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PHASE RETRIEVAL REVISITED

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Abstract. The main conclusion of this paper is the necessary and sufficient condition of the phase retrieval problem in general real normed space. The main tool is to prove

$$\{\|x+\varphi_n\|,\|x-\varphi_n\|\}=\{\|y+\varphi_n\|,\|y-\varphi_n\|\}$$

holds and equivalent conditions of the above equation. So we can find the equivalent condition of phase retrieval.

Keywords: phase retrieval; norm; $l^p(\Gamma)$ space.

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1. Introduction

In history, X-ray crystal structure analysis is the first important application of phase retrieval[1]. In addition, phase retrieval is widely used in signal recovery, astronomy, holographic imaging, adaptive optics, differential geometry and other fields. Therefore, phase retrieval is worth studying from different perspectives. It not only strengthens the understanding and application of phase retrieval, but also provides new ideas and methods to solve the problem of phase retrieval.

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Given a separable Hilbert space H, phase retrieval deals with the problem of recovering an unknown $f \in H$ from a set of intensity measurements $(|\langle f, \varphi_n \rangle|)_{n \in I}$ for some countable collection $\Phi = \{\varphi_n\}_{n \in I} \subseteq H$. Note that if $f = \alpha g$ with $|\alpha| = 1$, then $|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|$ for every $n \in I$ regardless of our choice of Φ; we say Φ does phase retrieval if the converse of statement is true, i.e., if the equalities $|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|$ for every n imply that there is a unimodular scalar α so that $f = \alpha g$.

In addition, the frame is a redundant vector system in Hilbert space. It satisfies the ideal reconstruction characteristic that any vector in Hilbert space can be synthesized by its inner product and frame vector. Therefore, the frame plays an critical role in the phase retrieval. We will generally assume that Φ forms a frame for a separable Hilbert space H, i.e., there are positive constants $0 < A \leq B < \infty$ so that

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x, \varphi_n \rangle|^2 \leq B\|x\|^2$$

for every x in H.

Consider a Hilbert space H, a finite sequence $F = \{f_i\}_{i=1}^N$ of H is called a frame if there are two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x, f_i \rangle|^2 \leq B\|x\|^2$$

holds for every $x \in H$, where A, B are the frame bounds and the numbers $\langle x, f_i \rangle$ are called frame coefficients.

There is a well-known necessary and sufficient condition for phase retrieval frame. It is given in terms of the so -called "complement property": We say a frame $F = \{f_i\}_{i=1}^N$ for a Hilbert space h has the complement property if for every $S \subseteq \{1, ..., N\}$ we have $\text{span}\{f_i\}_{i \in S} = H$ or $\text{span}\{f_i\}_{i \notin S} = H$

Theorem 1.1 (a) Let H be a separable Hilbert space and let Φ be a frame for H. If Φ does phase retrieval, then Φ has the complement property.

(b) Let H be a separable Hilbert space over the real numbers and let Φ be a frame for H. If Φ has the complement property, then Φ does phase retrieval.

This proposition was originally proved in [3] where it was only stated in the finite-dimensional case, but the proof still holds in infinite dimensions without any modifications. For further generalizations of this fundamental result, we mention the papers [2],[4],[5],[6],[7].

Problem 1. What is the necessary and sufficient condition for phase retrieval to hold when the space is a real $l^p(\Gamma)(1<p<\infty, p\neq 2)$ space?

In order to better solve the above problems, we will first study the following equation

$$\{\|x+\varphi_n\|,\|x-\varphi_n\|\}=\{\|y+\varphi_n\|,\|y-\varphi_n\|\} \quad (1)$$

Where $\Phi=\{\varphi_n\}_{n\in N}$ is the frame of real $l^p(\Gamma)$ space and x,y comes from the same space.

Indeed, if $\alpha\in\{1,-1\}$ and $x=\alpha y$, then

$$\|x\pm\varphi_n\|=\|\alpha y\pm\varphi_n\|=\|y\pm\alpha\varphi_n\|$$

which implies (1) because α is either equal to 1 or to -1.

2. Main Results

Throughout the remaining part of this paper, X denote real $l^p(\Gamma)$ space, $\Phi=\{\varphi_n\}_{n\in N}$ is the frame of X. The $l^p(\Gamma)(1<p<\infty, p\neq 2)$ space is

$$l^p(\Gamma)=\{x=\sum_\gamma \xi_\gamma e_\gamma : \|x\|=(\sum_\gamma|\xi|^p)^{\frac{1}{p}}<\infty, \xi_\gamma\in\mathbb{R}, \gamma\in\Gamma\}$$

Theorem 2.1. Let $\{\varphi_n\}_{n\in N}$ be a frame for X, the following three statements are equivalent:

(i) $\|x+\varphi_n\|=\|y+\varphi_n\|$

(ii) $\langle x,\varphi_n\rangle=\langle y,\varphi_n\rangle$

(iii) $x=y$

Proof. Suppose first that (1) holds. Because X is a real space, then it follows that

$$x=y$$

Now using (i), we get

$$\|x+\varphi_n\|^2=\|x\|^2+\|\varphi_n\|^2+2\langle x,\varphi_n\rangle$$

$$\|y+\varphi_n\|^2=\|y\|^2+\|\varphi_n\|^2+2\langle y,\varphi_n\rangle$$

which proves(ii)

Now suppose (ii). $\langle x,\varphi_n\rangle=\langle y,\varphi_n\rangle$, then $\langle x-y,\varphi_n\rangle=0$. Also $\{\varphi_n\}_{n\in N}$ is a frame we know that

$$A\|x-y\|^2\leq\sum_{n\in N}|\langle x-y,\varphi_n\rangle|^2\leq B\|x-y\|^2 \ (0<A\leq B<\infty)$$

so $\|x-y\|=0$, i.e. $x=y$.

The statement (iii) implies (i) obviously.

In the following theorem, we list four equivalent condition that are equivalent to (1).

Theorem 2.2. Let $\{\varphi_n\}_{n\in N}$ be a frame for X, the following five statements are equivalent:

(i) (1) holds;

(ii) $\|x+\varphi_n\|+\|x-\varphi_n\| = \|y+\varphi_n\|+\|y-\varphi_n\|$

(iii) $\|x+\varphi_n\|\|x-\varphi_n\| = \|y+\varphi_n\|\|y-\varphi_n\|$

(iv) $|\langle x,\varphi_n\rangle| = |\langle y,\varphi_n\rangle|$

(v) There exists a number $\alpha \in \{1,-1\}$ such that $x = \pm y$

Proof. The statement (i) implies (ii) obviously. It follows from (ii) that $\|x\| = \|y\|$ (2). Now we square the equation in (ii) to obtain

$$\|x\|^2+\|\varphi_n\|^2+2\langle x,\varphi_n\rangle+\|x\|^2+\|\varphi_n\|^2-2\langle x,\varphi_n\rangle+2\|x+\varphi_n\|\|x-\varphi_n\|$$

$$=\|y\|^2+\|\varphi_n\|^2+2\langle y,\varphi_n\rangle+\|y\|^2+\|\varphi_n\|^2-2\langle y,\varphi_n\rangle+2\|y+\varphi_n\|\|y-\varphi_n\|$$

so the above equality simplifies to the equality in (iii). Thus (ii) implies (iii).

Now suppose (iii). Squaring the second equation in (iii),we obtain that

$$(\|x+\varphi_n\|\|x-\varphi_n\|)^2 = (\|x\|^2+2\langle x,\varphi_n\rangle+\|\varphi_n\|^2)(\|x\|^2-2\langle x,\varphi_n\rangle+\|\varphi_n\|^2)$$

$$(\|y+\varphi_n\|\|y-\varphi_n\|)^2 = (\|y\|^2+2\langle y,\varphi_n\rangle+\|\varphi_n\|^2)(\|y\|^2-2\langle y,\varphi_n\rangle+\|\varphi_n\|^2)$$

we can also get (2) from (iii), this simplifies to

$$(\langle x,\varphi_n\rangle)^2 = (\langle y,\varphi_n\rangle)^2$$

If (iv) holds, we can obtain $\langle x,\varphi_n\rangle = \pm\langle y,\varphi_n\rangle$, i.e.$\langle x\pm y,\varphi_n\rangle = 0$.

Since the frame bounds $A,B>0$, so it can only be $\|x\pm y\| = 0$

Then $x = \pm y$

Finally, (v) implies (i) as we have seen it in the introduction.

Corollary. Let $\{\varphi_n\}_{n\in N}$ be a frame for X. The four equivalent statements $(i)-(iv)$ of Theorem 2 hold if and only if $\Phi = \{\varphi_n\}_{n\in N}$ does phase retrieval.

Proof. Assume that $\Phi = \{\varphi_n\}_{n\in N}$ be a frame satisfying any of the condition $(i)-(iv)$ of Theorem 2. So we can get $x = \alpha y$ from $|\langle x,\varphi_n\rangle| = |\langle y,\varphi_n\rangle|$, where $\alpha \in \{1,-1\}$.

If $\Phi = \{\varphi_n\}_{n\in N}$ does phase retrieval ,then obviously the four statements $(i)-(iv)$ hold.

Finally, we formulate a open problem.

Problem 2. Look for a set of $\Phi = \{\varphi_n\}_{n\in S}$ in the $l^p(\Gamma)(1 < p < \infty, p \neq 2)$ space, such that the following conditions are satisfied for $\forall x, y \in S(l^p(\Gamma))$:

$$\{\|x+\varphi_n\|, \|x-\varphi_n\|\} = \{\|y+\varphi_n\|, \|y-\varphi_n\|\}$$

then $x = \alpha y$, where $|\alpha| = 1$.

This problem can be regarded as the problem of phase recovery in $l^p(\Gamma)$ space, which needs further study.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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