0, \ otherwise. \end{cases}$$

Then $\varphi^b \in pap_0(L^p((0,1),\mathbb{X}))$ for $p > 1$. The function f defined by $f = g+\varphi \in PAA^p(\mathbb{X})$ while $f \notin PAPS^p(\mathbb{X})$.

From the Example 2.1, we can follow that S^p-pseudo-almost automorphic functions are more general and complicated than S^p-pseudo-almost periodic functions.

3. Composition theorem

In this section, we give a composition theorem for Stepanov-like pseudo-almost automorphic functions under uniform continuity condition. Let us give the following assumptions:

- (H$_1$) $f(t,\cdot)$ is uniformly continuous in each bounded subset $\mathbb{K} \subset \mathbb{Y}$ uniformly for $t \in \mathbb{R}$. More explicitly, given $\varepsilon > 0$ and $K \subset \mathbb{Y}$ bounded, there exists $\lambda > 0$ such that, $x, y \in K$ and $\|x-y\| < \lambda$ imply that $\|f(t,x) - f(t,y)\| < \varepsilon$ for all $t \in \mathbb{R}$.
- (H$_2$) For each bounded subset $J \subset \mathbb{Y}$, $\{f(\cdot,u) : u \in J\}$ is bounded in $BS^p(\mathbb{X})$.
- (H_3) $f_{aa}(t,\cdot)$ is uniformly continuous in any bounded subset $\mathbb{K} \subset \mathbb{Y}$ uniformly for $t \in \mathbb{R}$, where $f \in PAA(\mathbb{R} \times \mathbb{X})$ and f_{aa} is the almost automorphic components of f.
- (H$_4$) The function $f \in PAA^p(\mathbb{R} \times \mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ for $p > 1$.
- (H$_5$) Let $\{u_n\} \subset PAA(\mathbb{X})$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $f(\cdot, u_n(\cdot))$ is relatively compact in $BS^p(\mathbb{X})$.
- (H$_6$) There exists $L > 0$ such that

$$S_L := \sup_{t\in\mathbb{R}, \|u\|\le L} (\int_t^{t+1} \|f(\sigma,u)\|^p d\sigma)^{\frac{1}{p}} \le \frac{L}{M(\frac{e^{\delta q}-1}{q\delta})^{\frac{1}{q}} \sum_{n=1}^{\infty} e^{-\delta n}}.$$

The following lemma will be used in the proof of the composition theorem.

Lemma 3.1. [8] *Suppose $f \in BS^p(\mathbb{X})$. Then, $f^b \in PAP_0(L^p((0,1),\mathbb{X}))$ if and only if for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\varepsilon}(f)) = 0,$$

where $mes(\cdot)$ *denote the Lebegue measure and* $M_{T,\varepsilon}(f) = \{t \in [-T,T] : (\int_t^{t+1} \|f(\sigma)\|^p d\sigma)^{\frac{1}{p}} \geq \varepsilon\}$.

Theorem 3.1. *Let* $f : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ *be a* S^p*-pseudo-almost automorphic function. Assume that* (H_1)*-*(H_3) *hold. If* $\phi \in PAA(\mathbb{Y})$*, then* $f(\cdot, \phi(\cdot)) \in PAA^p(\mathbb{X})$.

Proof. Let $f = f_{aa}+h$, where $f_{aa}^b \in AA(\mathbb{R}\times L^p((0,1),\mathbb{X}))$ and $h^b \in AA_0(\mathbb{R}\times L^p((0,1),\mathbb{X}))$. Similarly, $\phi = h_1 + h_2$, where $h_1 \in AA(\mathbb{Y})$ and $h_2 \in PAP_0(\mathbb{Y})$. Let $f(t,\phi(t)) = f_{aa}(t,h_1(t)) + f(t,\phi(t)) - f(t,h_1(t)) + h(t,h_1(t))$.

Set

$$\Lambda_1(t) = f_{aa}(t,h_1(t)), \Lambda_2(t) = f(t,\phi(t)) - f(t,h_1(t)), \Lambda_3(t) = h(t,h_1(t)).$$

Now we show that $\Lambda_1^b \in AA(L^p((0,1),\mathbb{X}))$, $\Lambda_2^b \in PAP_0(L^p((0,1),\mathbb{X}))$ and $\Lambda_3^b \in PAP_0(L^p((0,1),\mathbb{X}))$.

We first show that $\Lambda_1^b \in AA(L^p((0,1),\mathbb{X}))$. Suppose that $\{s_n\}$ is a sequence of real numbers. Then we can extract a subsequence $\{\tau_n\}$ of $\{s_n\}$ such that

(i) $(\int_t^{t+1} \|f_{aa}(s+\tau_n,x) - g(s,x)\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty,$

(ii) $(\int_t^{t+1} \|g(s-\tau_n,x) - f_{aa}(s,x)\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty,$

(iii) $\lim_{n\to\infty} h_1(t+\tau_n) = \psi(t)$, for each $t \in \mathbb{R}$,

(iv) $\lim_{n\to\infty} \psi(t-\tau_n) = h_1(t)$, for each $t \in \mathbb{R}$.

If we define $G(t) : \mathbb{R} \to \mathbb{X}$ as $G(t) = g(t,\psi(t))$. Hence

$$\|\Lambda_1(t+\tau_n)-G(t)\| \leq \|f_{aa}(t+\tau_n,h_1(t+\tau_n))-f_{aa}(t+\tau_n,\psi(t))\|+\|f_{aa}(t+\tau_n,\psi(t))-g(t,\psi(t))\|.$$

Since $h_1(t)$ is almost automorphic, $h_1(t)$ and $\psi(t)$ are bounded. So we can choose a bounded subset $K \subset \mathbb{Y}$ such that $h_1(t) \subset K, \psi(t) \subset K$ for all $t \in \mathbb{R}$. Therefore, (iii) and (H_3) yield that

$$(\int_t^{t+1} \|f_{aa}(s+\tau_n,h_1(s+\tau_n)) - f_{aa}(s+\tau_n,\psi(s))\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty,$$

Moreover, from (i), it follows that

$$(\int_t^{t+1} \|f_{aa}(s+\tau_n,\psi(s)) - g(s,\psi(s))\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, we deduce that

$$(\int_t^{t+1} \|\Lambda_1(s+\tau_n) - G(s)\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

Using the same argument, we can prove that $(\int_t^{t+1} \|G(s-\tau_n)-\Lambda_1(s)\|^p ds)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty$. This proves that $\Lambda_1^b \in AA(L^p((0,1),\mathbb{X}))$.

We next show that $\Lambda_2^b \in PAP_0(L^p((0,1),\mathbb{X}))$. Let $J \subset \mathbb{Y}$ be bounded such that $\phi(\mathbb{R}), h_1(\mathbb{R}) \subset J$. It is easy to get from (H_2) that $\Lambda_2 \in BS^p(\mathbb{X})$. By (H_1), for $\varepsilon > 0$, there exists $\lambda > 0$ such that $u, v \in J$ and $\|u - v\| < \lambda$ imply that $\|f(t,u) - f(t,v)\| < \varepsilon$ for all $t \in \mathbb{R}$, and then

$$(\int_t^{t+1} \|f(\sigma,u) - f(\sigma,v)\|^p d\sigma)^{\frac{1}{p}} < \varepsilon, \quad t \in \mathbb{R}.$$

Hence, for each $t \in \mathbb{R}$, $\|h_2(\sigma)\| < \lambda, \sigma \in [t, t+1]$ implies that

$$(\int_t^{t+1} \|\Lambda_2(\sigma)\|^p d\sigma)^{\frac{1}{p}} = (\int_t^{t+1} \|f(\sigma,\phi(\sigma)) - f(\sigma,h_1(\sigma))\|^p d\sigma)^{\frac{1}{p}} < \varepsilon.$$

Let $M(T,\lambda,h_2) = \{t \in [-T,T] : \|h_2(\sigma)\| \geq \lambda\}$. So we get

$$M_{T,\varepsilon}(\Lambda_2) = M_{T,\varepsilon}(f(\cdot,\phi(\cdot)) - f(\cdot,h_1(\cdot))) \subset M(T,\lambda,h_2).$$

Since $h_2 \in PAP_0(\mathbb{Y})$, by an argument similar to the proof of Lemma 1.1 in [17], we can obtain that $\lim_{T\to\infty} \frac{1}{2T} mes(M(T,\lambda,h_2)) = 0$. Thus $\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\varepsilon}(\Lambda_2)) = 0$.

This implies that $\Lambda_2^b \in PAP_0(L^p((0,1),\mathbb{X}))$ by Lemma 3.1.

The left task is to prove $\Lambda_3^b \in PAP_0(L^p((0,1),\mathbb{X}))$. Since $h_1 \in AA(\mathbb{Y})$ and $f_{aa}^b \in AA(\mathbb{R} \times L^p((0,1),\mathbb{X}))$, $\overline{h_1(\mathbb{R})}$ is compact and f_{aa}^b is uniformly continuous in $\mathbb{R} \times \overline{h_1(\mathbb{R})}$. By an argument similar to the proof of Lemma 1.1 in [8], we can obtain that $\Lambda_3^b \in PAP_0(L^p((0,1),\mathbb{X}))$. The proof is complete.

Remark 3.1. (i) Theorem 3.1 improves the composition theorem given in [13] since $PAA(\mathbb{X}) \subset PAA^p(\mathbb{X})$ for $p > 1$(see [3], Theorem 3.2).

(ii) Theorem 3.1 improves Theorem 3.5 in [3]. In fact, the following Lipschitz condition for $F \in PAA^p(\mathbb{R} \times \mathbb{Y})$, is necessary for the composition theorem in [3]:

$$\|F(t,u) - F(t,v)\| \leq L\|u - v\| \quad for \quad all \quad u, v \in \mathbb{Y}, t \in \mathbb{R}. \tag{3.1}$$

It is easy to verify that (3.1) implies the conditions (H_1-H_3). On the other hand, Example 4.1 at the end of this paper shows that there are functions satisfying the conditions (H_1-H_3) but not satisfying (3.1).

4. Semilinear differential equations

Consider the semilinear differential equations

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), t \in \mathbb{R}, \tag{1}$$

where $f \in PAA^p(\mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

The following Lemma will be used in the proof of our main results in this section

Lemma 4.1. [3] Assume that the C_0-semigroup $(T(t))_{t\geq 0}$ associated with A is exponentially stable. If (H_4) hold, then equation

$$\frac{du(t)}{dt} = Au(t) + f(s), t \in \mathbb{R}, \tag{2}$$

has a unique mild solution $u \in PAA(\mathbb{X})$ given by

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds, t \in \mathbb{R}. \tag{3}$$

Theorem 4.1. *Assume that the C_0-semigroup $(T(t))_{t\geq 0}$ associated with A is exponentially stable. If* (H_1)-(H_6) *hold. Then equation (1) has a mild solution $u \in PAA(\mathbb{X})$ such that $\|u\| = \sup_{t\in\mathbb{R}} \|u(t)\| \leq L$.*

Proof. It is easy to see that each mild solution u to (2) is given by

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s, u(s))ds, t \in \mathbb{R}.$$

Let $\mathbb{B} = \{u \in PAA(\mathbb{X}) : \|u\| \leq L\}$. Clearly B is closed convex. Now consider the nonlinear operator on $BC(\mathbb{R}, \mathbb{X})$ defined by

$$(Vy)(t) = \int_{-\infty}^{t} T(t-s)f(s, y(s))ds, t \in \mathbb{R}.$$

We only need to prove that the existence of fixed points of V in B, and this can be approached by Schauder's fixed point theorem. By assumption (H_1), it is easy to verify that V is continuous. In fact, let $\{u_n\} \subset BC(\mathbb{R}, \mathbb{X}), u_n \to u$ in $BC(\mathbb{R}, \mathbb{X})$, as $n \to \infty$.

We may find a bounded subset $K \subset \mathbb{X}$ such that $u_n(t), u(t) \in K$ for $t \in \mathbb{R}, n = 1, 2, \cdots$. By assumption ($\mathrm{H}_1$), given $\varepsilon > 0$ there exists $\lambda > 0$ such that $x, u \in K$ and $\|x - u\| < \lambda$ imply that

$$\|f(t,x) - f(t,u)\| < \frac{\delta\varepsilon}{M} \quad \text{for all} \quad t \in \mathbb{R}.$$

. For the above $\lambda > 0$, there exists N such that $\|u_n(t) - u(t)\| < \lambda$ for $n > N$ and $t \in \mathbb{R}$. Then $\|f(t, u_n(t)) - f(t, u(t))\| < \frac{\delta\varepsilon}{M}$ for $n > N$, and $t \in \mathbb{R}$. Hence, for $n > N$, and $t \in \mathbb{R}$, we have

$$\begin{aligned}
& \|(Vu_n)(t) - (Vu)(t)\| \\
= & \|\int_{-\infty}^{t} T(t-s)[f(s, u_n(s)) - f(s, u(s))]ds\| \\
< & \int_{-\infty}^{t} Me^{-\delta(t-s)} \frac{\delta\varepsilon}{M} ds \\
< & \varepsilon.
\end{aligned}$$

First, for $u \in \mathbb{B}$ and $t \in \mathbb{R}$, by (H_6), we have

$$\begin{aligned}
\|(Vu)(t)\| &= \|\int_{-\infty}^{t} T(t-s) f(s, u(s)) ds\| \\
&\leq \sum_{n=1}^{\infty} \|\int_{t-n}^{t-n+1} T(t-s) f(s, u(s)) ds\| \\
&\leq \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} Me^{-\delta(t-s)} \|f(s, u(s))\| ds \\
&\leq \sum_{n=1}^{\infty} M(\int_{t-n}^{t-n+1} e^{-q\delta(t-s)} ds)^{\frac{1}{q}} (\int_{t-n}^{t-n+1} \|f(s, u(s)\|^p ds)^{\frac{1}{p}} \\
&\leq \sum_{n=1}^{\infty} M(\frac{e^{\delta q} - 1}{q\delta})^{\frac{1}{q}} e^{-\delta n} S_L \\
&\leq L,
\end{aligned}$$

which shows that $\|Vu\| \leq L$ for $u \in \mathbb{B}$.

Next, let $u(\cdot) \in PAA(\mathbb{X})$. From Theorem 3.1 and ($\mathrm{H}_4$), it follows that $f(\cdot, u(\cdot)) \in PAA^p(\mathbb{X})$. It is easy to check that $f(\cdot, u(\cdot)) \in C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. Applying Lemma 4.1, it is follows that the operator V maps $PAA(\mathbb{X})$ to $PAA(\mathbb{X})$, Therefore, we have $V(\mathbb{B}) \subset \mathbb{B}$. Now we show that the following statements are true.

(i) $\{(Vu)(t) : u \in \mathbb{B}\}$ is relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$.

(ii) $\{(Vu)(t) : u \in \mathbb{B}\} \subset PAA(\mathbb{X})$ is equicontinuous.

To prove (i), let $u(\cdot) \in PAA(\mathbb{X})$, it is easy to deduce that $f(\cdot, u(\cdot)) \in PAA^p(\mathbb{X})$. Given $0 < \varepsilon < 1$, let

$$\begin{aligned}(V_\varepsilon u)(t) &= \int_{-\infty}^{t-\varepsilon^q} T(t-s) f(s,u(s))ds \\ &= T(\varepsilon^q)\int_{-\infty}^{t-\varepsilon^q} T(t-\varepsilon^q-s) f(s,u(s))ds \\ &= T(\varepsilon^q)((Vu)(t-\varepsilon^q)),\end{aligned}$$

which implies $\{(V_\varepsilon u)(t) : u \in \mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$ since $T(\varepsilon^q)$ is compact. By(H_6), for each $u \in \mathbb{B}$ we have

$$\begin{aligned}\|(Vu)(t) - (V_\varepsilon u)(t)\| &= \|\int_{t-\varepsilon^q}^{t} T(t-s) f(s,u(s))ds\| \\ &\leq M\int_{t-\varepsilon^q}^{t} e^{-\delta(t-s)})\|f(s,u(s))\|ds \\ &\leq M(\int_{t-\varepsilon^q}^{t} e^{-\delta q(t-s)})ds)^{\frac{1}{q}}(\int_{t-\varepsilon^q}^{t} \|f(s,u(s))\|^p ds)^{\frac{1}{p}} \\ &\leq M(\varepsilon^q)^{\frac{1}{q}} S_L = MS_L\varepsilon,\end{aligned}$$

from which it follows that $\{(Vu)(t) : u \in \mathbb{B}\}$ is relatively compact subset of $\mathbb{X}$ for each $t \in \mathbb{R}$.

To prove (ii), suppose that $u \in \mathbb{B}, -\infty < t_1 < t_2 < +\infty$ and $0 < \varepsilon < 1$ such that

$$\rho \triangleq (\frac{\varepsilon}{6MS_L}) \leq 1.$$

Let $(V_1u)(t_2) - (V_1u)(t_1) = I_1 + I_2 + I_3$, where

$$\begin{aligned}I_1 &= \int_{-\infty}^{t_1-\rho} (T(t_2-s) - T(t_1-s)) f(s,u(s))ds, \\ I_2 &= \int_{t_1-\rho}^{t_1} (T(t_2-s) - T(t_1-s)) f(s,u(s))ds, \\ I_3 &= \int_{t_1}^{t_2} T(t_2-s) f(s,u(s))ds.\end{aligned}$$

Since $(T(t))_{t\geq 0}$ is a C_0-semigroup and $T(t)$ is compact for $t>0$, there exists $\lambda=\lambda(\varepsilon,\rho)<\rho$ such that $t_2-t_1<\lambda$ implies that

$$\|T(\frac{t}{2})-T(\frac{t}{2}+t_2-t_1)\|\leq\frac{\varepsilon}{\eta}\quad\text{for}\quad t>0,$$

where

$$\eta=3MS_L(\frac{2(e^{\frac{\delta q}{2}}-1)}{\delta q})^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{\frac{-\delta(n+\rho)}{2}}.$$

Here

$$\begin{aligned}I_1&=\int_{\rho}^{\infty}(T(t_2-t_1+t)-T(t))f(t_1-t,u(t_1-t))dt,\\&=\int_{\rho}^{\infty}(T(t_2-t_1+\frac{t}{2})-T(\frac{t}{2}))T(\frac{t}{2})f(t_1-t,u(t_1-t))dt.\end{aligned}$$

By (H_6) we then have

$$\begin{aligned}\|I_1\|&\leq\frac{\varepsilon}{\eta}M\int_{\rho}^{\infty}e^{\frac{-\delta t}{2}}\|f(t_1-t,u(t_1-t))\|dt\\&=\frac{\varepsilon}{\eta}M\sum_{n=1}^{\infty}\int_{n-1+\rho}^{n+\rho}e^{\frac{-\delta t}{2}}\|f(t_1-t,u(t_1-t))\|dt\\&\leq\frac{\varepsilon}{\eta}M\sum_{n=1}^{\infty}(\int_{n-1+\rho}^{n+\rho}e^{\frac{-\delta tq}{2}}dt)^{\frac{1}{q}}(\int_{n-1+\rho}^{n+\rho}\|f(t_1-t,u(t_1-t))\|^p dt)^{\frac{1}{p}}\\&\leq\frac{\varepsilon}{\eta}MS_L\sum_{n=1}^{\infty}(\int_{n-1+\rho}^{n+\rho}e^{\frac{-\delta tq}{2}}dt)^{\frac{1}{q}}\\&=\frac{\varepsilon}{\eta}MS_L(\frac{2(e^{\frac{q\delta}{2}}-1)}{q\delta})^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{\frac{-\delta(n+\rho)}{2}}=\frac{\varepsilon}{3}\end{aligned}$$

and

$$\begin{aligned}\|I_2\|&\leq\int_{t_1-\rho}^{t_1}(Me^{-\delta(t_2-s)}+Me^{-\delta(t_1-s)})\|f(s,u(s))\|ds\\&\leq M(\int_{t_1-\rho}^{t_1}(e^{-\delta(t_2-s)}+e^{-\delta(t_1-s)}))^q ds)^{\frac{1}{q}}(\int_{t_1-\rho}^{t_1}\|f(s,u(s))\|^p ds)^{\frac{1}{p}}\\&\leq 2M\rho^{\frac{1}{q}}S_L=2M((\frac{\varepsilon}{6MS_L})^q)^{\frac{1}{q}}S_L=\frac{\varepsilon}{3}\end{aligned}$$

and

$$\begin{aligned}\|I_3\| &\leq \int_{t_1}^{t_2} Me^{-\delta(t_2-s)}\|f(s,u(s))\|ds \\ &\leq M(\int_{t_1}^{t_2} e^{-\delta(t_2-s)q}ds)^{\frac{1}{q}}(\int_{t_1}^{t_2} \|f(s,u(s))\|^p ds)^{\frac{1}{p}} \\ &\leq M\lambda^{\frac{1}{q}}S_L < M\rho^{\frac{1}{q}}S_L = \frac{\varepsilon}{6}.\end{aligned}$$

Hence, for $u \in \mathbb{B}$ and $t_2 - t_1 < \lambda$, we get

$$\|(Vu)(t_2) - (Vu)(t_1)\| \leq \|I_1\| + \|I_2\| + \|I_3\| < \varepsilon.$$

This implies that the statement (ii) is true. Now using (H_5), by argument the same as the proof of ([17], Theorem 3.1), we can prove that V has a fixed point in $\overline{co}V(\mathbb{B})$ (here we omit the details). That is (1) has a mild pseudo almost antomorphic solution $u \in \mathbb{B}$. The Proof is complete.

Theorem 4.2. *Let $u(t)$ be a mild pseudo almost antomorphic solution of (1), and suppose that*

$$f(t,u(t)) \in D(A) \quad for \quad t \in \mathbb{R}, Af(\cdot,u(\cdot)) \in L^1(\mathbb{R},\mathbb{X})). \tag{4}$$

Then $u(t)$ is a pseudo almost antomorphic solution of (1).

Proof. Since $u(t)$ be a mild pseudo almost antomorphic solution of (1),

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s,u(s))ds.$$

By condition (4),

$$\begin{aligned}\tfrac{d_+u(t)}{dt} &= lim_{h\to 0_+}\frac{1}{h}(u(t+h)-u(t)) \\ &= lim_{h\to 0_+}[\int_{-\infty}^{t}\frac{1}{h}(T(t+h-s)-T(t-s))f(s,u(s))ds \\ &+\frac{1}{h}[\int_{t}^{t+h}T(t+h-s)f(s,u(s))ds] \\ &= \int_{-\infty}^{t}\frac{1}{h}T(t-s)f(s,u(s))ds + f(t,u(t)).\end{aligned}$$

This shows that $\frac{d_+ u(t)}{dt}$ is continuous, and then $u(t)$ is continuously differentiable in $t \in \mathbb{R}$. Noticing that A is closed,

$$u'(t) = Au(t) + f(t, u(t)).$$

That is, $u(t)$ is a pseudo almost antomorphic solution of (1).

To conclude this paper, we give an example involving the heat equation

Example 4.1. Consider the following heat equation given by the system

$$\begin{cases} \frac{\partial u}{\partial t} u(t,x) = \frac{\partial^2 u}{\partial^2 x} u(t,x) + x sin \frac{1}{cos^2 t + cos^2 \pi t} + \phi(t) sin x, \\ u(t,0) = u(t,1) = 0, t \in \mathbb{R} \end{cases} \tag{4.1}$$

Let $\mathbb{X} = L^2(0,1)$*, and define* $A : \mathbb{X} \to \mathbb{X}$ *by* $Au(\cdot) = u''(\cdot)$ *with domain* $D(A) := \{u(\cdot) \in \mathbb{X} : u'' \in \mathbb{X}, u(t,0) = u(t,1) = 0\}$.

It is well known that A *is the infinitesimal generator of compact* C_0*-semigroup* $T(t)_{t\geq 0}$ *satisfying* $\|T(t)\| \leq e^{-t}, t \geq 0$ *for* $t \geq 0$.

Write

$$f(t, u(t)) = u(t) sin \frac{1}{cos^2 t + cos^2 \pi t} + \phi(t) sin u(t),$$
$$\phi(t) = \max_{k\in\mathbb{Z}} \{e^{-(t \pm k^2)^2}\}.$$

for $(t, u(t)) \in \mathbb{R} \times \mathbb{X}$.

It is not hard to verify that $f \in PAA^p(\mathbb{R} \times \mathbb{X})$ *and* f *satisfying* (H_1)-(H_6) *with* $L = 2, M = 1, \delta = 1$. *Then, by Theorem 4.2, the above heat equation has a pseudo-almost automorphic mild solution* $u \in PAA(\mathbb{X})$ *such that* $\|u\| = \sup_{t\in\mathbb{R}} \|u(t)\| \leq 2$.

However, it is obvious that f *is not Lipschitz continuous. As a result, (*[3]*, Theorem 4.2)is not applicable.*

References

[1] T. Diagana, G.M Mahop, G.M. N'Guérékata, B. Toni, Existence and uniqueness of pseudo almost periodic solutions to some classes of semilinear differential equations and applications, Nonlinear Anal. TMA 64 (2006), 2442-2453.

[2] T. Diagana, G.M. N'Guérékata, Pseudo almost periodic mild solutions to hyperbolic evolution equations in abstract intermediate Banach spaces, Appl. Anal. 85 (6) (2006), 769-780.

[3] T. Diagana, Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients, Nonlinear Anal. 70(11) (2009), 3781-3790.

[4] T. Diagana, G.M Mahop, G.M. N'Guérékata, Existence of weighted pseudo-almost periodic solutions to some classes of differential equations with S^p-weighted pseudo-almost periodic coefficients, Appl. Anal. 72 (1) (2010), 430-438.

[5] J. Hong, R. Obaya, A. Sanz, Almost periodic type solutions of some of differential equations with piecewise constant argument, Nonlinear Anal. 45 (2001), 661-688.

[6] Z.R. Hu, Z. Jin, Stepanov-like pseudo almost automorphic mild solutions to nonautonomous evolution equations, Nonlinear Anal. 71 (2009), 2349-2360.

[7] Z.R. Hu, Z. Jin, Stepanov-like pseudo-almost periodic mild solutions to perturbed nonautonomous evolution equations with infinite delay, Nonlinear Anal. 71 (2009), 5381-5391.

[8] H.X. Li, L.L. Zhang, Stepanov-like pseudo-almost periodicity and semilinear differential equations with uniform continuity, Results. Math. 59 (2011), 43-61.

[9] T.J. Xiao, J. Liang, J. Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach space, Semigrop Forum 76 (2008), 518-524.

[10] T.J. Xiao, X.X. Zhu, J. Liang, Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications, Nonlinear Anal. 70 (2009), 4079-4085.

[11] C. Zhang, Pseudo almost-periodic solutions of some differential equations, J. Math. Anal. Appl. 181 (1994), 62-76.

[12] L.L. Zhang, H.X. Li, Weighted pseudo almost periodic solutions of second order neutral differential equations with piecewise constant argument, Nonlinear Anal. 74 (2011), 6770-6780.

[13] J. Liang, J. Zhang, T.J. Xiao, Composition of pseudo almost automorphic and asymptotically almost automorphic functions, J. Math. Anal. Appl. 340 (2008), 1493-1499.

[14] T. Diagana, Existence of ρ-almost automorphic mild solution to some abstract differential equations, Int. J. Evol.Equ. 1(1) (2005), 57-67.

[15] A. Pankov, Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equation, Kluwer. Dordrecht, 1990.

[16] G.M. N'Guérékata, A. Pankov. Stepanov-like almost automorphic functions and monotone evotion equations, Nonlinear Anal. 68(9) (2008), 2658-2667.

[17] H.X. Li, F.L. Huang, J.Y. Li, Composition of pseudo almost-periodic functions and semilinear differential equations, J. Math. Anal. Appl. 255 (2001), 436-446.