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FIXED POINT THEOREMS IN TWO COMPLETE INTUITIONISTIC GENERALIZED FUZZY METRIC SPACES

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Abstract: Fisher [4] proved a related fixed point theorem in two metric spaces. We generalized new results concerning the related fixed point theorems on two complete intuitionistic generalized fuzzy metric spaces are proved and deduced some corollaries.

Keywords: fixed point; G- metric space; G - fuzzy metric space; complete.

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1. INTRODUCTION

Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets. Park [11] using the idea of intuitionistic fuzzy sets defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t- norms and continuous t-conorms. George and Veeramani [5] showed

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that every metric induces fuzzy metric, every fuzzy metric induces an intuitionistic fuzzy metric. Fisher [4] proved a related fixed point theorem in two metric spaces.

Mustafa and Sims introduced [9] more appropriate notion of generalized metric space which called G- metric spaces, and obtained some topological properties. Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh [14], Mustafa, Shatanawi and Bataineh [8], Mustafa and Sims [9], Obtained some fixed point theorems for a single map in G- metric spaces. Then, Rao, Lakshmi, and Mustafa [12] obtained a unique common fixed point theorem for a six weakly compatible mappings in G-metric spaces and obtained some theorems of as corollaries also. Sun and Yang [5] introduced the concept of G-fuzzy metric spaces and proved two common fixed point theorems for four mappings. In this paper, new results concerning the related fixed point theorems on two complete intuitionistic generalized fuzzy metric spaces are proved and deduced some corollaries.

2. PRELIMINARIES

Definition: 2.1.

A 5-tuple (X, G, H, *, ◊) is said to be a Intuitionistic Generalized Fuzzy Metric Space (briefly IGFMS), if X is an arbitrary non-empty set, * is a continuous t-norm, ◊ is a continuous t- conorm, G and H are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions: for every x, y, z, a $\in$ X and t, s > 0

(i) $G(x, y, z, t) + H(x, y, z, t) \leq 1$,

(ii) $G(x, x, y, t) > 0$ for $x \neq y$,

(iii) $G(x, x, y, t) \geq G(x, y, z, t)$ for $y \neq z$,

(iv) $G(x, y, z, t) = 1$ if and only if $x = y = z$,

(v) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,

(vi) $G(x, a, a, t) * G(a, y, z, s) \leq G(x, y, z, t + s)$,

(vii) $G(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,

(viii) G is a non-decreasing of R^+, $\lim_{t\to\infty} G(x, y, z, t) = 1$,

$\lim_{t\to 0} G(x, y, z, t) = 0$ for all x, y, z $\in$X and t > 0,

(ix) $H(x, x, y, t) < 1$ for $x \neq y$,

(x) $H(x, x, y, t) \leq H(x, y, z, t)$ for $y \neq z$,

(xi) $H(x, y, z, t) = 0$ if and only if $x = y = z$,

(xii) H(x, y, z, t) = H(p(x, y, z), t) , where p is a permutation function,

(xiii) H(x, a, a, t) ◊ H(a, y, z, s) ≥ H(x, y, z, t + s),

(xiv) H(x, y, z, .) : $(0, \infty) \to [0, 1]$ is continuous,

(xv) G is a non-increasing of R^+, $\lim_{t\to\infty}$ H(x, y, z, t) = 0,

$\lim_{t\to 0}$ H(x, y, z, t) = 1, for all x, y, z ∈ X and t > 0.

In this case, the pair (G, H) is called an intuitionistic generalized fuzzy metric on X.

Definition: 2.2.

Let (X, G, H, *, ◊) be an intuitionistic generalized fuzzy metric space, then

(i) A sequence $\{x_n\}$ in X is said to be convergent to x if

$\lim_{n\to\infty} G(x_n, x_n, x, t) = 1$ and $\lim_{n\to\infty} H(x_n, x_n, x, t) = 0$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if

$\lim_{n,m\to\infty} G(x_n, x_n, x_m, t) = 1$ and $\lim_{n,m\to\infty} H(x_n, x_n, x_m, t) = 0$ that is, for any $\varepsilon > 0$

and for each t > 0, there exists $n_0 \in N$ such that $G(x_n, x_n, x_m, t) > 1 - \varepsilon$

and $H(x_n, x_n, x_m, t) < \varepsilon$ for n, m $\geq n_0$.

(iii) A intuitionistic generalized fuzzy metric space (X, G, H, *, ◊) is said to be complete if every Cauchy sequence in X is convergent.

3. MAIN RESULTS

In this section we prove the fixed point theorems in two complete intuitionistic generalized fuzzy metric spaces

Our main result follows:

Let Ψ be the set of all continuous real functions : $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that :

(i) $\psi(1, 1, 1) = 0$.

(ii) If $u^2 \geq \psi(uv, 1, 1)$ or $u^2 \geq \psi(1, 1, uv)$ or $u^2 \geq \psi(1, uv, 1)$, for all u, v $\in \mathbb{R}^+$,

then there exists $0 \leq k < 1$ such that $u \geq \frac{1}{4} kv$.

Let $\emptyset$ be the set of all continuous real functions φ : $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that :

(iii) $\varphi(0, 0, 0) = 0$.

(iv) If $u^2 \leq \varphi(uv, 0, 0)$ or $u^2 \leq \varphi(0, 0, uv)$ or $u^2 \leq \varphi(0, uv, 0)$, for all u, v $\in \mathbb{R}^+$, then there exists $0 \leq \lambda < 1$ such that $u \leq \frac{1}{4}\lambda v$.

Theorem: 3.1.

Let (X, G_1, H_1, $*$, $\Diamond$) and (Y, G_2, H_2, $*$, $\Diamond$) be two complete intuitionistic generalized fuzzy metric spaces and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities

$$G_2^2(Tx, TSy_1, TSy_2, t) \geq \psi(G_2(y_1, TSy_1, TSy_2, t)G_2(y_1, y_2, Tx, t), G_2(y_1, y_2, Tx, t)G_1(x, Sy_1, Sy_2, t), G_1(x, Sy_1, Sy_2, t)G_2(y_1, TSy_1, TSy_2, t)) \quad (3.1.1)$$

$$H_2^2(Tx, TSy_1, TSy_2, t) \leq \varphi(H_2(y_1, TSy_1, TSy_2, t)H_2(y_1, y_2, Tx, t), H_2(y_1, y_2, Tx, t)H_1(x, Sy_1, Sy_2, t), H_1(x, Sy_1, Sy_2, t)H_2(y_1, TSy_1, TSy_2, t)) \quad (3.1.2)$$

$$G_1^2(Sy_1, Sy_2, STx, t) \geq \psi(G_1(x, x, STx, t)G_1(x, Sy_1, Sy_2, t), G_1(x, Sy_1, Sy_2, t)G_2(y_1, y_2, Tx, t), G_2(y_1, y_2, Tx, t)G_1(x, x, STx, t)) \quad (3.1.3)$$

$$H_1^2(Sy_1, Sy_2, STx, t) \leq \varphi(H_1(x, x, STx, t)H_1(x, Sy_1, Sy_2, t), H_1(x, Sy_1, Sy_2, t)H_2(y_1, y_2, Tx, t), H_2(y_1, y_2, Tx, t)H_1(x, x, STx, t)) \quad (3.1.4)$$

for all x in X and y_1, y_2 in Y where $\psi \in \Psi$ and $\varphi \in \emptyset$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, Tz = w and Sw= z.

Proof:

We define the sequences (x_n) in X and (y_n) in Y by $x_n = (ST)^n x$, $y_n = T(ST)^{n-1}x$, for n = 1, 2.... We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n. Applying the inequalities (3.1.1), (3.1.2), using the properties (ii) and (iv), we have

$$\begin{aligned} G_2^2(y_n, y_{n+1}, y_{n+1}, t) &= G_2^2(Tx_{n-1}, TSy_n, TSy_n, t) \\ &\geq \psi(G_2(y_n, TSy_n, TSy_n, t)G_2(y_n, y_n, Tx_{n-1}, t), \\ &\qquad G_2(y_n, y_n, Tx_{n-1}, t)G_1(x_{n-1}, Sy_n, Sy_n, t), \\ &\qquad G_1(x_{n-1}, Sy_n, Sy_n, t)G_2(y_n, TSy_n, TSy_n, t)) \\ &\geq \psi(1, 1, G_1(x_{n-1}, Sy_n, Sy_n, t)G_2(y_n, TSy_n, TSy_n, t) \\ &\geq \psi(1, 1, G_1(x_{n-1}, x_n, x_n, t)G_2(y_n, y_{n+1}, y_{n+1}, t)), \end{aligned}$$

$$H_2^2(y_n, y_{n+1}, y_{n+1}, t) = H_2^2(Tx_{n-1}, TSy_n, TSy_n, t)$$

$$\begin{aligned}
&\le \varphi(H_2(y_n, TSy_n, TSy_n, t)H_2(y_n, y_n, Tx_{n-1}, t),\\
&\qquad H_2(y_n, y_n, Tx_{n-1}, t)H_1(x_{n-1}, Sy_n, Sy_n, t),\\
&\qquad H_1(x_{n-1}, Sy_n, Sy_n, t)H_2(y_n, TSy_n, TSy_n, t))\\
&\le \varphi(0, 0,\ H_1(x_{n-1}, Sy_n, Sy_n, t)H_2(y_n, TSy_n, TSy_n, t))\\
&\le \varphi(0,0, H_1(x_{n-1}, x_n, x_n, t)H_2(y_n, y_{n+1}, y_{n+1}, t)),
\end{aligned}$$

and it follows that

$$G_2^2(y_n, y_{n+1}, y_{n+1}, t) \ge \frac{1}{4}\ \mathrm{k}\,\{\, G_1(x_{n-1}, x_n, x_n, t)G_2(y_n, y_{n+1}, y_{n+1}, t)\}$$

$$G_2(y_n, y_{n+1}, y_{n+1}, t) \ge \frac{1}{4}\ \mathrm{k}\ \ G_1\,(x_{n-1}, x_n, x_n, t) \tag{3.1.5}$$

$$H_2^2(y_n, y_{n+1}, y_{n+1}, t) \le \frac{1}{4}\lambda\{\ H_1(x_{n-1}, x_n, x_n, t), H_2(y_n, y_{n+1}, y_{n+1}, t)\}$$

$$H_2(y_n, y_{n+1}, y_{n+1}, t) \le \frac{1}{4}\lambda(\,H_1(x_{n-1}, x_n, x_n, t)). \tag{3.1.6}$$

Similarly, applying the inequalities (3.1.3) and (3.1.4),

$$\begin{aligned}
G_1^2(x_n, x_n, x_{n+1}, t) &= \ G_1^2(Sy_n, Sy_n, STx_n, t)\\
&\ge \psi(G_1(x_n, x_n, x_{n+1}, t)G_1(x_n, Sy_n, Sy_n, t),\\
&\qquad G_1(x_n, Sy_n, Sy_n, t)G_2(y_n, y_n, Tx_n, t),\\
&\qquad G_2(y_n, y_n, Tx_n, t)G_1(x_n, x_n, x_{n+1}, t))\\
&\ge \psi(G_1(x_n, x_n, x_{n+1}, t)G_1(x_n, x_n, x_n, t),\\
&\qquad G_1(x_n, x_n, x_n, t)G_2(y_n, y_n, y_{n+1}, t),\\
&\qquad G_2(y_n, y_n, y_{n+1}, t)G_1(x_n, x_n, x_{n+1}, t))
\end{aligned}$$

$$\begin{aligned}
H_1^2(x_n, x_n, x_{n+1}, t) &= \ H_1^2(Sy_n, Sy_n, STx_n, t)\\
&\le \varphi(H_1(x_n, x_n, x_{n+1}, t)H_1(x_n, Sy_n, Sy_n, t),\\
&\qquad H_1(x_n, Sy_n, Sy_n, t)H_2(y_n, y_n, Tx_n, t),\\
&\qquad H_2(y_n, y_n, Tx_n, t)H_1(x_n, x_n, x_{n+1}, t))\\
&\le \varphi((H_1(x_n, x_n, x_{n+1}, t)H_1(x_n, x_n, x_n, t),\\
&\qquad H_1(x_n, x_n, x_n, t)H_2(y_n, y_n, y_{n+1}, t),\\
&\qquad H_2(y_n, y_n, y_{n+1}, t)H_1(x_n, x_n, x_{n+1}, t)).
\end{aligned}$$

Using properties (ii), (iv) and the definition (2.2.), we have

$$G_1^2(x_n, x_n, x_{n+1}, t) \ge \frac{1}{4}\ \mathrm{k}\,\psi(G_2(y_n, y_n, y_{n+1}, t), G_1(x_n, x_n, x_{n+1}, t))$$

$$\frac{1}{2}G_1(x_n, x_{n+1}, x_{n+1}, t) \ge G_1(x_n, x_n, x_{n+1}, t)$$

$$\geq \frac{1}{4}\,\mathrm{k}\;G_2(y_n, y_n, y_{n+1}, t) \geq \frac{1}{2}\,\mathrm{k}\;G_2(y_n, y_{n+1}, y_{n+1}, t)$$

$$G_1(x_n, x_{n+1}, x_{n+1}, t) \geq \mathrm{k}\;G_2(y_n, y_{n+1}, y_{n+1}, t) \tag{3.1.7}$$

$$H_1^2(x_n, x_n, x_{n+1}, t) \leq \frac{1}{4}\lambda\,\varphi(H_2(y_n, y_n, y_{n+1}, t), H_1(x_n, x_n, x_{n+1}, t))$$

$$\frac{1}{2}H_1(x_n, x_{n+1}, x_{n+1}, t) \leq H_1(x_n, x_n, x_{n+1}, t)$$

$$\leq \frac{1}{4}\lambda\,H_2(y_n, y_n, y_{n+1}, t)$$

$$\leq \frac{1}{2}\lambda\,H_2(y_n, y_{n+1}, y_{n+1}, t)$$

$$H_1(x_n, x_{n+1}, x_{n+1}, t) \leq \lambda\,H_2(y_n, y_{n+1}, y_{n+1}, t). \tag{3.1.8}$$

Now, it follows from the inequalities (3.1.5), (3.1.6), (3.1.7) and (3.1.8) that

$$G_1(x_n, x_{n+1}, x_{n+1}, t) \geq \frac{1}{4}\mathrm{k}^2\,G_1(x_{n-1}, x_n, x_n, t)$$

$$H_1(x_n, x_{n+1}, x_{n+1}, t) \leq \frac{1}{4}\lambda^2\,H_1(x_{n-1}, x_n, x_n, t).$$

Hence, by induction we get

$$G_1(x_n, x_{n+1}, x_{n+1}, t) \geq \left(\tfrac{1}{4}\right)^{\mathrm{n}}\mathrm{k}^{2\mathrm{n}}\,G_1(x, x_1, x_1, t) \quad \text{for n = 1,2,\ldots} \tag{3.1.9}$$

$$H_1(x_n, x_{n+1}, x_{n+1}, t) \leq \left(\tfrac{1}{4}\right)^{\mathrm{n}}\lambda^{2\mathrm{n}}\,H_1(x, x_1, x_1, t) \quad \text{for n = 1,2,\ldots} \tag{3.1.10}$$

So (x_n) and (y_n) are Cauchy sequences with limits z in X and w in Y.

Using the inequalities (3.1.1) and (3.1.2), we have

$$\begin{aligned}
G_2^2(Tz, y_n, y_n, t) = {} & G_2^2(Tz, TSy_{n-1}, TSy_{n-1}, t) \\
\geq {} & \psi(G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}, t)G_2(y_{n-1}, y_{n-1}, Tz, t), \\
& \quad G_2(y_{n-1}, y_{n-1}, Tz, t)G_1(z, Sy_{n-1}, Sy_{n-1}, t), \\
& \quad G_1(z, Sy_{n-1}, Sy_{n-1}, t)G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}, t)) \\
\geq {} & \psi(G_2(y_{n-1}, y_n, y_n, t)G_2(y_{n-1}, y_{n-1}, Tz, t), \\
& \quad G_2(y_{n-1}, y_{n-1}, Tz, t)G_1(z, x_{n-1}, x_{n-1}, t), \\
& \quad G_1(z, x_{n-1}, x_{n-1}, t)G_2(y_{n-1}, y_n, y_n, t)).
\end{aligned}$$

$$G_2^2(Tz, w, w, t) \geq \psi\,(1, 1, 1) = 1.$$

$$\begin{aligned}
H_2^2(Tz, y_n, y_n, t) = {} & H_2^2(Tz, TSy_{n-1}, TSy_{n-1}, t) \\
\leq {} & \varphi(H_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}, t)H_2(y_{n-1}, y_{n-1}, Tz, t), \\
& \quad H_2(y_{n-1}, y_{n-1}, Tz, t)H_1(z, Sy_{n-1}, Sy_{n-1}, t), \\
& \quad H_1(z, Sy_{n-1}, Sy_{n-1}, t)H_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}, t))
\end{aligned}$$

$$\leq \varphi(H_2(y_{n-1}, y_n, y_n, t)H_2(y_{n-1}, y_{n-1}, Tz, t),$$
$$H_2(y_{n-1}, y_{n-1}, Tz, t)H_1(z, x_{n-1}, x_{n-1}, t),$$
$$H_1(z, x_{n-1}, x_{n-1}, t)H_2(y_{n-1}, y_n, y_n, t)).$$
$$H_2^2(Tz, w, w, t) \leq \varphi\ (0, 0, 0) = 0.$$

It follows that $G_2(Tz, w, w, t) = 1$ and $H_2(Tz, w, w, t) = 0$, hence w = Tz.

Using the inequalities (3.1.3) and (3.1.4), we have

$$G_1^2(Sw, Sw, x_n, t) = G_1^2(Sw, Sw, STx_{n-1}, t)$$
$$\geq \psi(G_1(x_{n-1}, x_{n-1}, STx_{n-1}, t)G_1(x_{n-1}, Sw, Sw, t),$$
$$G_1(x_{n-1}, Sw, Sw, t)G_2(w, w, Tx_{n-1}, t),$$
$$G_2(w, w, Tx_{n-1}, t)G_1(x_{n-1}, x_{n-1}, STx_{n-1}, t)).$$
$$H_1^2(Sw, Sw, x_n, t) = H_1^2(Sw, Sw, STx_{n-1}, t)$$
$$\leq \varphi(H_1(x_{n-1}, x_{n-1}, STx_{n-1}, t)H_1(x_{n-1}, Sw, Sw, t),$$
$$H_1(x_{n-1}, Sw, Sw, t)H_2(w, w, Tx_{n-1}, t),$$
$$H_2(w, w, Tx_{n-1}, t)H_1(x_{n-1}, x_{n-1}, STx_{n-1}, t)).$$

Letting n tends to infinity, using (i) and (iii), we have

$G_1^2(Sw, Sw, x_n, t) \geq \psi\ (1, 1, 1) = 1$ and $H_1^2(Sw, Sw, x_n, t) \leq \varphi\ (0, 0, 0) = 0$ and it follows that z = S w. Thus STz = Sw = z, TSw = Tz = w and so ST has a fixed point z and TS has a fixed point w. To prove uniqueness, suppose that ST has a second fixed point z_1 and TS has a second fixed point w_1 . Then applying the inequalities (3.1.1), (3.1.2) and using properties (ii) and (iv), we have

$$G_2^2(w, w_1, w_1, t) = G_2^2(TSw, TSw_1, TSw_1, t) = G_2^2(Tz, TSw_1, TSw_1, t)$$
$$\geq \psi(G_2(w_1, TSw_1, TSw_1, t)G_2(w_1, w_1, Tz, t),$$
$$G_2(w_1, w_1, Tz, t)G_1(z, Sw_1, Sw_1, t),$$
$$G_1(z, Sw_1, Sw_1, t)\ \mathrm{G}_2(w_1, TSw_1, TSw_1, t))$$
$$\geq \psi(1, G_2(w_1, w_1, w, t)G_1(Sw, Sw_1, Sw_1, t), 1)).$$
$$H_2^2(w, w_1, w_1, t) = H_2^2(TSw, TSw_1, TSw_1, t) = H_2^2(Tz, TSw_1, TSw_1, t)$$
$$\leq \varphi(H_2(w_1, TSw_1, TSw_1, t)H_2(w_1, w_1, Tz, t),$$
$$H_2(w_1, w_1, Tz, t)H_1(z, Sw_1, Sw_1, t),$$
$$H_1(z, Sw_1, Sw_1, t)\ \mathrm{H}_2(w_1, TSw_1, TSw_1, t))$$
$$\leq \varphi(0, H_2(w_1, w_1, w, t)H_1(Sw, Sw_1, Sw_1, t), 0)).$$

It follows that

$$G_2^2(w,w_1,w_1,t) \geq \frac{1}{4}\, k\, \psi(\, G_1(Sw,Sw_1,Sw_1,t)\; G_2(w_1,w_1,w,t))$$

$$H_2^2(w,w_1,w_1,t) \leq \frac{1}{4}\lambda\, \varphi(H_1(Sw,Sw_1,Sw_1,t)\; H_2(w_1,w_1,w,t))$$

$$G_2(w,w_1,w_1,t) \geq \frac{1}{4}\, kG_1(Sw,Sw_1,Sw_1,t) \text{ and}$$

$$H_2(w,w_1,w_1,t) \leq \frac{1}{4}\lambda\, H_1(Sw,Sw_1,Sw_1,t). \tag{3.1.11}$$

Further, applying the inequalities (3.1.3), (3.1.4), using properties (ii) and (iv), we have

$$\begin{aligned} G_1^2(Sw,Sw,Sw_1,t) = \;& G_1^2(STSw,STSw,STSw_1,t) \\ &\geq \psi(G_1(Sw_1,Sw_1,STSw_1,t)G_1(Sw_1\,,STSw,STSw,t), \\ &\qquad G_1(Sw_1\,,STSw,STSw,t)G_2(TSw,TSw,TSw_1,t), \\ &\qquad G_2(TSw,TSw,TSw_1,t)G_1(Sw_1,Sw_1,STSw_1,t)) \\ &\geq \psi(1, G_1(Sw_1\,,Sw,Sw,t)G_2(w,w,w_1,t), 1). \end{aligned}$$

$$\begin{aligned} H_1^2(Sw,Sw,Sw_1,t) = \;& H_1^2(STSw,STSw,STSw_1,t) \\ &\leq \varphi(H_1(Sw_1,Sw_1,STSw_1,t)H_1(Sw_1\,,STSw,STSw,t), \\ &\qquad H_1(Sw_1\,,STSw,STSw,t)H_2(TSw,TSw,TSw_1,t), \\ &\qquad H_2(TSw,TSw,TSw_1,t)H_1(Sw_1,Sw_1,STSw_1,t)) \\ &\leq \varphi(0, H_1(Sw_1\,,Sw,Sw,t)H_2(w,w,w_1,t), 0)). \end{aligned}$$

Which implies that

$$G_1^2(Sw,Sw,Sw_1,t) \;\geq\; \frac{1}{4}\,\mathrm{k}\,\psi(G_2(w,w,w_1,t)\; G_1(Sw,Sw,Sw_1,t))$$

$$G_1(Sw,Sw,Sw_1,t) \;\geq\; \frac{1}{4}\,\mathrm{k}\;(G_2(w,w,w_1,t)) \tag{3.1.12}$$

$$H_1^2(Sw,Sw,Sw_1,t) \leq\; \frac{1}{4}\lambda\, \varphi(H_2(w,w,w_1,t)\; H_1(Sw,Sw,Sw_1,t))$$

$$H_1(Sw,Sw,Sw_1,t) \leq\; \frac{1}{4}\lambda\;(H_2(w,w,w_1,t)). \tag{3.1.13}$$

Again by using the definition (2.2.), we get,

$$\begin{aligned} \frac{1}{2}G_1(Sw,Sw_1,Sw_1,t) &\geq G_1(Sw,Sw,Sw_1,t) \\ &\geq\; \frac{1}{4}\,\mathrm{k}\;(G_2(w,w,w_1,t)) \geq \frac{1}{2}\,\mathrm{k}\;(G_2(w,w_1,w_1,t) \end{aligned}$$

$$G_1(Sw,Sw_1,Sw_1,t) \geq\; \mathrm{k}\;(G_2(w,w_1,w_1,t)) \tag{3.1.14}$$

$$\frac{1}{2}H_1(Sw,Sw_1,Sw_1,t) \leq H_1(Sw,Sw,Sw_1,t)$$

$$\leq \frac{1}{4}\lambda\ (H_2(w,w,w_1,t)) \quad \leq \frac{1}{2}\lambda\ (H_2(w,w_1,w_1,t))$$

$$H_1(Sw,Sw_1,Sw_1,t)\ \leq \lambda\ (H_2(w,w_1,w_1,t)). \tag{3.1.15}$$

Now it follows from the inequalities (3.1.11), (3.1.14) and (3.1.15) that

$$G_2(w,w_1,w_1,t) \geq \frac{1}{4}\mathrm{k}\ (G_1(Sw,Sw_1,Sw_1,t))$$

$$> \frac{1}{4}\mathrm{k}^2(G_2(w,w_1,w_1,t)) > G_2(w,w_1,w_1,t)$$

$$H_2(w,w_1,w_1,t) \leq \frac{1}{4}\mathrm{k}\ (H_1(Sw,Sw_1,Sw_1,t))$$

$$< \frac{1}{4}\lambda^2\ (H_2(w,w_1,w_1,t))\ < H_2(w,w_1,w_1,t)$$

and so w = w_1. Since k, λ < 1. The fixed point w of TS must be a unique.

Now TSz_1 = z_1 implies TSTz_1 = Tz_1 and so T z_1 = w.

Thus z = STz = Sw = STz_1= z_1, proving that z is a unique fixed point of ST.

Thus the proof of the theorem is completes.

Corollary 3.2.

Let (X, $G_1, H_1, *$, ◊) and (Y, $G_2, H_2, *$, ◊) be two complete intuitionistic generalized fuzzy metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:

$$G_2^2(Tx,TSy_1,TSy_2,t) \geq \frac{1}{4}\mathrm{k}\ min(G_2(y_1,TSy_1,TSy_2,t)G_2(y_1,y_2,Tx,t),$$
$$G_2(y_1,y_2,Tx,t)G_1(x,Sy_1,Sy_2,t),$$
$$G_1(x,Sy_1,Sy_2,t)G_2(y_1,TSy_1,TSy_2,t))$$

$$H_2^2(Tx,TSy_1,TSy_2,t) \leq \frac{1}{4}\lambda\,\mathrm{max}\ (H_2(y_1,TSy_1,TSy_2,t)H_2(y_1,y_2,Tx,t),$$
$$H_2(y_1,y_2,Tx,t)H_1(x,Sy_1,Sy_2,t),$$
$$H_1(x,Sy_1,Sy_2,t)H_2(y_1,TSy_1,TSy_2,t))$$

$$G_1^2(Sy_1,Sy_2,STx,t) \geq \frac{1}{4}k\ min(G_1(x,x,STx,t)G_1(x,Sy_1,Sy_2,t),$$
$$G_1(x,Sy_1,Sy_2,t)G_2(y_1,y_2,Tx,t),$$
$$G_2(y_1,y_2,Tx,t)G_1(x,x,STx,t))$$

$$H_1^2(Sy_1,Sy_2,STx,t) \leq \frac{1}{4}\lambda\ max(H_1(x,x,STx,t)H_1(x,Sy_1,Sy_2,t),$$
$$H_1(x,Sy_1,Sy_2,t)H_2(y_1,y_2,Tx,t),$$
$$H_2(y_1,y_2,Tx,t)H_1(x,x,STx,t))$$

for all x in X and y_1, y_2 in Y, $0 \leq k, \lambda < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw= z.

Proof:

It is immediate to see that, if we take a function $\psi, \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$,

$\psi(u, v, w) = \frac{1}{4}\, min\{uw, vu, wv\}$ and $\varphi(u, v, w) = \frac{1}{4}\, max\{uw, vu, wv\}$, for all u, v, w $\in \mathbb{R}^+$,

where $0 \leq k, \lambda < 1$.

Corollary 3.3.

Let (X, $G_1, H_1, *, \lozenge$) and (Y, $G_2, H_2, *, \lozenge$) be two complete intuitionistic generalized fuzzy metric spaces and T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities:

$$G_2^2(Tx, TSy_1, TSy_2, t) \geq \frac{1}{4}(a_1 G_2(y_1, TSy_1, TSy_2, t) G_2(y_1, y_2, Tx, t) + b_1 G_2(y_1, y_2, Tx, t) G_1(x, Sy_1, Sy_2, t) + c_1\, G_1(x, Sy_1, Sy_2, t) G_2(y_1, TSy_1, TSy_2, t))$$

$$H_2^2(Tx, TSy_1, TSy_2, t) \leq \frac{1}{4}(a_1 H_2(y_1, TSy_1, TSy_2, t) H_2(y_1, y_2, Tx, t) + b_1\, H_2(y_1, y_2, Tx, t) H_1(x, Sy_1, Sy_2, t) + c_1\, H_1(x, Sy_1, Sy_2, t) H_2(y_1, TSy_1, TSy_2, t))$$

$$G_1^2(Sy_1, Sy_2, STx, t) \geq \frac{1}{4}(a_2 G_1(x, x, STx, t) G_1(x, Sy_1, Sy_2, t) + b_2 G_1(x, Sy_1, Sy_2, t) G_2(y_1, y_2, Tx, t) + c_2 G_2(y_1, y_2, Tx, t) G_1(x, x, STx, t))$$

$$H_1^2(Sy_1, Sy_2, STx, t) \leq \frac{1}{4}(a_2 H_1(x, x, STx, t) H_1(x, Sy_1, Sy_2, t) + b_2 H_1(x, Sy_1, Sy_2, t) H_2(y_1, y_2, Tx, t) + c_2 H_2(y_1, y_2, Tx, t) H_1(x, x, STx, t)),$$

for all x in X and y_1, y_2 in Y, $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw= z.

Theorem: 3.4.

Let (X, $G_1, H_1, *, \lozenge$) and (Y, $G_2, H_2, *, \lozenge$) be two complete intuitionistic generalized fuzzy metric spaces, and T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities:

$$G_2^3(Tx, TSy_1, TSy_2, t)$$
$$\geq \frac{1}{4} k_1\, min(G_1(x_1, Sy_1, Sy_2, t) G_2(y_1, TSy_1, TSy_2, t) G_2(y_1, TSy_1, TSy_2, t),$$
$$G_2(y_1, y_2, Tx, t) G_1(x, Sy_1, Sy_2, t) G_2(y_1, y_2, Tx, t),$$
$$G_2(y_1, TSy_1, TSy_2, t) G_2\ (y_1, y_2, Tx, t)\ G_2(y_1, y_2, Tx, t)) \qquad (3.4.1)$$

$$H_2^3(Tx, TSy_1, TSy_2, t)$$
$$\leq \frac{1}{4} \lambda_1 \max(H_1(x_1, Sy_1, Sy_2, t) H_2(y_1, TSy_1, TSy_2, t) H_2(y_1, TSy_1, TSy_2, t),$$
$$H_2(y_1, y_2, Tx, t) H_1(x, Sy_1, Sy_2, t) H_2(y_1, y_2, Tx, t),$$
$$H_2(y_1, TSy_1, TSy_2, t) H_2(y_1, y_2, Tx, t)\ H_2(y_1, y_2, Tx, t)) \qquad (3.4.2)$$

$$G_1^3(Sy_1, Sy_2, STx, t) \geq \frac{1}{4} k_2\, min(G_2(y_1, y_2, Tx, t) G_1(x, x, STx, t)\ G_1(x, x, STx, t),$$
$$G_1(x, Sy_1, Sy_2, t) G_2(y_1, y_2, Tx, t) G_1(x, Sy_1, Sy_2, t),$$
$$G_1(x, x, STx, t) G_1(x, Sy_1, Sy_2, t)\ G_1(x, Sy_1, Sy_2, t)) \qquad (3.4.3)$$

$$H_1^3(Sy_1, Sy_2, STx, t) \leq \frac{1}{4} \lambda_2\, max(H_2(y_1, y_2, Tx, t) H_1(x, x, STx, t)\ H_1(x, x, STx, t),$$
$$H_1(x, Sy_1, Sy_2, t) H_2(y_1, y_2, Tx, t) H_1(x, Sy_1, Sy_2, t),$$
$$H_1(x, x, STx, t) H_1(x, Sy_1, Sy_2, t)\ H_1(x, Sy_1, Sy_2, t)) \qquad (3.4.4)$$

for all x in X and y_1, y_2 in Y, 0 $\leq k_1, k_2, \lambda_1, \lambda_2 < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw= z.

Proof:

We define the sequences (x_n) in X and (y_n) in Y by $x_n = (ST)^n x,\ y_n = T(ST)^{n-1}x$, for n = 1, 2, …. We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n.

Applying the inequalities (3.4.1) and (3.4.2), we have

$$G_2^3(y_n, y_{n+1}, y_{n+1}, t) = G_2^3(Tx_{n-1}, TSy_n, TSy_n, t)$$
$$\geq \frac{1}{4} k_1 \min(G_1(x_{n-1}, Sy_n, Sy_n, t) G_2(y_n, TSy_n, TSy_n, t)\ G_2(y_n, TSy_n, TSy_n, t),$$
$$G_2(y_n, y_n, Tx_{n-1}, t) G_1(x_{n-1}, Sy_n, Sy_n, t)\ G_2(y_n, y_n, Tx_{n-1}, t),$$
$$G_2(y_n, TSy_n, TSy_n, t)\ G_2(y_n, y_n, Tx_{n-1}, t)\ G_2(y_n, y_n, Tx_{n-1}, t))$$
$$\geq \frac{1}{4} k_1 \min(G_1(x_{n-1}, x_n, x_n, t) G_2(y_n, y_{n+1}, y_{n+1}, t) G_2(y_n, y_{n+1}, y_{n+1}, t), 1, 1),$$

$$H_2^3(y_n, y_{n+1}, y_{n+1}, t) = H_2^3(Tx_{n-1}, TSy_n, TSy_n, t)$$
$$\leq \frac{1}{4} \lambda_1 \max(H_1(x_{n-1}, Sy_n, Sy_n, t) H_2(y_n, TSy_n, TSy_n, t)\ H_2(y_n, TSy_n, TSy_n, t),$$
$$H_2(y_n, y_n, Tx_{n-1}, t) H_1(x_{n-1}, Sy_n, Sy_n, t)\ H_2(y_n, y_n, Tx_{n-1}, t),$$

$$H_2(y_n, TSy_n, TSy_n, t)\ H_2(y_n, y_n, Tx_{n-1}, t)\ H_2(y_n, y_n, Tx_{n-1}, t))$$

$$\leq \frac{1}{4}\ \lambda_1 \max(\ H_1(x_{n-1}, x_n, x_n, t) H_2(y_n, y_{n+1}, y_{n+1}, t) H_2(y_n, y_{n+1}, y_{n+1}, t), 0, 0).$$

It follows that

$$G_2^3(y_n, y_{n+1}, y_{n+1}, t) \geq \frac{1}{4} \mathrm{k}_1(\ G_1(x_{n-1}, x_n, x_n, t) G_2(y_n, y_{n+1}, y_{n+1}, t) G_2(y_n, y_{n+1}, y_{n+1}, t))$$

$$G_2(y_n, y_{n+1}, y_{n+1}, t) \geq \frac{1}{4}\ \mathrm{k}\ \ G_1(x_{n-1}, x_n, x_n, t) \tag{3.4.5}$$

$$H_2^3(y_n, y_{n+1}, y_{n+1}, t) \leq \frac{1}{4}\ \lambda_1(\ H_1(x_{n-1}, x_n, x_n, t) H_2(y_n, y_{n+1}, y_{n+1}, t) H_2(y_n, y_{n+1}, y_{n+1}, t))$$

$$H_2(y_n, y_{n+1}, y_{n+1}, t) \leq \frac{1}{4}\ \lambda\ \ H_1(x_{n-1}, x_n, x_n, t). \tag{3.4.6}$$

Applying the inequalities (3.4.3), (3.4.4) and using the definition (2.1.), we get

$$G_1^3(x_n, x_n, x_{n+1}, t) =\ G_1^3(Sy_n, Sy_n, STx_n, t)$$

$$\geq \frac{1}{4} k_2 \min(G_2(y_n, y_n, Tx_n, t) G_1(x_n, x_n, x_{n+1}, t)\ \ G_1(x_n, x_n, x_{n+1}, t),$$

$$G_1(x_n, Sy_n, Sy_n, t) G_2(y_n, y_n, Tx_n, t) G_1(x_n, Sy_n, Sy_n, t),$$

$$G_1(x_n, x_n, x_{n+1}, t) G_1(x_n, Sy_n, Sy_n, t) G_1(x_n, Sy_n, Sy_n, t)),$$

$$\geq \frac{1}{4} k_2 \min(G_2(y_n, y_n, y_{n+1}, t) G_1(x_n, x_n, x_{n+1}, t)\ G_1(x_n, x_n, x_{n+1}, t),$$

$$G_1(x_n, x_n, x_n, t)\ G_2(y_n, y_n, y_{n+1}, t) G_1(x_n, x_n, x_n, t),$$

$$G_1(x_n, x_n, x_{n+1}, t) G_1(x_n, x_n, x_n, t)\ G_1(x_n, x_n, x_n, t))$$

$$G_1^3(x_n, x_n, x_{n+1}, t) \geq \frac{1}{4} \mathrm{k}_2(G_2(y_n, y_n, y_{n+1}, t)\ G_1(x_n, x_n, x_{n+1}, t)\ G_1(x_n, x_n, x_{n+1}, t))$$

$$\frac{1}{2} G_1(x_n, x_{n+1}, x_{n+1}, t) \geq G_1(x_n, x_n, x_{n+1}, t)$$

$$\geq \frac{1}{4} \mathrm{k}_2\ G_2(y_n, y_n, y_{n+1}, t) \geq \frac{1}{2} \mathrm{k}_2\ G_2(y_n, y_{n+1}, y_{n+1}, t) \tag{3.4.7}$$

$$H_1^3(x_n, x_n, x_{n+1}, t) =\ H_1^3(Sy_n, Sy_n, STx_n, t)$$

$$\leq\ \frac{1}{4}\ \lambda_2 \max(H_2(y_n, y_n, Tx_n, t) H_1(x_n, x_n, x_{n+1}, t)\ H_1(x_n, x_n, x_{n+1}, t),$$

$$H_1(x_n, Sy_n, Sy_n, t)\ H_2(y_n, y_n, Tx_n, t) H_1(x_n, Sy_n, Sy_n, t),$$

$$H_1(x_n, x_n, x_{n+1}, t)\ H_1(x_n, Sy_n, Sy_n, t) H_1(x_n, Sy_n, Sy_n, t))$$

$$\leq \frac{1}{4}\ \lambda_2 \max(H_2(y_n, y_n, y_{n+1}, t) H_1(x_n, x_n, x_{n+1}, t)\ H_1(x_n, x_n, x_{n+1}, t),$$

$$H_1(x_n, x_n, x_n, t)\ H_2(y_n, y_n, y_{n+1}, t)\ H_1(x_n, x_n, x_n, t),$$

$$H_1(x_n, x_n, x_{n+1}, t) H_1(x_n, x_n, x_n, t)\ H_1(x_n, x_n, x_n, t)$$

$$H_1^3(x_n,x_n,x_{n+1},t) \le \frac{1}{4}\ \lambda_2(H_2(y_n,y_n,y_{n+1},t)\ H_1(x_n,x_n,x_{n+1},t)\ H_1(x_n,x_n,x_n,t))$$

$$\frac{1}{2}H_1(x_n,x_{n+1},x_{n+1},t) \le H_1(x_n,x_n,x_{n+1},t) \le \frac{1}{4}\ \lambda_2\ H_2(y_n,y_n,y_{n+1},t)$$

$$\le \frac{1}{2}\ \lambda_2\ H_2(y_n,y_{n+1},y_{n+1},t). \qquad (3.4.8)$$

Now, it follows from the inequalities (3.4.5), (3.4.6), (3.4.7) and (3.4.8) that

$$G_1(x_n,x_{n+1},x_{n+1},t) \ge \mathrm{k}_2\ G_2(y_n,y_{n+1},y_{n+1},t) \ge \frac{1}{4}\mathrm{k}_1\mathrm{k}_2\ G_1(x_{n-1},x_n,x_n,t)$$

$$H_1(x_n,x_{n+1},x_{n+1},t) \le\ \lambda_2\ H_2(y_n,y_{n+1},y_{n+1},t) \le \frac{1}{4}\ \lambda_1\lambda_2\ H_1(x_{n-1},x_n,x_n,t).$$

Hence, by induction, we get

$$G_1(x_n,x_{n+1},x_{n+1},t) \ge (\tfrac{1}{4})^{\mathrm{n}}(\ \mathrm{k}_2\mathrm{k}_1)^{\mathrm{n}}\ G_1(x,x_1,x_1,t)\ \text{ for n = 1,2,…}$$

$$H_1(x_n,x_{n+1},x_{n+1},t) \le (\tfrac{1}{4})^{\mathrm{n}}(\ \lambda_2\lambda_1)^{\mathrm{n}}\ H_1(x,x_1,x_1,t)\ \text{ for n = 1,2,…}$$

Since $\mathrm{k}_2\mathrm{k}_1 < 1$ and $\lambda_2\lambda_1 < 1$, it follows that (x_n) and (y_n) are Cauchy sequences with limits z in X and w in Y. Using the inequalities (3.4.1) and (3.4.2), we have

$$G_2^3(Tz,y_n,y_n,t) =\ G_2^3(Tz,TSy_{n-1},TSy_{n-1},t)$$

$$\ge \frac{1}{4}\mathrm{k}_1\min(G_1(z,Sy_{n-1},Sy_{n-1},t)G_2(y_{n-1},TSy_{n-1},TSy_{n-1},t)\ G_2(y_{n-1},TSy_{n-1},TSy_{n-1},t),$$
$$G_2(y_{n-1},y_{n-1},Tz,t)G_1(z,Sy_{n-1},Sy_{n-1},t)G_2(y_{n-1},y_{n-1},Tz,t),$$
$$G_2(y_{n-1},TSy_{n-1},TSy_{n-1},t)\ G_2(y_{n-1},y_{n-1},Tz,t)G_2(y_{n-1},y_{n-1},Tz,t))$$

$$\ge \frac{1}{4}\mathrm{k}_1\min(G_1(z,x_{n-1},x_{n-1},t)G_2(y_{n-1},y_n,y_n,t)G_2(y_{n-1},y_n,y_n,t),$$
$$G_2(y_{n-1},y_{n-1},Tz,t)G_1(z,x_{n-1},x_{n-1},t)G_2(y_{n-1},y_{n-1},Tz,t),$$
$$G_2(y_{n-1},y_n,y_n,t)G_2(y_{n-1},y_{n-1},Tz,t)G_2(y_{n-1},y_{n-1},Tz,t))$$

$$H_2^3(Tz,y_n,y_n,t) =\ H_2^3(Tz,TSy_{n-1},TSy_{n-1},t)$$

$$\le \frac{1}{4}\lambda_1\ \max\{(H_1(z,Sy_{n-1},Sy_{n-1},t)H_2(y_{n-1},TSy_{n-1},TSy_{n-1},t)H_2(y_{n-1},TSy_{n-1},TSy_{n-1},t),$$
$$H_2(y_{n-1},y_{n-1},Tz,t)H_1(z,Sy_{n-1},Sy_{n-1},t)H_2(y_{n-1},y_{n-1},Tz,t),$$
$$H_2(y_{n-1},TSy_{n-1},TSy_{n-1},t)H_2(y_{n-1},y_{n-1},Tz,t)H_2(y_{n-1},y_{n-1},Tz,t)$$

$$\le \frac{1}{4}\lambda_1\max(H_1(z,x_{n-1},x_{n-1},t)H_2(y_{n-1},y_n,y_n,t)\ H_2(y_{n-1},y_n,y_n,t),$$
$$H_2(y_{n-1},y_{n-1},Tz,t)H_1(z,x_{n-1},x_{n-1},t)H_2(y_{n-1},y_{n-1},Tz,t),$$
$$H_2(y_{n-1},y_n,y_n,t)H_2(y_{n-1},y_{n-1},Tz,t)H_2(y_{n-1},y_{n-1},Tz,t)).$$

Letting $n \to \infty$, we have $G_2^3(Tz,w,w,t) \ge 1$ and $H_2^3(Tz,w,w,t) \le 0$, it follows that

$G_2(Tz, w, w, t) = 1$ and $H_2(Tz, w, w, t) = 0$, hence w = T z.

Using the inequalities (3.4.3) and (3.4.4), we obtain

$$G_1^3(Sw, Sw, x_n, t) = G_1^3(Sw, Sw, STx_{n-1}, t)$$
$$\geq \frac{1}{4} k_2 \min(G_2(w, w, Tx_{n-1}, t) G_1(x_{n-1}, x_{n-1}, STx_{n-1}, t) G_1(x_{n-1}, x_{n-1}, STx_{n-1}, t),$$
$$G_1(x_{n-1}, Sw, Sw, t) G_2(w, w, Tx_{n-1}, t) G_1(x_{n-1}, Sw, Sw, t),$$
$$G_1(x_{n-1}, x_{n-1}, STx_{n-1}, t) G_1(x_{n-1}, Sw, Sw, t) G_1(x_{n-1}, Sw, Sw, t)$$

$$H_1^3(Sw, Sw, x_n, t) = H_1^3(Sw, Sw, STx_{n-1}, t)$$
$$\leq \frac{1}{4} \lambda_2 \max(H_2(w, w, Tx_{n-1}, t) H_1(x_{n-1}, x_{n-1}, STx_{n-1}, t)\, H_1(x_{n-1}, x_{n-1}, STx_{n-1}, t),$$
$$H_1(x_{n-1}, Sw, Sw, t) H_2(w, w, Tx_{n-1}, t) H_1(x_{n-1}, Sw, Sw, t),$$
$$H_1(x_{n-1}, x_{n-1}, STx_{n-1}, t) H_1(x_{n-1}, Sw, Sw, t) H_1(x_{n-1}, Sw, Sw, t)).$$

Letting n tends to infinity, we have $G_1^3(Sw, Sw, x_n, t) \geq 1$ and $H_1^3(Sw, Sw, x_n, t) \leq 0$, and it follows that z = Sw. Thus STz = Sw = z, TSw = Tz = w and so ST has a fixed point z and TS has a fixed point w. Now, suppose that ST has a second fixed point z_1 and TS has a second fixed point w_1. Then using the inequalities (3.4.1), (3.4.2) and using the properties (iii) and (iv), we have

$$G_2^3(w, w_1, w_1, t) = G_2^3(TSw, TSw_1, TSw_1, t) = G_2^3(Tz, TSw_1, TSw_1, t)$$
$$\geq \frac{1}{4} k_1 \min(G_1(z, Sw_1, Sw_1, t) G_2(w_1, TSw_1, TSw_1, t)\, G_2(w_1, TSw_1, TSw_1, t),$$
$$G_2(w_1, w_1, Tz, t)\, G_2(w_1, w_1, Tz, t)\, G_1(z, Sw_1, Sw_1, t),$$
$$G_2(w_1, TSw_1, TSw_1, t)\, G_2(w_1, w_1, Tz, t)\, G_2(w_1, w_1, Tz, t))$$
$$\geq \frac{1}{4} k_1 \min(1, G_2(w_1, w_1, w, t) G_1(Sw, Sw_1, Sw_1, t) G_2(w_1, w_1, w, t), 1)$$

$$H_2^3(w, w_1, w_1, t) = H_2^3(TSw, TSw_1, TSw_1, t) = H_2^3(Tz, TSw_1, TSw_1, t)$$
$$\leq \frac{1}{4} \lambda_1 \max(H_1(z, Sw_1, Sw_1, t) H_2(w_1, TSw_1, TSw_1, t)\, H_2(w_1, TSw_1, TSw_1, t),$$
$$H_2(w_1, w_1, Tz, t)\, H_2(w_1, w_1, Tz, t)\, H_1(z, Sw_1, Sw_1, t),$$
$$H_2(w_1, TSw_1, TSw_1, t)\, H_2(w_1, w_1, Tz, t)\, H_2(w_1, w_1, Tz, t))$$
$$\leq \frac{1}{4} \lambda_1 \max(0, H_2(w_1, w_1, w, t) H_1(Sw, Sw_1, Sw_1, t) H_2(w_1, w_1, w, t), 0) \text{ and so}$$

$$G_2^3(w, w_1, w_1, t) \geq \frac{1}{4} k_1 \min(G_1(Sw, Sw_1, Sw_1, t)\, G_2(w_1 w_1, w, t)\, G_2(w_1 w_1, w, t))$$

$$G_2(w, w_1, w_1, t) \geq \frac{1}{4} k_1 G_1(Sw, Sw_1, Sw_1, t) \tag{3.4.9}$$

$H_2^3(w, w_1, w_1, t) \leq \frac{1}{4}\lambda_1 max(H_1(Sw, Sw_1, Sw_1, t)\ H_2(w_1 w_1, w, t)\ H_2(w_1 w_1, w, t)$

$H_2(w, w_1, w_1, t)\ \leq \frac{1}{4}\lambda_1 H_1(Sw, Sw_1, Sw_1, t).$ (3.4.10)

Applying the inequalities (3.4.3), (3.4.4) and using definition (2.2.), we have

$$G_1^3(Sw, Sw, Sw_1, t) = G_1^3(STSw, STSw, STSw_1, t)$$

$$\geq \frac{1}{4}k_2 \min(G_2(TSw, TSw, TSw_1, t)G_1(Sw_1\ , Sw_1, STSw_1, t)\ G_1(Sw\ , Sw, STSw, t),$$

$$G_1(Sw_1\ , STSw, STSw, t)G_2(TSw, TSw, TSw_1, t)G_1(Sw_1, STSw, STSw, t),$$

$$G_1(Sw_1, Sw_1, STSw_1, t)G_1(Sw_1, STSw, STSw, t)G_1(Sw_1, STSw, STSw, t)),$$

$$\geq\ \frac{1}{4}k_2 \min(1, G_1(Sw_1\ , Sw, Sw, t)G_1(Sw_1\ , Sw, Sw, t), G_2(w, w, w_1, t), 1)$$

$$G_1^3(Sw, Sw, Sw_1, t)\ \geq \frac{1}{4}\mathrm{k}_2(G_2(w, w, w_1, t)G_1(\ Sw, Sw, Sw_1, t)\ G_1(Sw, Sw, Sw_1, t)$$

$$\frac{1}{2}G_1(Sw, Sw_1, Sw_1, t) \geq G_1(Sw, Sw, Sw_1, t)\ \geq\ \frac{1}{4}k_2 G_2(w, w, w_1, t) \geq \frac{1}{2}\mathrm{k}_2\ \ G_2(w, w_1, w_1, t)$$

$$H_1^3(Sw, Sw, Sw_1, t) = H_1^3(STSw, STSw, STSw_1, t)$$

$$\leq \frac{1}{4}\lambda_2 \max(H_2(TSw, TSw, TSw_1, t)H_1(Sw_1\ , Sw_1, STSw_1, t)\ H_1(Sw\ , Sw, STSw, t),$$

$$H_1(Sw\ , STSw, STSw, t)\ \ H_2(TSw, TSw, TSw_1, t)H_1(Sw_1, STSw, STSw, t),$$

$$H_1(Sw_1, Sw_1, STSw_1, t)H_1(Sw_1, STSw, STSw, t)H_1(Sw_1, STSw, STSw, t))$$

$$\leq\ \frac{1}{4}\lambda_2 \max(0, H_1(Sw_1\ , Sw, Sw, t)H_1(Sw_1\ , Sw, Sw, t)H_2(w, w, w_1, t), 0)$$

$$H_1^3(Sw, Sw, Sw_1, t)\ \leq\ \frac{1}{4}\mathrm{k}_2(H_2(w, w, w_1, t)H_1(\ Sw, Sw, Sw_1, t)\ H_1(Sw, Sw, Sw_1, t))$$

$$\frac{1}{2}H_1(Sw, Sw_1, Sw_1, t) \leq H_1(Sw, Sw, Sw_1, t) \leq\ \frac{1}{4}\lambda_2\ H_2(w, w, w_1, t) \leq\ \frac{1}{2}\lambda_2\ H_2(w, w_1, w_1, t)$$

$$G_1(Sw, Sw_1, Sw_1, t) \geq\ \mathrm{k}_2\ \ G_2(w, w_1, w_1, t) \qquad (3.4.11)$$

$$H_1(Sw, Sw_1, Sw_1, t)\ \leq \lambda_2\ \ H_2(w, w_1, w_1, t). \qquad (3.4.12)$$

Now it follows from the inequalities (3.4.9), (3.4.10), (3.4.11) and (3.4.12) that

$$G_2(w, w_1, w_1, t) \geq \frac{1}{4}\mathrm{k}_1\ \ G_1(Sw, Sw_1, Sw_1, t)\ > \frac{1}{4}\mathrm{k}_1\mathrm{k}_2\ G_2(w, w_1, w_1, t) > G_2(w, w_1, w_1, t)$$

$$H_2(w, w_1, w_1, t) \leq \frac{1}{4}\lambda_1\ \ H_1(Sw, Sw_1, Sw_1, t)\ < \frac{1}{4}\lambda_1\lambda_2\ H_2(w, w_1, w_1, t)\ < H_2(w, w_1, w_1, t),$$

and so w = w_1. Since $k_1 k_2 < 1$ and $\lambda_1\lambda_2 < 1$. The fixed point w of TS must be a unique.

Now TSz_1 = z_1 implies TSTz_1 = Tz_1 and so T z_1 = w.

Thus z = STz = Sw = STz_1 = z_1, proving that z is the unique fixed point of ST.

This completes the proof of the theorem.

Corollary 3.5.

Let (X, $G_1, H_1, *, \Diamond$) and (Y, $G_2, H_2, *, \Diamond$) be two complete intuitionistic generalized fuzzy metric spaces and T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities:

$$G_2^3(Tx,TSy_1,TSy_2,t) \geq \frac{1}{4}(a_1G_1(x,Sy_1,Sy_2,t)G_2(y_1,TSy_1,TSy_2,t)G_2(y_1,TSy_1,TSy_2,t) + b_1G_2(y_1,y_2,Tx,t)G_1(x,Sy_1,Sy_2,t)\, G_2(y_1,y_2,Tx,t) + c_1\, G_2(y_1,TSy_1,TSy_2,t)G_2\,(y_1,y_2,Tx,t)\, G_2(y_1,y_2,Tx,t))$$

$$H_2^3(Tx,TSy_1,TSy_2,t) \leq \frac{1}{4}(a_1H_1(x,Sy_1,Sy_2,t)H_2(y_1,TSy_1,TSy_2,t)H_2(y_1,TSy_1,TSy_2,t) + b_1H_2(y_1,y_2,Tx,t)H_1(x,Sy_1,Sy_2,t)\, H_2(y_1,y_2,Tx,t) + c_1\, G_2(y_1,TSy_1,TSy_2,t)H_2\,(y_1,y_2,Tx,t)\, H_2(y_1,y_2,Tx,t))$$

$$G_1^3(Sy_1,Sy_2,STx,t) \geq \frac{1}{4}(a_2G_2(y_1,y_2,Tx,t)G_1(x,x,STx,t)G_1(x,x,STx,t) + b_2G_1(x,Sy_1,Sy_2,t)G_2(y_1,y_2,Tx,t)G_1(x,Sy_1,Sy_2,t) + c_2G_1(x,x,STx,t)G_1(x,Sy_1,Sy_2,t)\, G_1(x,Sy_1,Sy_2,t))$$

$$H_1^3(Sy_1,Sy_2,STx,t) \leq \frac{1}{4}(a_2H_2(y_1,y_2,Tx,t)H_1(x,x,STx,t)H_1(x,x,STx,t) + b_2H_1(x,Sy_1,Sy_2,t)H_2(y_1,y_2,Tx,t)H_1(x,Sy_1,Sy_2,t) + c_2H_1(x,x,STx,t)H_1(x,Sy_1,Sy_2,t)\, H_1(x,Sy_1,Sy_2,t)$$

for all x in X and y_1, y_2 in Y, $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1+b_1+c_1)(a_2+b_2+c_2) < 1$.
Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y.
Further, Tz = w and Sw= z.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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