Available online at http://scik.org
Adv. Fixed Point Theory, 2020, 10:11
https://doi.org/10.28919/afpt/4655
ISSN: 1927-6303

COMMON FIXED POINT THEOREMS OF FOUR MAPS ON b-METRIC SPACES WITH wt-DISTANCE

MEIMEI SONG, FEIRI AZI*

Science of College, Tianjin University of Technology, 300384, Tianjin, China

Abstract. In this paper, some common fixed point theorems of four maps on b-metric spaces with wt-distance are proved, which extend some results in the literature.

Keywords: b-compatible; common fixed point; wt-distance; b-metric space.

2010 AMS Subject Classification: 47H10.

1. Introduction and Preliminaries

Since the concept of b-metric space as a generalization of metric space was given by Czerwik [1], many fixed point results in metric spaces were generalized in b-metric spaces (see [9, 10], etc.). In 2014, the concept of wt-distance on b-metric spaces was given by N. Hussain et al. [2], we shall use wt-distance on b-metric spaces to extend some results by others.

In the section 1, we give some elementary definitions and lemmas. In the section 2, inspired by J. R. Roshan et al. [7], Nawab Hussain et al. [8], Mirko Jovanović et al. [9] and Liya Liu and Feng Gu [10], we prove the main theorem on b-metric spaces with wt-distance and get some related fixed point results.

*Corresponding author
E-mail address: azifeiri@163.com
Received April 24, 2020

Throughout, we denote all natural number by $\mathbb{N}$.

Definition 1.1[1] Let X be a nonempty set and constant $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to [0,\infty)$ satisfies the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.

Then (X,d) is called a b-metric space with coefficient s.

Definition 1.2[2, 3] Let (X,d) be a b-metric space with constant $s \geq 1$, then a function $p : X \times X \to [0,\infty)$ is called a wt-distance on X if the following conditions are satisfied:

(1) $p(x,z) \leq s[p(x,y)+p(y,z)]$ for any $x,y,z \in X$;

(2) $p(x,\cdot) : X \to [0,\infty)$ is s-lower semi-continuous for any $x \in X$, if

$$\liminf_{n\to\infty} p(x,x_n) = \infty,\ or\ p(x,x_0) \leq \liminf_{n\to\infty} sp(x,x_n),$$

where $\lim_{n\to\infty} d(x_0,x_n) = 0$;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

The wt-distance p is called symmetric if $p(x,y) = p(y,x)$ for any $x,y \in X$.

We say that

(a) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n,x) = 0$, i.e., $x_n \to x$;

(b) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n,x_m) = 0$;

(c) (X,d) is complete if and only if any Cauchy sequence in X is convergent.

Lemma 1.3[2, 3] Let (X,d) be a b-metric space with constant $s \geq 1$ and p be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero. Then for any $x,y,z \in X$, the following properties hold:

(1) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;

(2) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(y_n,z) = 0$;

(3) If $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(4) If $p(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.4[4, 5] We say that f and g are b-compatible on b-metric space (X,d) with constant $s \geq 1$, and p be a wt-distance on X if

$$\lim_{n\to\infty} p(fgx_n, gfx_n) = 0,$$

when $\{x_n\}$ is a sequence such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$, for some t in X.

2. Main Results

In this part, we will show our main results.

Lemma 2.1[6] Let (X,d) be a b-metric space with constant $s \geq 1$ and p be a wt-distance on X, $\{x_n\}$ be sequence in X, we say the $\{x_n\}$ is a Cauchy sequence if there exists $c \in [0,1)$, such that $p(x_n, x_{n+1}) \leq cp(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$.

Theorem 2.2 Let (X,d) be a complete b-metric space with $s \geq 1$ and p be a wt-distance on X, $p(x,x) = 0$ for any $x \in X$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$p(Sx,Ty) \leq \lambda max\{p(Ix,Jy), p(Ix,Sx), p(Jy,Ty), \frac{1}{2s}[p(Sx,Jy) + p(Ix,Ty)]\} \tag{2.1}$$

for any $x, y \in X$, where $\lambda \in [0,1)$ and $\lambda s^2 < 1$, if $S(x) \subseteq J(x)$, $T(x) \subseteq I(x)$ and I, J, S and T are continuous maps, then I, J, S and T have a unique common fixed point in X.

Proof. If $\lambda = 0$, then $p(Sx,Ty) = 0$, $p(Sx,Ix) = 0$ and $p(Ty,Jy) = 0$, we have that $Ty = Ix = Sx = Jy$.

Now, we construct $\{x_n\} \subset X$. Let $\forall x_0 \in X$, $Sx_0 \in J(X)$, there is any $x_1 \in X$, such that $Jx_1 = Sx_0$. $Tx_1 \in I(X)$, then there is $x_2 \in X$, such that $Tx_1 = Ix_2$. In general, we chosen $x_{2n+1} \in X$, such that $Jx_{2n+1} = Sx_{2n}$, and $x_{2n+2} \in X$, such that $Ix_{2n+2} = Tx_{2n+1}$, for $n = 0, 1, 2, \cdots$.

Denote a sequence $\{y_n\}$ with

$$y_{2n} = Jx_{2n+1} = Sx_{2n}, y_{2n+1} = Ix_{2n+2} = Tx_{2n+1}.$$

We show that $\{y_n\}$ is a Cauchy sequence. If not, we suppose that there is a constant n_0, such that $p(y_{2n}, y_{2n+1}) > 0$ for any $2n > n_0$, then for some constant k, by (2.1),

$$\begin{aligned}
& p(y_{2k}, y_{2k+1}) = p(Sx_{2k}, Tx_{2k+1}) \\
\leq\ & \lambda max\{p(Ix_{2k}, Jx_{2k+1}), p(Ix_{2k}, Sx_{2k}), p(Jx_{2k+1}, Tx_{2k+1}), \frac{1}{2s}[p(Sx_{2k}, Jx_{2k+1}) + p(Ix_{2k}, Tx_{2k+1})]\} \\
=\ & \lambda max\{p(y_{2k-1}, y_{2k}), p(y_{2k-1}, y_{2k}), p(y_{2k}, y_{2k+1}), \frac{1}{2s}[p(y_{2k}, y_{2k}) + p(y_{2k-1}, y_{2k+1})]\} \\
=\ & \lambda max\{p(y_{2k-1}, y_{2k}), p(y_{2k}, y_{2k+1}), \frac{1}{2s}p(y_{2k-1}, y_{2k+1})\}.
\end{aligned}$$

Since $\frac{1}{2s}p(y_{2k-1},y_{2k+1}) \leq p(y_{2k-1},y_{2k})$ or $\frac{1}{2s}p(y_{2k-1},y_{2k+1}) \leq p(y_{2k},y_{2k+1})$, we only need to think about the following two cases.

For the first case, if

$$p(y_{2k},y_{2k+1}) \leq \lambda p(y_{2k-1},y_{2k}),$$

which $\lambda \in (0,1)$, then by Lemma 2.1, we have that $lim_{k\to\infty}p(y_{2k},y_{2k+1}) = 0$, it is a contradiction.

For the second case, if

$$p(y_{2k},y_{2k+1}) \leq \lambda p(y_{2k},y_{2k+1}) < p(y_{2k},y_{2k+1}),$$

which $\lambda \in (0,1)$, it is a contradiction.

Thus, $\{y_n\}$ is a Cauchy sequence, by Cauchy sequence, we have that $lim_{m,n\to\infty}p(y_m,y_n) = 0$, then for any $\varepsilon > 0$, there exists a $n > N_\varepsilon - 1$, such that $p(y_{N_\varepsilon},y_n) < \frac{\varepsilon}{s}$.

Since X is complete, there exists $u \in X$, such that

$$u = lim_{n\to\infty}Ix_{2n+2} = lim_{n\to\infty}Tx_{2n+1} = lim_{n\to\infty}Jx_{2n+1} = lim_{n\to\infty}Sx_{2n}. \tag{2.2}$$

$p(x,\cdot)$ is s-lower semi-continuous, thus we have

$$p(y_{N_\varepsilon},u) \leq lim_{n\to\infty}inf sp(y_{N_\varepsilon},y_n) \leq \varepsilon.$$

When $n \to \infty$ we have,

$$p(y_{2n},u) < \varepsilon. \tag{2.3}$$

Since T and J are continuous and b-compatible, we have

$$\begin{aligned} p(Tu,Ju) &\leq sp(Tu,TJx_{2n+1}) + sp(TJx_{2n+1},Ju) \\ &\leq sp(Tu,TJx_{2n+1}) + s^2p(TJx_{2n+1},JTx_{2n+1}) + s^2p(JTx_{2n+1},Ju). \end{aligned}$$

There are $lim_{n\to\infty}p(Tu,TJx_{2n+1}) = 0$, $lim_{n\to\infty}p(TJx_{2n+1},JTx_{2n+1}) = 0$ and $lim_{n\to\infty}p(JTx_{2n+1},Ju) = 0$, thus we have $p(Tu,Ju) = 0$ and $p(Ju,Tu) = 0$. Similarly, we have $p(Iu,Su) = 0$.

Since I and S are continuous and b-compatible, we have

$$\begin{aligned} p(Iu,Su) &\leq sp(Iu,ISx_{2n})+sp(ISx_{2n},Su) \\ &\leq sp(Iu,ISx_{2n})+s^2p(ISx_{2n},SIx_{2n})+s^2p(SIx_{2n},Su). \end{aligned}$$

There are $lim_{n\to\infty}p(Iu,ISx_{2n})=0$, $lim_{n\to\infty}p(ISx_{2n},SIx_{2n})=0$ and $lim_{n\to\infty}p(SIx_{2n},Su)=0$, thus we have $p(Iu,Su)=0$.

We shall prove that $p(Su,Tu)=0$, if not, we suppose that $p(Su,Tu)>0$,

$$\begin{aligned} p(Su,Tu) &\leq \lambda max\{p(Iu,Ju),p(Iu,Su),p(Ju,Tu),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\} \\ &= \lambda max\{p(Iu,Ju),0,0,\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\} \\ &= \lambda max\{p(Iu,Ju),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\}. \end{aligned}$$

If $max\{p(Iu,Ju),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\}=\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]$, then

$$\begin{aligned} p(Su,Tu) &\leq \frac{\lambda}{2s}[p(Su,Ju)+p(Iu,Tu)] \\ &\leq \frac{\lambda}{2s}[sp(Su,Tu)+sp(Tu,Ju)+sp(Iu,Su)+sp(Su,Tu)] \\ &= \lambda p(Su,Tu) \\ &< p(Su,Tu), \end{aligned}$$

which $\lambda\in(0,1)$, it is a contradiction.

If $max\{p(Iu,Ju),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\}=p(Iu,Ju)$, then

$$\begin{aligned} p(Su,Tu) &\leq \lambda p(Iu,Ju) \\ &\leq \lambda[sp(Iu,Su)+sp(Su,Ju)] \\ &\leq \lambda[sp(Iu,Su)+s^2p(Su,Tu)+s^2p(Tu,Ju)] \\ &= (\lambda\cdot s^2)p(Su,Tu), \end{aligned}$$

by $\lambda s^2<1$, we have $p(Su,Tu)\leq(\lambda s^2)p(Su,Tu)<p(Su,Tu)$, it is a contradiction. Then we have $Ju=Su=Tu=Iu$.

Next, we shall prove that $lim_{n\to\infty}p(y_{2n},Ty_{2n})=0$. If not, we suppose that $lim_{n\to\infty}p(y_{2n},Ty_{2n})>0$, when $n\to\infty$, we have

$$\begin{aligned}
p(y_{2n},Ty_{2n})=p(Sx_{2n},Ty_{2n}) &\le \lambda max\{p(Ix_{2n},Jy_{2n}),p(Ix_{2n},Sx_{2n}),\\
&\quad p(Jy_{2n},Ty_{2n}),\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\}\\
&= \lambda max\{p(Ix_{2n},Jy_{2n}),0,0,\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\}\\
&= \lambda max\{p(Ix_{2n},Jy_{2n}),\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\}.
\end{aligned}$$

If $max\{p(Ix_{2n},Jy_{2n}),\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\}=p(Ix_{2n},Jy_{2n})$, then

$$\begin{aligned}
p(Sx_{2n},Ty_{2n}) &\le \lambda p(Ix_{2n},Jy_{2n})\\
&\le \lambda s[p(Ix_{2n},Sx_{2n})+p(Sx_{2n},Jy_{2n})]\\
&\le \lambda s[p(Ix_{2n},Sx_{2n})+sp(Sx_{2n},Ty_{2n})+sp(Ty_{2n},Jy_{2n})]\\
&= \lambda s^2 p(Sx_{2n},Ty_{2n})\\
&< p(Sx_{2n},Ty_{2n}),
\end{aligned}$$

which $\lambda s^2<1$, it is a contradiction, then we have $lim_{n\to\infty}p(y_{2n},Ty_{2n})=0$.

If $max\{p(Ix_{2n},Jy_{2n}),\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\}=\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]$, then

$$\begin{aligned}
p(Sx_{2n},Ty_{2n}) &\le \lambda\frac{1}{2s}[p(Sx_{2n},Jy_{2n})+p(Ix_{2n},Ty_{2n})]\\
&\le \frac{\lambda}{2s}[sp(Sx_{2n},Ty_{2n})+sp(Ty_{2n},Jy_{2n})+sp(Ix_{2n},Sx_{2n})+sp(Sx_{2n},Ty_{2n})]\\
&= \lambda p(Sx_{2n},Ty_{2n})\\
&< p(Sx_{2n},Ty_{2n}),
\end{aligned}$$

which $\lambda<1$, it is a contradiction, then we have $lim_{n\to\infty}p(y_{2n},Ty_{2n})=0$.

Thus, we have $lim_{n\to\infty}p(y_{2n},Ty_{2n})=0$. For any $\varepsilon>0$, we have

$$p(y_{2n},Ty_{2n})<\varepsilon. \tag{2.4}$$

where $n\to\infty$.

By (2.3), (2.4) and Lemma 1.3, we have $lim_{n\to\infty}d(Ty_{2n},u)=0$.

By the continuity of T, we have $u = lim_{n\to\infty} Ty_{2n} = Tu$, thus we have $u = Tu$, We get that $u = Tu = Ju = Su = Iu$. Then we have u is a common fixed point of S, I, T and J. Now, we shall prove the uniqueness of the common fixed point.

If not, there is another common fixed point w of S, I, T and J, and $p(Sw,Tu) \neq 0$,

$$\begin{aligned} p(Sw,Tu) &\leq \lambda max\{p(Iw,Ju), p(Iw,Sw), p(Ju,Tu), \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]\} \\ &= \lambda max\{p(Iw,Ju), 0, 0, \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]\} \\ &= \lambda max\{p(Iw,Ju), \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]\}. \end{aligned}$$

If $max\{p(Iw,Ju), \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]\} = p(Iw,Ju)$, then

$$\begin{aligned} p(Sw,Tu) \leq \lambda p(Iw,Ju) &\leq \lambda s[p(Iw,Sw)+p(Sw,Ju)] \\ &\leq \lambda s[p(Iw,Sw)+sp(Sw,Tu)+sp(Tu,Ju)] \\ &= \lambda s^2 p(Sw,Tu) \\ &< p(Sw,Tu), \end{aligned}$$

which $\lambda s^2 < 1$, it is a contradiction, then we have $p(Sw,Tu) = 0$.

If $max\{p(Iw,Ju), \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]\} = \frac{1}{2s}[p(Sw,Ju)+p(Iw,Tu)]$, then

$$\begin{aligned} p(Sw,Tu) &\leq \frac{\lambda}{2s}[p(Sw,Ju)+p(Iw,Tu)] \\ &= \frac{\lambda}{2s}[sp(Sw,Tu)+sp(Tu,Ju)+sp(Iw,Sw)+sp(Sw,Tu)] \\ &= \lambda p(Sw,Tu) \\ &< p(Sw,Tu), \end{aligned}$$

which $\lambda \in (0,1)$, it is a contradiction, then we have $p(Sw,Tu) = 0$. Thus we have $Sw = Tu$ and $w = Sw = Tu = u$, then $w = u$. S, I, T and J have a unique common fixed point.

By Theorem 2.2, we have the following result.

Theorem 2.3 Let (X,d) be a complete b-metric space with $s \geq 1$ and p be a wt-distance on X, $p(x,x) = 0$ for any $x \in X$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$p(Sx,Ty) \leq \lambda p(Ix,Jy) \tag{2.5}$$

for any $x,y \in X$, where $\lambda \in [0,1)$, $\lambda s^2 < 1$, if $S(x) \subseteq J(x)$, $T(x) \subseteq I(x)$ and I,J,S and T are continuous maps, then I,J,S and T have a unique common fixed point in X.

We can get two corollaries if wt-distance p is symmetric.

Corollary 2.4 Let (X,d) be a complete b-metric space with $s \geq 1$ and p be a symmetric wt-distance on X, $p(x,x) = 0$ for any $x \in X$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$p(Sx,Ty) \leq \lambda max\{p(Ix,Jy), p(Ix,Sx), p(Jy,Ty), \frac{1}{2s}[p(Sx,Jy) + p(Ix,Ty)]\} \tag{2.6}$$

for any $x,y \in X$, where $\lambda \in [0,1)$ and $\lambda s^2 < 1$, if $S(x) \subseteq J(x)$, $T(x) \subseteq I(x)$, then the four maps I,J,S and T have a unique common fixed point in X.

Proof. By observing the proof of Theorem 2.2, we only need to prove the existence of common fixed point. We continue using the similar notations in Corollary 2.4.

We have that $\{y_n\}$ is a Cauchy sequence, by Cauchy sequence, we have that $lim_{m,n\to\infty} p(y_m,y_n) = 0$. For any $\varepsilon > 0$, there exists a $n > N_\varepsilon - 1$, such that $p(y_{N_\varepsilon}, y_n) < \frac{\varepsilon}{s}$.

Since X is complete, there exists $u \in X$, such that

$$u = lim_{n\to\infty} Ix_{2n+2} = lim_{n\to\infty} Tx_{2n+1} = lim_{n\to\infty} Jx_{2n+1} = lim_{n\to\infty} Sx_{2n}. \tag{2.7}$$

$p(x,\cdot)$ is s-lower semi-continuous, thus we have

$$p(y_{N_\varepsilon}, u) \leq lim_{n\to\infty} inf sp(y_{N_\varepsilon}, y_n) \leq \varepsilon.$$

When $n \to \infty$ we have,

$$p(y_{2n}, u) < \varepsilon. \tag{2.8}$$

By the symmetry of wt-distance p, we have

$$lim_{n\to\infty} p(u, y_{2n}) < \varepsilon. \tag{2.9}$$

Since T and J are b-compatible, by (2.7), we have

$$\begin{aligned} p(Tu,Ju) &\leq sp(Tu,TJx_{2n+1})+sp(TJx_{2n+1},Ju) \\ &\leq sp(Tu,TJx_{2n+1})+s^2p(TJx_{2n+1},JTx_{2n+1})+s^2p(JTx_{2n+1},Ju). \end{aligned}$$

Since $lim_{n\to\infty}p(Tu,TJx_{2n+1})=lim_{n\to\infty}p(TJx_{2n+1},JTx_{2n+1})=lim_{n\to\infty}p(JTx_{2n+1},Ju)=0$, thus we have $p(Tu,Ju)=0$ and $p(Ju,Tu)=0$. Similarly, we have $p(Iu,Su)=0$.

Since I and S are b-compatible, by (2.7), we have

$$\begin{aligned} p(Iu,Su) &\leq sp(Iu,ISx_{2n})+sp(ISx_{2n},Su) \\ &\leq sp(Iu,ISx_{2n})+s^2p(ISx_{2n},SIx_{2n})+s^2p(SIx_{2n},Su). \end{aligned}$$

There are $lim_{n\to\infty}p(Iu,ISx_{2n})=0$, $lim_{n\to\infty}p(ISx_{2n},SIx_{2n})=0$ and $lim_{n\to\infty}p(SIx_{2n},Su)=0$, thus we have $p(Iu,Su)=0$.

We shall prove that $p(Su,Tu)=0$, if not, we suppose that $p(Su,Tu)>0$.

$$\begin{aligned} p(Su,Tu) &\leq \lambda max\{p(Iu,Ju),p(Iu,Su),p(Ju,Tu),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\} \\ &= \lambda max\{p(Iu,Ju),0,0,\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\} \\ &= \lambda max\{p(Iu,Ju),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\}. \end{aligned}$$

If $max\{p(Iu,Ju),\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\}=\frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]$, then

$$\begin{aligned} p(Su,Tu) &\leq \frac{\lambda}{2s}[p(Su,Ju)+p(Iu,Tu)] \\ &\leq \frac{\lambda}{2s}[sp(Su,Tu)+sp(Tu,Ju)+sp(Iu,Su)+sp(Su,Tu)] \\ &= \lambda p(Su,Tu) \\ &< p(Su,Tu), \end{aligned}$$

which $\lambda\in(0,1)$, it is a contradiction.

If $max\{p(Iu,Ju), \frac{1}{2s}[p(Su,Ju)+p(Iu,Tu)]\} = p(Iu,Ju)$, then

$$\begin{aligned} p(Su,Tu) &\leq \lambda p(Iu,Ju) \\ &\leq \lambda[sp(Iu,Su)+sp(Su,Ju)] \\ &\leq \lambda[sp(Iu,Su)+s^2p(Su,Tu)+s^2p(Tu,Ju)] \\ &= (\lambda \cdot s^2)p(Su,Tu), \end{aligned}$$

by $\lambda s^2 < 1$, we have $p(Su,Tu) \leq (\lambda s^2)p(Su,Tu) < p(Su,Tu)$, it is a contradiction. Then we have $Ju = Su = Tu = Iu$.

Next, we shall prove that $p(u,Tu) = 0$. If not, we suppose that $p(u,Tu) > 0$, when $n \to \infty$, by (2.9), we have

$$\begin{aligned} p(u,Tu) &\leq s[p(u,y_{2n})+p(y_{2n},Tu)] = s[p(u,y_{2n})+p(Sx_{2n},Tu)] \\ &= sp(Sx_{2n},Tu) \\ &\leq (\lambda s)max\{p(Ix_{2n},Ju), p(Ix_{2n},Sx_{2n}), \\ &\quad p(Ju,Tu), \frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]\} \\ &= (\lambda s)max\{p(Ix_{2n},Ju), 0, 0, \frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]\} \\ &= (\lambda s)max\{p(Ix_{2n},Ju), \frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]\}. \end{aligned}$$

Since $s > 1$, if $max\{p(Ix_{2n},Ju), \frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]\} = p(Ix_{2n},Ju)$, then

$$\begin{aligned} p(Sx_{2n},Tu) &\leq \lambda p(Ix_{2n},Ju) \\ &\leq \lambda s[p(Ix_{2n},Sx_{2n})+p(Sx_{2n},Ju)] \\ &\leq \lambda s[p(Ix_{2n},Sx_{2n})+sp(Sx_{2n},Tu)+sp(Tu,Ju)] \\ &= \lambda s^2 p(Sx_{2n},Tu) \\ &< p(Sx_{2n},Tu), \end{aligned}$$

which $\lambda s^2 < 1$, it is a contradiction, then we have $p(u,Tu) = 0$.

If $max\{p(Ix_{2n},Ju),\frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]\}=\frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)]$, then

$$\begin{aligned} p(Sx_{2n},Tu) &\leq \lambda\frac{1}{2s}[p(Sx_{2n},Ju)+p(Ix_{2n},Tu)] \\ &\leq \frac{\lambda}{2s}[sp(Sx_{2n},Tu)+sp(Tu,Ju)+sp(Ix_{2n},Sx_{2n})+sp(Sx_{2n},Tu)] \\ &= \lambda p(Sx_{2n},Tu) \\ &< p(Sx_{2n},Tu), \end{aligned}$$

which $\lambda<1$, it is a contradiction, then we have $p(u,Tu)=0$. Thus, we have $p(u,Tu)=0$, then $u=Tu$. We get that $u=Tu=Ju=Su=Iu$. Then we have u is a common fixed point of S, I, T and J.

Corollary 2.5 Let (X,d) be a complete b-metric space with $s\geq 1$ and p be a symmetric wt-distance on X, $p(x,x)=0$ for any $x\in X$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$p(Sx,Ty)\leq\lambda p(Ix,Jy) \tag{2.10}$$

for any $x,y\in X$, where $\lambda\in[0,1)$, $\lambda s^2<1$, if $S(x)\subseteq J(x)$, $T(x)\subseteq I(x)$, then the four maps I,J,S and T have a unique common fixed point in X.

Since b-metric d is also a wt-distance on (X,d), let $p=d$ in (2.6) of corollary 2.4 and $p=d$ in (2.10) of corollary 2.5. We obtain the Theorem 2.1 by given by J. R. Roshan et al. [7] and the Theorem 2.3 by given by Nawab Hussain et al. [8].

Corollary 2.6[7] Let (X,d) be a complete b-metric space with $s\geq 1$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$d(Sx,Ty)\leq\lambda max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),\frac{1}{2s}[d(Sx,Jy)+d(Ix,Ty)]\}$$

for any $x,y\in X$, where $\lambda\in[0,1)$ and $\lambda s^2<1$, if $S(x)\subseteq J(x)$, $T(x)\subseteq I(x)$, then the four maps I,J,S and T have a unique common fixed point in X.

Corollary 2.7[8] Let (X,d) be a complete b-metric space with $s\geq 1$, the pairs (S,I) and (T,J) be b-compatible defined on (X,d) and satisfying

$$d(Sx,Ty)\leq\lambda d(Ix,Jy)$$

for any $x,y \in X$, where $\lambda \in [0,1)$, $\lambda s^2 < 1$, if $S(x) \subseteq J(x)$, $T(x) \subseteq I(x)$, then the four maps I,J,S and T have a unique common fixed point in X.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[2] N. Hussain, R. Saadati, R. P Agrawal, On the topology and *wt*-distance on the metric type spaces, Fixed Point Theory Appl. 2014 (2014), 88.

[3] H. Lakziana, B.E. Rhoades, Some fixed point theorems using weaker Meir-Keeler function in metric spaces with *w*-distance, Appl. Math. Comput. 342 (2019), 18-25.

[4] Borimandafu Wu, Fei He, Tao Xu, Common fixed point theorems for Ćirić type mappings in *b*-metric spaces without any completeness assumption, J. Nonlinear Sci. Appl. 10 (2017), 3180-3190.

[5] Zhongzhi Yang, Hassan Sadati, Shaban Sedghi, Nabi Shobe, Common fixed point theorems for non-compatible self-maps in *b*-metric spaces, J. Nonlinear Sci. Appl. 8 (2015), 1022-1031.

[6] MeiMei Song, FeiRi AZi, Fixed point theorems for cyclic contraction on *b*-metric spaces with *wt*-distance, Adv. Fixed Point Theory, 9 (4) (2019), 365-380.

[7] J. R. Roshan, N. Shobkolaei, S. Sedghi, M. Abbas, Common fixed point of four maps in *b*-metric spaces, Hacet. J. Math. Stat. 43 (4) (2014), 613-624.

[8] N. Hussain, Z.D. Mitrovic, S. Radenovic, A common fixed point theorem of Fisher in b-metric spaces, RACSAM, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 113 (2019), 949?956.

[9] M. Jovanovic, Z. Kadelburg, S. Radenovic, Common fixed point results in metric-type spaces, Fixed Point Theory Appl. 2010 (2010), 978121.

[10] L. Liu, F. Gu, Common fixed point theorems for six self-maps in *b*-metric spaces with nonlinear contractive conditions, J. Nonlinear Sci. Appl. 9 (2016), 5909-5930.