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FIXED POINT RESULTS FOR GENERALIZED GERAGHTY AND QUASI (*α,β*)-ADMISSIBLE TYPE CONTRACTION MAPPINGS IN b-METRIC-LIKE SPACES

MUSTAFA MUDHESH[1,*], MUHAMMAD ARSHAD[1], ESKANDAR AMEER[2]

[1]Department of Mathematics, International Islamic University, H-10, Islamabad-44000, Pakistan

[2]Department of Mathematics, Taiz University, Taiz, Yemen

Abstract: In this paper, we extend the concept of generalized (α,β)-quasi contraction and obtain some fixed point theorems for such contractions on b-metric like spaces. We introduce the notion of a pair of generalized (α,β)-Geraghty type contractive mappings and establish some new common fixed point theorems satisfying a pair of generalized (α,β)-Geraghty type contractive mapping in complete b-metric-like space. Examples are included to illustrate that our result is a proper generalization of previous results. An application is also provided.

Keywords: fixed point; b-metric like space; generalized (α,β)-quasi contraction; generalized (α,β)-Geraghty type contractive mapping.

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1. INTRODUCTION AND PRELIMINARIES

In 1922 the Banach contraction principle [1]. was appeared in explicit form in Banach's thesis to

*Corresponding author
E-mail address: mustfa.rajh@gmail.com
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be a fundamental result in metric fixed point theory, which has applications in many branches of mathematical analysis. Later, Banach contraction principle was improved and generalized in different spaces. Many authors obtained interesting generalized results of the classical Banach contraction principle in metric spaces see[3, 4, 6, 7, 8, 12, 14, 15]. On the other hand, in 1993, the concept of a b-metric space was introduced by Czerwik [5] as generalization of the metric space with introducing new fixed point results, Pant and Panicker [2] extended some fixed point theorems from metric spaces to b-metric spaces. In 2013, Alghamdi et al [9] generalized the notion of a b-metric space by introduction of the concept of a b-metric-like space and proved some related fixed point results. After that, Hussain et al. [10], Chen et al [11] and Aydi et al. [13] proved some fixed point theorems in the setting of b-metric-like spaces. Geraghty [12] and Ćirić [3] obtained two important results which generalized the Banach Contraction principle in metric spaces. Pant and Panicker [2] extended results of Geraghty and Ćirić. [12, 3] and established new fixed point theorems for such contractive mappings in b-metric spaces.

In this paper, we extend and generalize the results of Pant and Panicker [2] into Generalized (α,β)-Geraghty type contractive and (α,β)-quasi contractive mappings in b-metric-like spaces.

Definition 1.1 [5]. Let X be a non-empty set and $d: X\times X \to [0,\infty)$ be a function. Then d is called a b-metric on X, if for all $x,y,z \in X$;

(1) $d(x,y) = 0$ *if and only if* $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq s[\, d(x,z) + d(z,y)\,]$, *where* $s>1$.

Definition 1.2 [9]. Let X be a non-empty set and $d:X\times X\to [0,\infty)$ be a function, called a b-metric-like if there exists a real number $s \geq 1$ such that the following conditions hold for every $x,y,z \in X$,

(1) $d(x,y) = 0$ *then* $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq s[\, d(x,z) + d(z,y)\,]$.

Example 1.3 [13]. Let $X = [0,\infty)$ and $d:X\times X\to[0,\infty)$ defined by $d(x,y) = (x^3+y^3)^2$, then (X,d) is a b-metric-like space with $s = 2$, but is not a b-metric space since $d(1,1) = 4$.

Definition 1.4 [3, 6]. Let X be a metric space and $T:X\to X$ be a self-mapping. For $A \subseteq X$, let $\delta(A) = sup\{ d(a,b): a,b \in A \}$ and for each $x \in X$, let $O(x,n) = \{ x,Tx,T^2x,\ldots,T^nx \}$, $n = 0,1,2,\ldots$, $O(x,\infty) = \{ x,Tx,T^2x,\ldots\}$. The set $O(x,\infty)$ is called the orbit of T and the metric space X is said to be T-orbitally complete, if every Cauchy sequence in $O(x,\infty)$ is convergent in X.

Definition 1.5 [9]. Let (X,d) be a b-metric-like space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{ x_n \}$ converges to x if and only if $\lim_{n\to\infty} d(x_n,x) = d(x,x)$.

Remark 1.6 [9]. In a b-metric-like space, the limit for a convergent sequence is not unique in general.

Definition 1.7 [9]. Let (X,d) be a b-metric-like space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n,x_m)$ exists and is finite.

Definition 1.8 [9]. Let (X,d) be a b-metric-like space. We say that (X,d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 1.9 [14]. Let $\alpha: X\times X \to [0,\infty)$ be a functional. A mapping $T: X \to X$ is said to be α-admissible, if for all $x,y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Definition 1.10 [15]. The mapping $T: X\to X$ is said to be triangular α-admissible, if for all $x,y,z \in X$,

(i) $\alpha(x,y)\geq 1$ implies $\alpha(Tx,Ty)\geq 1$;

(ii) $\alpha(x,z)\geq 1$ and $\alpha(z,y)\geq 1$ implies $\alpha(x,y)\geq 1$.

Definition 1.11. [8]. Let $S,T: X\to X$ and $\alpha: X\times X\to[0,\infty)$. We say that the pair (S,T) is α-admissible if $x,y \in X$ such that $\alpha(x,y)\geq 1$, then we have $\alpha(Sx,Ty)\geq 1$ and $\alpha(Tx,Sy)\geq 1$.

Definition 1.12. [8]. Let $S,T: X\to X$ and $\alpha: X\times X\to[0,\infty)$. We say that a pair (S,T) is triangular α-admissible if

(i) $\alpha(x,y)\geq 1$, implies $\alpha(Sx,Ty)\geq 1$ and $\alpha(Tx,Sy)\geq 1$, $x,y \in X$.

(ii) $\alpha(x,z) \geq 1$, and $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$, $x,y,z \in X$.

Theorem 1.13 [3]. Let X be a T-orbitally complete metric space and $T: X\to X$ be a quasi-contraction, that is, there exists a real number $q \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq q.m(x,y),$$

where

$$m(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

Then T has a unique fixed point $x^* \in X$.

Theorem 1.14 [6]. Let X be a T-orbitally complete metric space and $T:X\to X$ be a generalized quasi-contraction, that is, there exists a real number $q\in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty)\leq q.M(x,y),$$

where,

$$M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),d(T^2x,x),d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\}.$$

Then T has a unique fixed point $x^*\in X$.

Theorem 1.15 [12]. Let (X,d) be a complete metric space and $T: X\to X$ be a mapping such that for all $x,y \in X$,

$$d(Tx,Ty)\leq \beta(d(x,y))d(x,y),$$

where β: $[0,\infty)\to[0,1)$ is a function satisfying $\beta(t_n)\to 1$ implies $t_n\to 0$ as $n\to\infty$. Then T has a unique fixed point $x^* \in X$.

Definition 1.16. [2]. Let X be a b-metric space. A mapping $T: X\to X$ is said to be generalized α-quasi contraction, if there exists a functional $\alpha: X\times X\to[0,\infty)$ and $q< \frac{1}{s^2}$ such that

$$\alpha(x,y)d(Tx,Ty)\leq qM(x,y),$$

where

$$M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),d(T^2x,x),d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\}$$

Lemma 1.17. [2]. Let X be a b-metric space and $T: X\to X$ be a generalized α-quasi contraction satisfying the following conditions:

(A) T is triangular α-admissible;

(B) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Then for all positive integers $i,j \in \{1,2,\cdots,n\},(i < j)$

$$d(T^i x_0,T^j x_0) \leq q.\delta[O(x_0,n)].$$

Theorem 1.18 [2]. Let X be a T-orbitally complete b-metric space (with constant $s \geq 1$) and $T: X \to X$ a generalized α-quasi contraction satisfying conditions (A) and (B) of Lemma 1.14. Then T has a fixed point in X.

Definition 1.19 [7]. Let X be a b-metric space, $T: X \to X$ and $\alpha,\beta: X \times X \to [0,\infty)$. The mapping T is said to be an (α,β)-admissible mapping, if $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ and $\beta(Tx,Ty) \geq 1$ for all $x,y \in X$.

Definition 1.20 [7]. Let $\alpha,\beta: X \times X \to [0,\infty)$. A b-metric space X is (α,β)-regular, if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$, $\alpha(x_n,x_{n+1}) \geq 1$ and $\beta(x_n,x_{n+1}) \geq 1$ for all n and there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\alpha(x_{nk},x_{nk+1}) \geq 1$, $\alpha(x_{nk},x_{nk+1}) \geq 1$ for all $k \in N$. Also $\alpha(x,Tx) \geq 1$, $\beta(x,Tx) \geq 1$.

Definition 1.21 [2]. Let X be a b-metric space, $T: X \to X$ and $\alpha,\beta: X \times X \to [0,\infty)$. A mapping T is said to be (α,β)-Geraghty type contractive mapping, if there exists $\theta \in \Theta$ such that for all $x,y \in X$, we have:

$$\alpha(x,Tx)\beta(y,Ty)\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)),$$

where,

$$N(x,y) = max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s}\}$$

and $\psi \in \Psi$. However the following class of functions are defined as

(1) Θ is a family of functions θ: $[0,\infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ implies $t_n \to 0$;

(2) Ψ is a family of functions ψ: $[0,\infty) \to [0,\infty)$ such that ψ is continuous, strictly increasing and $\psi(0) = 0$.

Theorem 1.22 [2]. Let (X,d) be a complete b-metric space, $T: X \to X$ and $\alpha,\beta: X \times X \to [0,\infty)$. Suppose the following conditions hold:

(A) T is an (α,β)-admissible mapping;

(B) T is an (α,β)-Geraghty type contractive mapping;

(C) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$;

(D) either T is continuous or X is (α,β)-regular. Then T has a unique fixed point.

2. GENERALIZED *(α,β)*-GERAGHTY TYPE CONTRACTIVE MAPPING

In this section we obtain some common fixed point theorems for a pair of generalized (α,β)-Geraghty type contractive mappings in complete b-metric-like spaces.

Definition 2.1. [16]. Let X be a non-empty set, $S,T: X\to X$ be two mappings and $\alpha,\beta: X\times X\to[0,\infty)$ be functions such that (S,T) is called a pair of (α,β)-admissible mappings, if $\forall\ x,y \in X$.

$$\alpha(x,y) \geq 1 \text{ and } \beta(x,y) \geq 1$$

implies,

$$\alpha(Sx,Ty) \geq 1,\ \alpha(Tx,Sy) \geq 1 \text{ and } \beta(Sx,Ty) \geq 1,\ \beta(Tx,Sy) \geq 1.$$

Example 2.2. Let $X = [0,\infty)$, and define the self mappings $S,T : X\to X$ and $\alpha,\beta : X\times X\to[0,\infty)$ by $Sx=2x,\ Tx=x^2\ \forall x,y \in X$,

$$\alpha(x,y) = \begin{cases} e^{2xy}, & if\ x,y > 0 \\ 0, & otherwise \end{cases} \quad \text{and} \quad \beta(x,y) = \begin{cases} a^{2xy}, & if\ x,y > 0,\ a > 1 \\ 0, & otherwise \end{cases}.$$

Then the pair (S,T) is (α,β)-admissible

Definition 2.3. Let (X,d) be a b-metric-like space and $\alpha,\beta : X\times X\to [0,\infty)$. X is said to be (α,β)-regular, if $\{x_n\}$ is a sequence in X such that $x_n\to x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1\ \forall$ $n\in\mathbb{N}$ and there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\alpha(x_{nk}, x_{nk+1}) \geq 1$, $\beta(x_{nk}, x_{nk+1}) \geq 1\ \forall$ $k\in\mathbb{N}$. Also $\alpha(x,Sx) \geq 1$ and $\beta(x,Tx) \geq 1$.

Definition 2.4. Let X be a b-metric-like space, $S,T : X\to X$ be two mappings and $\alpha,\beta : X\times X\to[0,\infty)$ be two functions. Then (S,T) is called a pair of generalized (α,β)-Geraghty type contractive mappings, if $\exists\ \theta \in \Theta$ and $\psi \in \Psi$ such that $\forall x,y \in X$

$$\alpha(x,Sx)\beta(y,Ty)\psi(s^3d(Sx,Ty)) \leq \theta(\psi(M(x,y)))\psi(M(x,y)), \tag{2.1}$$

where,

$$M(x,y)=max\{d(x,y),d(x,Sx),d(y,Ty),\frac{d(x,Ty)+d(y,Sx)}{4s}\}.$$

Theorem 2.5. Let (X,d) be a complete b-metric-like space, $\alpha,\beta: X\times X\to[0,\infty)$ be two functions and $S,T: X\to X$ be two mappings. Suppose the following conditions hold:

(1) (S,T) is a pair of (α,β)-admissible mappings;

(2) (S,T) is a pair of generalized (α,β)-Geraghty type contractive mappings;

(3) $\exists\ x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$;

(4) the pair (S,T) is continuous or X is (α,β)-regular.

Then S,T have a unique common fixed point in X.

Proof: By presumption $\exists\, x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by letting $x_1 \in X$ such that $x_1 = Sx_0,\ x_2 = Tx_1,\ x_3 = Sx_2,\ x_4 = Tx_3$. Continuing this process, we get

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{where } i = 0,1,2,\ldots$$

Since (S,T) is a pair of (α,β)-admissible, so

$$\alpha(x_0,Sx_0) = \alpha(x_0,x_1) \geq 1,$$

$$\alpha(Sx_0,Tx_1) = \alpha(x_1,x_2) \geq 1 \text{ and } \alpha(Tx_1,Sx_2) = \alpha(x_2,x_3) \geq 1$$

continuing this manner, we obtain,

$$\alpha(x_n,x_{n+1}) \geq 1 \ \forall n \geq 0.$$

Similarly,

$$\beta(x_n,x_{n+1}) \geq 1 \ \forall n \geq 0.$$

Also from (2.1), we have

$$\begin{aligned}\psi(d(x_{2i+1},x_{2i+2})) &= \psi(d(Sx_{2i},Tx_{2i+1}))\\ &\leq \psi(s^3\, d(Sx_{2i},Tx_{2i+1}))\\ &\leq \alpha(x_{2i},Sx_{2i})\beta(x_{2i+1},Tx_{2i+1})\ \psi\,(s^3\, d(Sx_{2i},Tx_{2i+1}))\\ &\leq \theta(\psi(M(x_{2i},x_{2i+1})))\psi(M(x_{2i},x_{2i+1})),\end{aligned}$$

where,

$$M(x_{2i},x_{2i+1}) = \max\left\{\begin{matrix} d(x_{2i},x_{2i+1}), d(x_{2i},Sx_{2i}), d(x_{2i+1},Tx_{2i+1}), \\ \frac{d(x_{2i},Tx_{2i+1})+d(x_{2i+1},Sx_{2i})}{4s}\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2}), \\ \frac{d(x_{2i},x_{2i+2})+d(x_{2i+1},x_{2i+1})}{4s}\end{matrix}\right\}.$$

Note that,

$$\begin{aligned}\frac{d(x_{2i},x_{2i+2})+d(x_{2i+1},x_{2i+1})}{4s} &\leq \frac{s[d(x_{2i},x_{2i+1})+3d(x_{2i+1},x_{2i+2})]}{4s}\\ &= \frac{d(x_{2i},x_{2i+1})+3d(x_{2i+1},x_{2i+2})}{4}\\ &\leq \max\{d(x_{2i},x_{2i+1}), d(x_{2i+1},x_{2i+2})\}.\end{aligned}$$

Consequently, we find that

$$max\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2})\} \le M(x_{2i},x_{2i+1}) \le max\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2})\},$$

therefore, we conclude that

$$M(x_{2i},x_{2i+1}) = \max\{d(x_{2i},x_{2i+1}),d(x_{2i+1},x_{2i+2})\}.$$

If $M(x_{2i},x_{2i+1}) = d(x_{2i+1},x_{2i+2})$, then

$$\begin{aligned}\Psi(d(x_{2i+1},x_{2i+2})) &\le \theta(\psi(M(x_{2i},x_{2i+1})))\psi(M(x_{2i},x_{2i+1}))\\ &\le \theta\left(\psi\big(M(x_{2i},x_{2i+1})\big)\right)\psi\big(d(x_{2i+1},x_{2i+2})\big)\\ &< \psi\big(d(x_{2i+1},x_{2i+2})\big),\end{aligned}$$

which is a contradiction, therefore $M(x_{2i},x_{2i+1}) = d(x_{2i},x_{2i+1})$. Now

$$\begin{aligned}\psi(d(x_{2i+1},x_{2i+2})) &\le \theta(\psi(M(x_{2i},x_{2i+1})))\psi(M(x_{2i},x_{2i+1}))\\ &\le \theta\left(\psi\big(M(x_{2i},x_{2i+1})\big)\right)\psi\big(d(x_{2i},x_{2i+1})\big)\\ &< \psi\big(d(x_{2i},x_{2i+1})\big),\end{aligned} \tag{2.2}$$

that is $d(x_{2i+1},x_{2i+2}) < d(x_{2i},x_{2i+1})$ $\forall n \in \mathbb{N}\cup\{0\}$. Since ψ is strictly increasing, the sequence $\{d(x_n,x_{n+1})\}$ is decreasing and bounded from below, so $\exists\, r > 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Now, we verify that $r = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, from (2.2), we have

$$\frac{\psi(d(x_{n+1},x_{n+2}))}{\psi(d(x_n,x_{n+1}))} \le \theta\left(\psi\big(M(x_n,x_{n+1})\big)\right) < 1 \tag{2.3}$$

Letting $n\to\infty$ in (2.3), we get

$$1 \le \lim_{n\to\infty}\theta\left(\psi\big(M(x_n,x_{n+1})\big)\right) \le 1.$$

That is, $\lim_{n\to\infty}\theta\left(\psi\big(M(x_n,x_{n+1})\big)\right) = 1$, implies that $\lim_{n\to\infty}\psi\big(M(x_n,x_{n+1})\big) = 0$, which yields that

$$r = \lim_{n\to\infty} M(x_n,x_{n+1}) = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \tag{2.4}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that $\{x_n\}$ is not Cauchy sequence. Then $\exists\ \varepsilon > 0$ and the subsequences $\{x_{nk}\}$ and $\{x_{mk}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{nk}, x_{mk}) \geq \epsilon, \tag{2.5}$$

and n_k is the smallest such that (2.5) holds, then we have

$$d(x_{nk-1}, x_{mk}) < \epsilon \tag{2.6}$$

By triangular inequality and from (2.4), (2.5), we obtain

$$\begin{aligned} \epsilon &\leq d(x_{nk}, x_{mk}) \\ &\leq s[d(x_{nk}, x_{nk-1}) + d(x_{nk-1}, x_{mk})] \\ &< s[d(x_{nk}, x_{nk-1}) + \epsilon]. \end{aligned} \tag{2.7}$$

Taking the upper limit as $k \to \infty$ in (2.7) and using (2.4), we get

$$\epsilon \leq \lim_{k\to\infty} supd(x_{nk}, x_{mk}) < s\epsilon. \tag{2.8}$$

Again, by triangular inequality

$$d(x_{nk}, x_{mk}) \leq s[d(x_{nk}, x_{nk+1}) + d(x_{nk+1}, x_{mk})], \tag{2.9}$$

and,

$$d(x_{nk+1}, x_{mk}) \leq s[d(x_{nk+1}, x_{nk}) + d(x_{nk}, x_{mk})]. \tag{2.10}$$

Taking the upper limit as $k \to \infty$ in (2.9) and applying (2.4), (2.8), we have

$$\epsilon \leq \lim_{k\to\infty} supd(x_{nk}, x_{mk}) \leq s \lim_{k\to\infty} supd(x_{nk+1}, x_{mk}),$$

which implies, $\frac{\epsilon}{s} \leq \lim_{k\infty} supd(x_{nk+1}, x_{mk})$, and taking the upper limit as $k \to \infty$ in (2.10), gives

$$\lim_{k\to\infty} supd(x_{nk+1}, x_{mk}) \leq s \lim_{k\to\infty} supd(x_{nk}, x_{mk}) \leq s.s\epsilon = s^2\epsilon.$$

Thus,

$$\frac{\epsilon}{s} \leq \lim_{k\infty} supd(x_{nk+1}, x_{mk}) \leq s^2\epsilon. \tag{2.11}$$

Similarly,

$$\frac{\epsilon}{s} \leq \lim_{k\infty} supd(x_{nk}, x_{mk+1}) = \lim_{k\infty} supd(x_{nk+1}, x_{mk+2}) \leq s^2\epsilon. \tag{2.12}$$

By triangular inequality, we have

$$d(x_{nk+1}, x_{mk}) \leq s[d(x_{nk+1}, x_{mk+1}) + supd(x_{mk+1}, x_{mk})], \tag{2.13}$$

taking the upper limit as $k\to\infty$ in (2.13), from (2.4), (2.11), we get

$$\frac{\epsilon}{s^2} \leq \lim_{k\to\infty} sup d(x_{nk+1}, x_{mk+1}). \tag{2.14}$$

By triangular inequality again

$$d(x_{nk+1}, x_{mk+1}) \leq s[d(x_{nk+1}, x_{nk}) + d(x_{nk}, x_{mk+1})],$$

taking the upper limit as $k\to\infty$ from (2.4) and (2.12)

$$\lim_{k\to\infty} sup d(x_{nk+1}, x_{mk+1}) \leq s^3\epsilon. \tag{2.15}$$

from (2.14) and (2.15), we get

$$\frac{\epsilon}{s^2} \leq \lim_{k\to\infty} sup d(x_{nk+1}, x_{mk+1}) \leq s^3\epsilon. \tag{2.16}$$

Since X is (α,β)-regular, so by (2.1), we have

$$\psi\big(s^3 d(x_{nk+1}, x_{mk+2})\big) \leq \psi\big(s^3 d(Sx_{nk}, Tx_{mk+1})\big) \tag{2.17}$$

$$\leq \alpha(x_{nk}, Sx_{nk})\beta(x_{mk+1}, Tx_{mk+1})\psi\big(s^3 d(Sx_{nk}, Tx_{mk+1})\big)$$

$$\leq \theta\left(\psi\big(M(x_{nk}, x_{mk+1})\big)\right)\psi\big(M(x_{nk}, x_{mk+1})\big),$$

where,

$$M(x_{nk}, x_{mk+1}) = max\left\{\begin{matrix} d(x_{nk}, x_{mk+1}), d(x_{nk}, Sx_{nk}), d(x_{mk+1}, Tx_{mk+1}) \\ \frac{d(x_{nk}, Tx_{mk+1}) + M(x_{mk+1}, Sx_{nk})}{4s} \end{matrix}\right\} \tag{2.18}$$

$$= max\left\{\begin{matrix} d(x_{nk}, x_{mk+1}), d(x_{nk}, x_{nk+1}), d(x_{mk+1}, x_{mk+2}) \\ \frac{d(x_{nk}, x_{mk+2}) + M(x_{mk+1}, x_{nk+1})}{4s} \end{matrix}\right\}.$$

Taking upper limit as $k\to\infty$ and using (2.4), (2.11), (2.12) and (2.16), we get

$$\frac{\epsilon}{s} = max\left\{\frac{\epsilon}{s}, \frac{\epsilon + \frac{\epsilon}{s^2}}{4s}\right\} \leq \lim_{k\to\infty} sup M(x_{nk}, x_{mk+1}) \leq max\left\{s^2\epsilon, \frac{s^2\epsilon}{2}\right\} = s^2\epsilon. \tag{2.19}$$

Similarly,

$$\frac{\epsilon}{s} = max\left\{\frac{\epsilon}{s}, \frac{\epsilon + \frac{\epsilon}{s^2}}{4s}\right\} \leq \lim_{k\to\infty} inf M(x_{nk}, x_{mk+1}) \leq max\left\{s^2\epsilon, \frac{s^2\epsilon}{2}\right\} = s^2\epsilon. \tag{2.20}$$

Hence, from (2.12) it follows

$$\psi(s^2\epsilon) = \psi\left(s^3\left(\frac{\epsilon}{s}\right)\right) = \psi\left(s^3 \lim_{k\to\infty} sup d(x_{nk+1}, x_{mk+2})\right)$$

$$\leq \alpha(x_{nk}, Sx_{nk})\beta(x_{mk+1}, Tx_{mk+1})\psi\left(s^3 \lim_{k\to\infty} sup d(x_{nk+1}, x_{mk+2})\right)$$

$$\leq \theta\left(\psi\left(\lim_{k\to\infty} sup M(x_{nk}, x_{mk+1})\right)\right)\psi\left(\lim_{k\to\infty} sup M(x_{nk}, x_{mk+1})\right)$$

$$\leq \theta(\psi(s^2\epsilon))\psi(s^2\epsilon)$$

$$< \psi(s^2\epsilon),$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\exists\, x^* \in X$ such that $x_n \to x$ as $n\to\infty$, $x_{nk+1} \to x^*$ and $x_{nk+2} \to x^*$ as $k\to\infty$. To show that $x^*=Sx^*=Tx^*$. First postulate that S and T are continuous. Then we have

$$x^* = \lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} Sx_{2k} = S\lim_{k\to\infty} x_{2k} = Sx^*.$$

Similarly,

$$x^* = \lim_{k\to\infty} x_{2k+2} = \lim_{k\to\infty} Tx_{2k+1} = T\lim_{k\to\infty} x_{2k+1} = Tx^*.$$

Thus,

$$x^* = Sx^* = Tx^*.$$

Hence the pair (S,T) has a common fixed point $x^* \in X$. Now, presume that X is (α,β)-regular. Then there is a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\alpha(x_{nk}, x_{nk+1}) \geq 1$and $\beta(x_{nk}, x_{nk+1}) \geq 1$ $\forall$ $k \in \mathbb{N}$ and $\alpha(x^*, Sx^*) \geq 1$ and $\beta(x^*, Tx^*) \geq 1$. Now by (2.1) with $x = x_{nk}$ and $y = x^*$, we get

$$\psi(d(x_{nk+1}, Tx^*)) = \psi(d(Sx_{nk}, Tx^*)) \tag{2.21}$$

$$\leq \alpha(x_{nk}, Sx_{nk})\beta(x^*, Tx^*)\psi(s^3 d(Sx_{nk}, Tx^*))$$

$$\leq \theta\left(\psi(M(x_{nk}, x^*))\right)\psi(M(x_{nk}, x^*)),$$

where,

$$M(x_{nk}, x^*) = max\left\{d(x_{nk}, x^*), d(x_{nk}, Sx_{nk}), d(x^*, Tx^*), \frac{d(x_{nk}, Tx^*) + d(x^*, Sx_{nk})}{4s}\right\}$$

$$= max\left\{d(x_{nk}, x^*), d(x_{nk}, x_{nk+1}), d(x^*, Tx^*), \frac{d(x_{nk}, Tx^*) + d(x^*, x_{nk+1})}{4s}\right\}$$

$$\leq max\left\{\begin{array}{c} d(x_{nk}, x^*), s[d(x_{nk}, x^*) + d(x_{nk+1}, x^*)], d(x^*, Tx^*), \\ \dfrac{d(x_{nk}, Tx^*) + d(x^*, Sx_{nk})}{4s} \end{array}\right\}.$$

Letting $k\to\infty$ in above inequality, we get

$$M(x_{nk},x^*)=max\left\{d(x^*,Tx^*),\frac{d(x^*,Tx^*)}{4s}\right\}$$
$$=d(x^*,Tx^*).$$

By taking the limit as $k\to\infty$ in (2.21), we get

$$\psi\big(d(x^*,Tx^*)\big)\le\lim_{k\to\infty}\theta\left(\psi\big(M(x_{nk},x^*)\big)\right)\psi\big(d(x^*,Tx^*)\big)<\psi\big(d(x^*,Tx^*)\big),$$

implies that $\lim_{k\to\infty}\theta\left(\psi\big(M(x_{nk},x^*)\big)\right)=1$. So we obtain $\lim_{k\to\infty}M(x_{nk},x^*)=0$, therefore $d(x^*,Tx^*)=0$ that is, $x^*=Tx^*$. Similarly, we conclude that $x^*=Sx^*$. Hence the pair (S,T) has a common fixed point $x^*\in X$.

For uniqueness postulate that x^* and y^* are two common fixed point of S,T such that $x^*\neq y^*$ and $\alpha(x^*,Sx^*)\ge 1$, $\alpha(y^*,Sy^*)\ge 1$ and $\beta(x^*,Tx^*)\ge 1$, $\beta(y^*,Ty^*)\ge 1$. By (2.1), we have

$$\psi\big(d(x^*,y^*)\big)=\psi\big(d(Sx^*,Ty^*)\big)$$
$$\le\alpha(x^*,Sx^*)\beta(y^*,Ty^*)\psi\big(s^3d(Sx^*,Ty^*)\big)$$
$$\le\theta\left(\psi\big(M(x^*,y^*)\big)\right)\psi\big(M(x^*,y^*)\big)$$
$$<\psi\big(M(x^*,y^*)\big)$$

where,

$$M(x^*,y^*)=max\left\{d(x^*,y^*),d(x^*,Sx^*),d(y^*,Ty^*),\frac{d(x^*,Ty^*)+d(y^*,Sx^*)}{4s}\right\}$$
$$=d(x^*,y^*).$$

Hence, $\psi\big(d(x^*,y^*)\big)\le\theta\left(\psi\big(M(x^*,y^*)\big)\right)\psi\big(d(x^*,y^*)\big)\le\psi\big(d(x^*,y^*)\big)$,

which is a contradiction. Therefore $x^*=y^*$ and the pair (S,T) has a unique common fixed point $x^*\in X$.

Corollary 2.6. Let (X,d) be a complete b-metric-like space, $T:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$. Presume that the following conditions hold:

(1) T is an (α,β)-admissible mapping;

(2) T is an (α,β)-Geraghty type contraction mapping;

(3) $\exists\, x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,

(4) either T is continuous or X is (α,β)-regular.

Then T has a unique fixed point.

Example 2.7. Let $X = [0,\infty)$ and $d : X\times X\to[0,\infty)$ defined by $d(x,y) = (x^3+y^3)^2$ for all $x,y\in X$. Then (X,d) is a b-metric-like space with $s=2$, but is not a b-metric space since $d(1,1)=4$. Let $S,T : X\to X$ be defined by,

$$Sx = \begin{cases} \frac{x}{\sqrt[6]{45}}, & if\ x \in [0,1] \\ 3x-1, & otherwise \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{\sqrt[6]{45}}, & if\ x \in [0,1] \\ 2x, & otherwise \end{cases}.$$

Define $\alpha,\beta : X\times X\to[0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & if\ x,y \in [0,1] \\ 2, & otherwise \end{cases} \quad \text{and} \quad \beta(x,y) = \begin{cases} 1, & if\ x,y \in [0,1] \\ 0, & otherwise \end{cases}.$$

Also we define θ: $[0,\infty)\to[0,1)$ and ψ:$[0,\infty)\to[0,\infty)$ such that $\theta(t) = \frac{2}{5}$ and $\psi(t)=t$.

Now we show that (S,T) is a pair of (α,β)-admissible mapping. For $x,y \in [0,1]$, then $\alpha(x,y) \geq 1$, $\beta(x,y) \geq 1$, $Sx \leq 1$, $Sy \leq 1$, $Tx \leq 1$, and $Ty \leq 1$. So it follows that $\alpha(Sx,Ty) \geq 1$, $\alpha(Tx,Sy) \geq 1$ and $\beta(Sx,Ty) \geq 1$, $\beta(Tx,Sy) \geq 1$. Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$ $\forall\, n \in \mathbb{N}\cup \{0\}$ and $x_n\to x$ as $n\to\infty$, then $x_n \subseteq [0,1]$ and hence $x \in [0,1]$. $\alpha(x,Sx) \geq 1$ and $\beta(x,Tx) \geq 1$. Also for $x,y \in [0,1]$, we have

$$\begin{aligned} \alpha(x,Sx)\beta(y,Ty)\psi\big(s^3 d(Sx,Ty)\big) &= 8\left(\left(\frac{x}{\sqrt[6]{45}}\right)^3 + \left(\frac{y}{\sqrt[6]{45}}\right)^3\right)^2 \\ &= \frac{8}{45}(x^3+y^3)^2 \\ &\leq \frac{2}{5}d(x,y) \\ &\leq \theta\big(\psi(M(x,y))\big)\psi(M(x,y)). \end{aligned}$$

Otherwise, we have

$$\alpha(x,Sx)\beta(y,Ty)\psi\big(s^3 d(Sx,Ty)\big) = 0 \leq \theta\big(\psi(M(x,y))\big)\psi\big(M(x,y)\big).$$

Therefore, all conditions of theorem 2.5 are satisfied and the maps S & T have a unique common fixed point $x^* = 0$.

3. GENERALIZED (α,β)-QUASI CONTRACTIONS

In this section we give the notion of generalized (α,β)-quasi contraction on complete b-metric-like space.

Definition 3.1. Let (X,d) be a b-metric-like space. A mapping $J: X\to X$ is called generalized (α,β)-quasi contraction if $\exists$ $\alpha,\beta: X\times X\to[0,\infty)$, and $q \leq \frac{1}{s^2}$ such that $\forall$ $x,y \in X$

$$\alpha(x,Jx)\beta(y,Jy)d(Jx,Jy) \leq q.M(x,y),$$

where,

$$\mathrm{M(x,y)} = \left\{ \begin{matrix} d(x,y), d(x,Jx), d(y,Jy), d(x,Jy), d(y,Jx), \\ d(J^2x,x), d(J^2x,Jx), d(J^2x,y), d(J^2x,Jy), d(x,x), d(y,y) \end{matrix} \right\}.$$

Lemma 3.2. Let (X,d) be a b-metric-like space and $J: X\to X$ be generalized (α,β)-quasi contraction satisfying the following conditions:

(1) J is triangular (α,β)-admissible,

(2) $\exists$ $x_1\in X$ such that $\alpha(x_1,Jx_1)\geq 1$ and $\beta(x_1,Jx_1)\geq 1$.

Then $\forall$ $i,j \in \{1,2,...,n\}$, $(i<j)$

$$d\left(J^i x_1, J^j x_1\right) \leq q.\delta[O(x_1,n)].$$

Proof: By presumption $\exists$ $x_1\in X$, such that $\alpha(x_1,Jx_1)\geq 1$ and $\beta(x_1,Jx_1)\geq 1$. Define a sequence $\{x_n\}$ by $x_n = J^n x_1$ $\forall$ $n\in \mathbb{N}$. Since J is triangular (α,β)-admissible, $\alpha\left(J^i x_1, J^j x_1\right) = \alpha\left(x_i,x_j\right) \geq 1$ and $\beta\left(J^i x_1, J^j x_1\right) = \beta\left(x_i,x_j\right) \geq 1$, for $i,j\in \mathbb{N}\cup\{0\}$, $I\leq j$. Then $J^{i-1}x_1, J^i x_1, J^{j-1}x, J^j x \in O(x_1,n)$. Since J is a generalized (α,β)-quasi contraction, we have $d\left(J^i x_1, J^j x_1\right) = d\left(JJ^{i-1}x_1, JJ^{j-1}x_1\right)$

$$\leq \alpha\left(J^{i-1}x_1, J^i x_1\right)\beta\left(J^{j-1}x_1, J^j x_1\right)d\left(JJ^{i-1}x_1, JJ^{j-1}x_1\right)$$

$$\leq q.max\left\{ \begin{matrix} d\left(J^{i-1}x_1, J^{j-1}x_1\right), d\left(J^{i-1}x_1, J^i x_1\right), d\left(J^{j-1}x_1, J^j x_1\right), \\ d\left(J^{i-1}x_1, J^j x_1\right), d\left(J^{j-1}x_1, J^i x_1\right), d\left(J^{i+1}x_1, J^{i-1}x_1\right), \\ d\left(J^{i+1}x_1, J^i x_1\right), d\left(J^{i+1}x_1, J^{j-1}x_1\right), d\left(J^{i+1}x_1, J^j x_1\right), \\ d\left(J^{i-1}x_1, J^{i-1}x_1\right), d\left(J^{j-1}x_1, J^{j-1}x_1\right) \end{matrix} \right\}$$

$$\leq q.\delta[O(x_1,n)],$$

where,

$$\delta[O(x_1,n)] = max\{d(J^i x_1, J^j x_1) : 0 \leq i,j \leq n\}.$$

Remark 3.3. If J is a generalized (α,β)-quasi contraction mapping and $x_1 \in X$, from previous Lemma 3.2 for every $n \in \mathbb{N}$, $\exists\ k \in \mathbb{N}$,and $k \leq n$ such that

$$d(x_1, J^k x_1) = \delta[O(x_1,n)].$$

Theorem 3.4. Let (**X,d**) be a J-orbitally complete b-metric-like space and $J: X \to X$ be a generalized (α,β)-quasi contraction satisfying conditions (1) and (2) of Lemma 3.2.

Then J has a fixed point $w \in X$.

Proof: By assumption $\exists\ x_1 \in X$ such that $\alpha(x_1,Jx_1) \geq 1$. and $\beta(x_1,Jx_1) \geq 1$ Define a sequence $\{x_n\}$ by $x_n = Jx_{n-1} = J^n x_1\ \forall\ n \in \mathbb{N}$. To show that $\{J^n x_1\}$ is Cauchy Sequence in X, by trianguler inequality, Lemma 3.2 and Remark 3.3, we get

$$\begin{aligned} d(x_1, J^k x_1) &\leq s[d(x_1, Jx_1) + d(Jx_1, J^k x_1)] \\ &\leq s[d(x_1, Jx_1) + q.\delta[O(x_1,n)]] \\ &\leq sd(x_1, Jx_1) + q.s.d(x_1, J^k x_1). \end{aligned}$$

Therefore,

$$\delta[O(x_1,n)] = d(x_1, J^k x_1) \leq \frac{s}{1-qs} d(x_1, Jx_1).$$

For any $m,n \in \mathbb{N}$ with $n \leq m$. Since J is generalized (α,β)-quasi contraction, we get

$$\begin{aligned} d(J^n x_1, J^m x_1) &= d(JJ^{n-1}x_1, JJ^{m-1}x_1) \\ &\leq \alpha(J^{n-1}x_1, J^n x_1)\beta(J^{m-1}x_1, J^m x_1) d(JJ^{n-1}x_1, JJ^{m-1}x_1) \\ &\leq q.max\left\{\begin{matrix} d(J^{n-1}x_1, J^{m-1}x_1), d(J^{n-1}x_1, JJ^{n-1}x_1), d(J^{m-1}x_1, JJ^{m-1}x_1), \\ d(J^{n-1}x_1, JJ^{m-1}x_1), d(J^{m-1}x_1, JJ^{n-1}x_1), d(J^2J^{n-1}x_1, J^{n-1}x_1), \\ d(J^2J^{n-1}x_1, JJ^{n-1}x_1), d(J^2J^{n-1}x_1, J^{m-1}x_1), d(J^2J^{n-1}x_1, JJ^{m-1}x_1), \\ d(J^{n-1}x_1, J^{n-1}x_1), d(J^{m-1}x_1, J^{m-1}x_1) \end{matrix}\right\} \end{aligned}$$

$$\leq q.max\left\{\begin{matrix} d(J^{n-1}x_1, J^{m-n}J^{n-1}x_1), d(J^{n-1}x_1, JJ^{n-1}x_1), d(J^{m-n}J^{n-1}x_1, J^{m-n+1}J^{n-1}x_1), \\ d(J^{n-1}x_1, J^{m-n+1}J^{n-1}x_1), d(J^{m-n}J^{n-1}x_1, JJ^{n-1}x_1), d(J^2J^{n-1}x_1, J^{n-1}x_1), \\ d(J^2J^{n-1}x_1, JJ^{n-1}x_1), d(J^2J^{n-1}x_1, J^{m-n}J^{n-1}x_1), d(J^2J^{n-1}x_1, J^{m-n+1}J^{n-1}x_1), \\ d(J^{n-1}x_1, J^{n-1}x_1), d(J^{m-n}J^{n-1}x_1, J^{m-n}J^{n-1}x_1) \end{matrix}\right\}$$

$$\leq q.\delta[O(J^{n-1}x_1, m-n+1)].$$

Using Remark 3.3, ∃ k_1, 1≤ k_1 ≤ m-n+1 such that

$$\delta[O(J^{n-1}x_1, m-n+1)] = d(J^{n-1}x_1, J^{k_1}J^{n-1}x_1).$$

Now by Lemma 3.2, we have

$$\begin{aligned} d(J^{n-1}x_1, J^{k_1}J^{n-1}x_1) &= d(JJ^{n-2}x_1, J^{k_1+1}J^{n-2}x_1) \\ &\leq q.\delta[O(J^{n-2}x_1, k_1+1)] \\ &\leq q.\delta[O(J^{n-2}x_1, m-n+2)]. \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} d(J^n x_1, J^m x_1) &\leq q.\delta[O(J^{n-1}x_1, m-n+1)] \\ &\leq q^2.\delta[O(J^{n-2}x_1, m-n+2)] \\ &\;. \\ &\;. \\ &\;. \\ &\leq q^n.\delta[O(x_1, m)]. \end{aligned}$$

Which implies,

$$d(J^n x_1, J^m x_1) \leq \frac{q^n s}{1-qs} d(x_1, Jx_1).$$

Since $\lim_{n\to\infty} q^n = 0$, then $\{J^n x_1\}$ is a Cauchy sequence in a J-orbitally complete b-metric-like space X. So, ∃ $w \in X$ such that $\lim_{n\to\infty} d(J^n x_1, w) = 0$. Now by triangular inequality we show that $Jw = w$,

$$\begin{aligned} d(w, Jw) &\leq s[d(w, J^{n+1}x_1) + d(J^{n+1}x_1, Jw)] \\ &\leq s[d(w, J^{n+1}x_1) + d(JJ^n x_1, Jw)] \\ &\leq s[d(w, J^{n+1}x_1) + \alpha(J^n x_1, J^{n+1}x_1)\beta(w, Jw)d(JJ^n x_1, Jw)] \\ &\leq S\left[d(w, J^{n+1}x_1) + q.max\left\{\begin{matrix} d(J^n x_1, w), d(J^n x_1, J^{n+1}x_1), d(w, Jw), \\ d(J^n x_1, Jw), d(w, J^{n+1}x_1), d(J^{n+2}x_1, J^n x_1), \\ d(J^{n+2}x_1, J^{n+1}x_1), d(J^{n+2}x_1, w), d(J^{n+2}x_1, Jw), \\ d(w, w), d(J^n x_1, J^n x_1) \end{matrix}\right\}\right]. \end{aligned}$$

This implies by using triangular inequality,

$$d(w,Jw) \leq s\left[d(w,J^{n+1}x_1) + q.max\begin{Bmatrix} d(J^n x_1,w), s[d(J^n x_1,w) + d(w,J^{n+1}x_1)], \\ d(w,Jw), s[d(J^n x_1,w) + d(w,Jw)], \\ d(w,J^{n+1}x_1), s[d(J^{n+2}x_1,w) + d(w,J^n x_1)], \\ s[d(J^{n+2}x_1,w) + d(w,J^{n+1}x_1)], \\ d(J^{n+2}x_1,w), s[d(J^{n+2}x_1,w) + d(w,Jw)], \\ s[d(w,J^n x_1) + d(J^n x_1,w)], \\ s[d(J^n x_1,w) + d(w,J^n x_1)] \end{Bmatrix}\right].$$

By taking $n\to\infty$, we obtain

$$\begin{aligned} d(w,Jw) &\leq qs.mxa\{d(w,Jw), sd(w,Jw)\} \\ &= qs^2 d(w,Jw) \\ &< \frac{1}{s^2}s^2 d(w,Jw) \\ &< d(w,Jw), \end{aligned}$$

which is a contradiction. Therefore w is a fixed point of J.

Corollary 3.5. Let (X,d) be a J-orbitally complete b-metric-like space and $J: X\to X$ be a mapping. Suppose that ∃ $\alpha: X\times X\to[0,\infty)$, $p\in \mathbb{N}$, $q < \frac{1}{s^2}$, and $s \geq 1$ such that ∀ $x,y\in X$,

$$\alpha(x,y)d(J^p x, J^p y) \leq q.max\begin{Bmatrix} d(x,y), d(x,J^p x), d(y,J^p y), d(x,J^p y), \\ d(y,J^p x), d(J^{2p}x,x), d(J^{2p}x,J^p x), \\ d(J^{2p}x,y), d(J^{2p}x,J^p y), d(x,y), d(x,y) \end{Bmatrix}.$$

Then J has a fixed point $w\in X$.

Proof. By the same way of proof of theorem 3.4 we conclude that J^k has a fixed point $w\in X$ and $J^k(Jw) = J(J^k) = Jw \Rightarrow Jw = w$. Then J has a fixed point $w\in X$.

Corollary 3.6. Let (X,d) be a J-orbitally complete b-metric-like space and $J: X\to X$ be a mapping. Postulate that ∃ $\alpha: X\times X\to[0,\infty)$, $p\in \mathbb{N}$ and $q < \frac{1}{s^2}$, $s \geq 1$ such that ∀ $x,y\in X$

$$\alpha(x,y)d(J^p x, J^p y) \leq q.max\begin{Bmatrix} d(x,y), d(x,J^p x), d(y,J^p y), d(x,J^p y), \\ d(y,J^p x), d(x,y), d(x,y) \end{Bmatrix}.$$

Then J has a fixed point $w\in X$.

Corollary 3.7. Let X be a J-orbitally complete b-metric-like space and J: $X\to X$ be a generalized α-quasi contraction satisfying the following conditions:

(1) J is triangular α-admissible;

(2) ∃ $x_1\in X$ such that $\alpha(x_1,Jx_1) \geq 1$.

Then J has a fixed point in X.

Example 3.8. Let $X = \mathbb{R}$ and $p > 1$ be a real number. Define $d: X \times X \to [0,\infty)$ by $d(x,y) = (|x| + |y|)^p$ $\forall$ $x,y \in X$. Let $J: X \to X$ be a mapping defined by

$$Jx = \begin{cases} \frac{x}{8}, & if\ x \in [0,1] \\ 2x, & otherwise \end{cases},$$

and $\alpha,\beta: X \times X \to [0,\infty)$ be functions such that

$$\alpha(x,y) = \begin{cases} 4, & if\ x,y \in [0,1] \\ 2, & otherwise \end{cases} \quad \text{and} \quad \beta(x,y) = \begin{cases} 2, & if\ x,y \in [0,1] \\ 0, & otherwise \end{cases}.$$

Then (X,d) is a J-orbitally complete b-metric-like space with constant $s=2^{p-1}$, but is neither a b-metric space since $d(1,1)=2^p$ nor a metric-like space since $d(-1,1) = 2^p > 2 = 1+1 = d(-1,0) + d(0,1)$. If $x,y \in [0,1]$, then

$$\begin{aligned} \alpha(x,Jx)\beta(y,Jy)d(Jx,Jy) &= 4 \times 2\left(\left|\frac{x}{8}\right| + \left|\frac{y}{8}\right|\right)^p \\ &= \frac{8}{8^p}(|x| + |y|)^p \\ &= \frac{2^3}{2^{3p}}(|x| + |y|)^p \\ &= \frac{1}{2^{3p-3}}(|x| + |y|)^p \\ &= qd(x,y) \\ &\le q.M(x,y), \end{aligned}$$

where,

$$q = \frac{1}{2^{3(p-1)}} < \frac{1}{2^{2(p-1)}} = \frac{1}{s^2}.$$

Otherwise, we have

$$\alpha(x,Jx)\beta(y,Jy)d(Jx,Jy) = 0 \le q.M(x,y).$$

Also we check that J is triangular (α,β)-admissible mapping, $\forall x,y,z \in [0,1]$, $\alpha(x,y) \ge 1$, $\beta(x,y) \ge 1$ and $Jx \le 1$, $Jy \le 1$, $\alpha(Jx,Jy) \ge 1$ and $\beta(Jx,Jy) \ge 1$; $\alpha(x,z) \ge 1$ and $\alpha(z,y) \ge 1$ $\alpha(x,y) \ge 1$. Furthermore $\alpha(x,Jx) \ge 1$, and $\beta(y,Jy) \ge 1$. Hence all conditions of theorem 3.4 are satisfied, and $x = 0$ is a fixed point of J. But for $x,y \in \mathbb{R}/[0,1]$ $\alpha(x,y)d(Jx,Jy) = 2(|2x| + |2y|)^p = 2^{p+1}d(x,y) \ge qM(x,y)$. Therefore Corollary 3.7 is not satisfied.

4. Applications to Nonlinear Integral Equations

We consider existence of a solution for the following integral equation

$$x(t) = \int_0^1 h(t,s)g\big(s,x(s)\big)ds, \tag{4.1}$$

$t \in I = [0,1]$. Let $X = C(I)$ be the space of all continuous real functions defined from I to $\mathbb{R}$, also let X be endowed with the b-metric-like $d(x,y) = max(|x(t)|+|y(t)|)^p$ for all $x,y \in X$, where $p>1$. Obviously, (X,d) is a complete b-metric-like space with the constant $s=2^{p-1}$.

Theorem 4.1. Assume that the following conditions hold:

(i) $g: I\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $g(t,s) \geq 0$ and there exists a constant $0\leq \lambda <1$ such that for all $x,y \in X$

$$|g(t,x(t))| + |g(t,y(t)| < \lambda(|x(t)| + |y(t)|);$$

(ii) $h: I\times I\to\mathbb{R}$ is a continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $h(t,x) \geq 0$ and $\int_0^1 h(t,s)ds \leq L$;

(iii) $\lambda^p L^p \leq \frac{1}{2^{2p-3}}$;

(iv) Define a function $\alpha: X\times X\to[0,\infty)$ by $\alpha(x,y) = \frac{1}{2}$, for all $x,y \in X$.

Then the integral equation (4.1) has a unique solution in X.

Proof: Define a mapping $T: X\to X$ by

$$Tx(t) = \int_0^1 h(t,s)g\big(s,x(s)\big)ds,$$

$t \in I=[0,1]$ and for all $x,y \in X$, we have

$$(|Tx(t)| + |Ty(t)|)^p = \left(\left|\int_0^1 h(t,s)g\big(s,x(s)\big)ds\right| + \left|\int_0^1 h(t,s)g\big(s,y(s)\big)ds\right|\right)^p$$

$$\leq \left(\int_0^1 \big|h(t,s)g\big(s,x(s)\big)\big|ds + \int_0^1 \big|h(t,s)g\big(s,y(s)\big)\big|ds\right)^p$$

$$= \left(\int_0^1 \left(\big|h(t,s)g\big(s,x(s)\big)\big| + \big|h(t,s)g\big(s,y(s)\big)\big|\right) ds\right)^p$$

$$= \left(\int_0^1 h(t,s)\lambda(|x(s)| + |y(s)|)\, ds \right)^p$$

$$= \left(\int_0^1 h(t,s)\lambda((|x(s)| + |y(s)|)^p)^{\frac{1}{p}}\, ds \right)^p$$

$$\leq \lambda^p d\big(x(t), y(t)\big) \left(\int_0^1 h(t,s)\, ds \right)^p$$

$$\leq \lambda^p L^p d(x,y)$$

$$\leq \frac{1}{2^{2p-3}} M(x,y)$$

$$= \frac{2}{2^{2p-2}} M(x,y).$$

Therefore,

$$\frac{1}{2} d(Tx, Ty) \leq \frac{1}{s^2} M(x,y),$$

which implies that,

$$\alpha(x,y) d(Tx, Ty) \leq q. M(x,y).$$

Hence corollary 2.7 is satisfied and the equation (4.1) has a unique solution in X.

5. CONCLUSION

We conclude that the above results are real generalizations for the results of Pant and Panicker [2] in b-Metric-Like Space utilizing the pair of triangular (α,β)-admissible mappings. Some examples to support our results, we showed that results of Pant and Panicker [2] are not applicable with such examples. An application to nonlinear integral equations is discussing.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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