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SOME COMMON FIXED POINT THEOREMS IN $\mathcal{F}$-METRIC SPACES

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Abstract. In this paper, we prove a new common fixed point theorem for four mappings in $\mathcal{F}$-metric spaces and give its results. Our main results are $\mathcal{F}$-metric version of Fisher's main theorems [5] and [6]. Furthermore, our main result is a generalization of Theorem 4 in [12].

Keywords: fixed point; $\mathcal{F}$-metric spaces; commuting mappings.

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1. Introduction and Preliminaries

Fixed point theorems are tools in many fields in mathematics, physics, and computer science. The notion of metric spaces has been generalized by several authors, such as Czerwik [4], Khamsi and Hussain [9], Mlaiki et al. [13], Abdeljawad et al. [1], and so on. In 2018, Jleli and Samet [7] introduced a new concept, named an $\mathcal{F}$-metric space, as a generalization of the notion of a metric space. Mitrovi'c et al. [12] proved certain common fixed point theorems in $\mathcal{F}$-metric spaces. They obtained the well results of Banach, Jungck, Reich, and Berinde in these spaces. Many authors studied the concept of $\mathcal{F}$-metric spaces and its applications [2, 3, 10, 11, 14].

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Let $\mathcal{F}$ be the set of function $f:(0,+\infty)\longrightarrow\mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0<s<t\Longrightarrow f(s)\leq s(t)$.

($\mathcal{F}_2$) For every sequence $\{t_n\}\subset(0,+\infty)$, we have

$$\lim_{n\longrightarrow+\infty}t_n=0\Longleftrightarrow\lim_{n\longrightarrow+\infty}f(t_n)=-\infty.$$

We generalize the concept of metric spaces as follows:

Definition 1.1. ([7]) Let X be a nonempty set, and let $D:X\times X\longrightarrow[0,+\infty)$ be a given mapping. Suppose that there exist $(f,\alpha)\in\mathcal{F}\times[0,+\infty)$ such that

(D1) $(x,y)\in X\times X$, $D(x,y)=0\Longleftrightarrow x=y$.

(D2) $D(x,y)=D(y,x)$, for all $(x,y)\in X\times X$.

(D3) For every $(x,y)\in X\times X$, for every $N\in\mathbb{N}$, $N\geq2$, and for every $(u_i)_{i=1}^N\subset X$ with $(u_1,u_N)=(x,y)$, we have

$$D(x,y)>0\Longrightarrow f\Big(D(x,y)\Big)\leq f\Big(\sum_{i=1}^{N-1}D(u_i,u_{i+1})\Big)+\alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X,D) is said to be an F-metric space.

Observe that any metric on X is an $\mathcal{F}$-metric on X. Indeed, if d is a metric on X, then it satisfies $(D1)$ and $(D2)$. On the other hand, by the triangle inequality, for every $(x,y)\in X\times X$, for every $N\in\mathbb{N}$, $N\geq2$, and for every $(u_i)_{i=1}^N\subset X$ with $(u_1,u_N)=(x,y)$, we have

$$d(x,y)\leq\sum_{i=1}^{N-1}d(u_i,u_{i+1}),$$

which yields,

$$d(x,y)>0\Longrightarrow\ln\Big(d(x,y)\Big)\leq\ln\Big(\sum_{i=1}^{N-1}d(u_i,u_{i+1})\Big)$$

Then d satisfies $(D3)$ with $f(t)=\ln t$, $t>0$, and $\alpha=0$.

In the following, some example of $\mathcal{F}$-metric spaces which are not metric spaces are presented.

Definition 1.2. ([7]) Let $\{x_n\}$ be sequence in an $\mathcal{F}$-metric space (X,D). Then

(i) $\{x_n\}$ is $\mathcal{F}$-convergent $x \in X$ if $\{x_n\}$ is convergent to x with respect to the $\mathcal{F}$-metric D; that is,

$$\lim_{n\to+\infty} D(x_n,x) = 0.$$

(ii) $\{x_n\}$ is $\mathcal{F}$-Cauchy if $\lim_{n,m\to+\infty} D(x_n,x_m) = 0$.

(iii) (X,D) is $\mathcal{F}$-complete if each $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some element in X.

Definition 1.3. ([8]) Let K be a mapping of a set X into itself. Then K has a fixed point if there is a constant mapping $L : X \longrightarrow X$ which commutes with K $\Big($ i.e., $L\Big(K(x)\Big) = K\Big(L(x)\Big)$ for all $x \in X\Big)$.

Example 1.4. ([7]) Let $X = \mathbb{N}$, and let $D : X \times X \longrightarrow [0+\infty)$ be the mapping defined by

$$D(x,y) = \begin{cases} (x-y)^2, & (x,y) \in [0,3]\times[0,3] \\ |x-y|, & (x,y) \notin [0,3]\times[0,3] \end{cases}$$

for all $(x,y) \in X \times X$. It can be easily seen that D satisfies $(D1)$ and $(D2)$. However, D doesn't satisfy the triangle inequality, since

$$d(1,3) = 4 > 1+1 = d(1,2)+d(2,3).$$

Hence, D is not a metric on X. But, D is an $\mathcal{F}$-metric on X.

Definition 1.5. ([7]) Let X be a nonempty set, and let $D : X \times X \longrightarrow [0,+\infty)$ be a given mapping satisfying $(D1)$ and $(D2)$. We say that the pair (X,D) is $\mathcal{F}$-metric bounded with respect to $(f,\alpha) \in \mathcal{F} \times [0,+\infty)$, if there exists a metric d on X such that

$$(x,y) \in X \times X,\ D(x,y) > 0 \Rightarrow f\Big(d(x,y)\Big) \le f\Big(D(x,y)\Big) \le f\Big(d(x,y)\Big) + \alpha.$$

Proposition 1.6. ([7]) *Let (X,D) be an $\mathcal{F}$-metric space. Let $\{x_n\}$ be a sequence in X, and $x \in X$. The following statements are equivalent:*

(i) $\{x_n\}$ *is $\mathcal{F}$-convergent to x.*

(ii) $\lim_{n\to+\infty} D(x_n,x) = 0$.

Definition 1.7. ([7]) Let (X,D) be an $\mathcal{F}$-metric space. Let $\{x_n\}$ be a sequence in X.

(i) We say they $\{x_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{m,n\to+\infty} D(x_n,x_m)=0.$$

(ii) We say that (X,D) is $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a certain element in X.

2. $\mathcal{F}$-Metric Version of Fisher's Fixed Point Theorems

Theorem 2.1. *Let K and L be commuting mappings an L and M be commuting mappings of a $\mathcal{F}$-complete (X,D) into self satisfying*

$$D(Kx,Ny)\leq \lambda D(Lx,My) \qquad (1)$$

for all x, $y\in X$, where $0\leq\lambda<1$. If $Kx\in M(X)$ and $Nx\in L(X)$ for each $x\in X$ and if L and M are continuous, then all K, L, N and M have unique common fixed point.

Proof. Let $x_0\in X$ be arbitrary. Since $Kx_0\in M(X)$ there is some $x_1\in X$ such that $Mx_1=Kx_0$ and also as $Nx\in L(X)$, there is some $x_2\in X$ such that $Lx_2=Nx_1$.
In general $x_{2k+1}\in X$ is chosen such that $Mx_{2k+1}=Kx_{2k}$ and $x_{2k+2}\in X$ such that
$Lx_{2k+2}=Nx_{2k+1},\ k=0,1,2,...$
We denote a sequence such that

$$y_{2k}=Mx_{2k+1}=Kx_{2k}$$

$$y_{2k+1}=Lx_{2k+2}=Nx_{2k+1} \quad k\geq 0.$$

Now, we show that $\{y_n\}$is a Cauchy sequence. Let $(f,\alpha)\in\mathcal{F}\times[0,+\infty)$ be such that $(D3)$ is satisfied. Let $\varepsilon>0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta>0$ such that

$$0<t<\delta\Rightarrow f(t)<f(\varepsilon)-\alpha. \qquad (2)$$

From (1) we have

$$\begin{aligned} D(y_{2k}, y_{2k+1}) = D(Kx_{2k}, Nx_{2k+1}) &\leq \lambda D(Lx_{2k}, Mx_{2k+1}) \\ &= \lambda D(y_{2k-1}, y_{2k}) \\ &\leq \lambda^2 D(y_{2k-1}, y_{2k-2}). \end{aligned}$$

Repeating this argument n-times we get

$$\begin{aligned} D(y_{2k}, y_{2k+1}) &\leq \lambda D(y_{2k-1}, y_{2k}) \\ &\leq \lambda^2 D(y_{2k-2}, y_{2k-1}) \\ \ldots &\leq \lambda^n D(y_0, y_1). \end{aligned}$$

Which yields

$$\sum_{i=n}^{m-1} D(y_i, y_{i+1}) \leq \frac{\lambda^n}{1-\lambda} D(y_0, y_1), \quad m > n.$$

Since

$$\lim_{n \to +\infty} \frac{\lambda^n}{1-\lambda} D(y_0, y_1) = 0.$$

There exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} D(y_0, y_1) < \delta, \quad n \geq N.$$

Hence by (2) and $(\mathcal{F}_1)$ we have

$$f\Big(\sum_{i=n}^{m-1} D(y_i, y_{i+1})\Big) \leq f\Big(\frac{\lambda^n}{1-\lambda} D(y_0, y_1)\Big) < f(\varepsilon) - \alpha. \quad m > n > N. \tag{3}$$

Using $(D3)$ and (3), we obtain

$$D(y_n, y_m) > 0, \ (m > n > N) \Rightarrow f\Big(D(y_n, y_m)\Big) \leq f\Big(\sum_{i=n}^{m-1} D(y_i, y_{i+1})\Big) + \alpha < f(\varepsilon)$$

which implies by $(\mathcal{F}_1)$ that

$$D(y_n, y_m) < \varepsilon, \quad m > n \geq N.$$

This proves than $\{y_n\}$ is Cauchy sequence. From $\mathcal{F}$-completeness of (X, D), there exists, $u \in X$ such that

$$\lim_{k \to +\infty} Kx_{2k} = \lim_{k \to +\infty} Mx_{2k+1} = \lim_{k \to +\infty} Nx_{2k+1} = \lim_{k \to +\infty} Lx_{2k+2} = u.$$

Since L continuous and K and N commute,

$$Lu = L(\lim_{k\to+\infty} Kx_{2k}) = \lim_{k\to+\infty} LKx_{2k} = \lim_{k\to+\infty} KLx_{2k} = K\lim_{k\to+\infty} Lx_{2k} = Ku.$$

Since M continuous and N and M commute,

$$\begin{aligned} Mu = M(\lim_{k\to+\infty} Nx_{2k+1}) &= \lim_{k\to+\infty} MNx_{2k+1} \\ &= \lim_{k\to+\infty} NMx_{2k+1} \\ &= N(\lim_{k\to+\infty} Mx_{2k+1}) \\ &= Nu. \end{aligned}$$

$$\begin{aligned} D(Ku,Nu) &\leq \lambda D(Lu,Mu) \\ &\leq \lambda D(Ku,Nu) \\ &\Rightarrow Ku = Nu. \end{aligned}$$

$$\begin{aligned} D(KLx_{2k},Nx_{2k+1}) &\leq \lambda D(L^2x_{2k},Mx_{2k+1}) \\ &\Rightarrow D(Lu,u) \leq \lambda D(Lu,u) \\ &\Rightarrow Lu = u \\ &\Rightarrow Ku = Lu = Nu = Mu = u. \end{aligned}$$

Now, we show that the uniqueness of common fixed point. We assume that v is another common fixed point of L, K, N and M such that $u = v$. That is $Ku = Lu = Nu = Mu = u$ and $Kv = Lv = Nv = Mv = v$. If thwn $D(u,v) > 0$. From (1) we get

$$\begin{aligned} D(u,v) &= D(Ku,Nv) \\ &\leq \lambda D(Lu,Mv) \\ &= \lambda D(u,v). \end{aligned}$$

Which is a contradiction and so the fixed point is unique. □

Corollary 2.2. *Theorem* 2.1 *is* F*-metric version of the main theorem of* [6]*. Then we get a generalization of the main result of* [6].

Corollary 2.3. *Our main result is a generalization of Theorem* 4 *in* [12]. *Note that condition* (1) *implies that K is a continuous map. To obtain Theorem* 4 *in* [12], *it suffices to take* $K = N$ *and* $L = M = I$ *(I = Identity map) in Theorem* 2.1.

Theorem 2.4. (X,D) *be an* $\mathcal{F}$*-metric space, and let* $K : X \longrightarrow X$ *be an given mapping suppose that the following conditions are satisfying*

(i) (X,D) *is* $\mathcal{F}$*-complete,*

(ii) There exists $a \leq b,c,b+c < 1$ *such that*

$$D(Kx,Ky) \leq bD(x,y) + c\max\{D(x,Kx),D(y,Ky)\} \qquad (4)$$

for all x, y in X, then K has a unique fixed point.

Proof. Let $(f,\alpha) \in \mathcal{F} \times [0,+\infty)$ be such that $(D3)$ is satisfied. Let $\varepsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha \qquad (5)$$

From (4) have

$$D(K^n x, K^{n+1}x) \leq bD(K^{n-1}x, K^n x) + c\max\{D(K^{n-1}x, K^n x), D(K^n x, K^{n+1}x)\}$$

and so either

$$D(K^n x, K^{n+1}x) \leq (b+c)D(K^{n-1}x, K^n x)$$

or

$$D(K^n x, K^{n+1}x) \leq \frac{b}{1-c}D(K^{n-1}x, K^n x).$$

Putting

$$\lambda = \max\{(b+c), \frac{b}{1-c}\} < 1.$$

Repeating this argument n-times we get

$$D(K^n x, K^{n+1}x) \leq \lambda D(K^{n-1}x, K^n x) + \lambda^2 D(K^{n-2}x, K^{n-1}x) + \ldots + \lambda^n D(x, Kx)$$

which yields

$$\sum_{i=n}^{m-1} D(K^i x, K^{i+1}x) \leq \frac{\lambda^n}{1-\lambda} D(x, Kx), \quad m > n.$$

Since $\lim_{n\to+\infty}\frac{\lambda^n}{1-\lambda}D(x,Kx)=0$, then exists some $N\in\mathbb{N}$ such that

$$0<\frac{\lambda^n}{1-\lambda}D(x,Kx)<\delta,\ n\geq N. \qquad (6)$$

From (5) and $(\mathcal{F}_1)$ we have

$$f\Big(\sum_{i=n}^{m-1}D(K^ix,K^{i+1}x)\Big)\leq f\Big(\frac{\lambda^n}{1-\lambda}D(x,Kx)\Big)<f(\varepsilon)-\alpha, \quad m>n\geq N. \qquad (7)$$

Using $(D3)$ and (7), we obtain

$$D(K^nx,K^mx)<\varepsilon, \quad m>n\geq N.$$

This proves that $\{K^nx\}$ is $\mathcal{F}$-Cauchy. Since (X,D) is $\mathcal{F}$-complete, there exists $z\in X$ such that $\{K^nx\}$ is $\mathcal{F}$-convergent to z, i.e.

$$\lim_{n\to+\infty}D(K^nx,z)=0. \qquad (8)$$

We shall prove that z is a fixed point of K. We argue by contradiction by supposing that

$$D(Kz,z)>0.$$

By $(D3)$, we have

$$f\Big(D(Kz,z)\Big)\leq f\Big(D(Kz,K^nx)+D(K^nx,z)\Big)+\alpha,\ n\in\mathbb{N}.$$

Using (ii) and $(\mathcal{F}_1)$, we obtain

$$f\Big(D(Kz,z)\Big)\leq f\Big(\lambda D(z,K^nx)+D(K^{n+1}x,z)\Big)+\alpha,\ \ n\in\mathbb{N}.$$

On the other hand, using $(\mathcal{F}_2)$ and (8), we have

$$\lim_{n\to+\infty}f\Big(\lambda D(z,K^nx)+D(K^{n+1}x,z)\Big)+\alpha=-\infty,$$

which is a contradiction. Therefore, we have

$$D(Kz,z)=0$$

i.e.

$$Kz=z.$$

As a consequence, $z \in X$ is the unique fixed point z. That at most one fixed point. Indeed, if $(u,v) \in X \times X$ are

$$D(u,v) > 0, \quad Ku = u, \quad Tv = v.$$

Then from (ii), we have

$$\begin{aligned} D(u,v) = D(Ku,Kv) &\leq bD(u,v) + c\max\{D(u,Ku), D(v,Tv)\} \\ &\leq (b+c)D(u,v) \\ &< D(u,v) \end{aligned}$$

which is a contradiction. □

Corollary 2.5. *Theorem* 2.4 *is F-metric version of the main theorem of* [5]. *Then we get a generalization of the main result of* [5].

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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