Available online at http://scik.org
Adv. Fixed Point Theory, 2020, 10:14
https://doi.org/10.28919/afpt/4702
ISSN: 1927-6303

SOLUTIONS OF HAMMERSTEIN EQUATIONS IN THE SPACE $(\Lambda_1,\Lambda_2)BV(I_a^b)$

MEIMEI SONG, JINPENG LIU*

College of Science, Tianjin University of Technology, Tianjin, 300384, China

Abstract. In this paper, we study the form of Hammerstein integral equations

$$u(x) = v(x) + \lambda \int_{I_a^b} k(x,y) f(y,u(y))dy, (\lambda \in \mathbb{R})$$

and Volterra Hammerstein integral equations in the condition of two-variables. Show the definition of (Λ_1,Λ_2) bounded variation, write as $(\Lambda_1,\Lambda_2)BV(I_a^b)$. If v, k are $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$ functions and f is a locally Lipschitz function, there exists a number $\rho > 0$ such that when $|\lambda| < \rho$, Hammerstein integral equations has a unique solution. Give the proof and extend.

Keywords: Hammerstein integral equation; Volterra Hammerstein integral equations; (Λ_1,Λ_2)-bounded variation; $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$-solution.

2010 AMS Subject Classification: 45G10, 45D05.

1. Introduction

The concept of bounded variation was first proposed by Jordon in [1] and applied it to the theory of Fourier series. Since then, bounded variation has come into view, and related research has begun. Solutions of many integral equations which describe concrete physical phenomena

*Corresponding author
E-mail address: 810192567@qq.com
Received May 11, 2020

from the real world are often functions of bounded variation. Recall that some superposition operator in the space of functions of bounded variation was investigated in [2], [3], [4]. Waterman introduced the notion of Λ-variation in [5] in 1972. In 1986, Sahakian extended Λ-variation from the case of one variable to that of two variables[6].

The problem of solving Hammerstein integral equations has a long history and is always popular. In [7], the author gave the condition of existence and uniqueness of bounded variation (shortly: BV) solutions and continuous BV-solutions of the following Hammerstein

$$x(t) = g(t) + \lambda \int_I K(t,s) f(x(s)) ds, t \in I = [\alpha, \beta], \lambda \in \mathbb{R}$$

and Volterra-Hammerstein

$$x(t) = g(t) + \int_I K(t,s) f(x(s)) ds, t \in I = [\alpha, \beta], \lambda \in \mathbb{R}$$

integral equations, where $g : I \to \mathbb{R}$ is a $BV(I)$-function, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $K : I \times I \to \mathbb{R}$. It is the first to combine the theorem of bounded variation with the integral equation. The authors prove under some conditions on λ, K and f, this equation have only one solution in the space $BV(I)$ for each $g \in BV(I)$. Then in the paper [8], [9], the integral space is extended from bounded variation space to ΛBV space, ϕBV space, and theorems in [6] are also extended.

Particularly in [10], the authors study the existence conditions of solutions for the Hammerstein equations firstly in a two variables space, that is, the space of two-variables bounded variation functions BV(I_a^b). The form of two-variable Hammerstein integral equation

$$u(x) = v(x) + \lambda \int_{I_a^b} k(x,y) f(y, u(y)) dy, (\lambda \in \mathbb{R}) \tag{1.1}$$

and Volterra-Hammerstein

$$u(x) = v(x) + \int_{I_a^b} k(x,y) f(y, u(y)) dy, (\lambda \in \mathbb{R}) \tag{1.2}$$

can be seen in [11], where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$, $I_a^b = [a_1, b_1] \times [a_2, b_2]$, $x = (x_1, x_2), y = (y_1, y_2) \in I_a^b$, $v : I_a^b \to \mathbb{R}$, $k : I_a^b \times I_a^b \to \mathbb{R}$ and $f : I_a^b \times \mathbb{R} \to \mathbb{R}$. Under some conditions on λ, k and f, we prove that this equation admits only one solution in the space $\Lambda BV(I_a^b)$ for each $v \in \Lambda BV(I_a^b)$.

In this paper, we will give the general conditions for the existence of unique solutions and continuous unique solutions of the Hammerstein and Volterra-Hammerstein integral equation in the space of two-variables (Λ_1,Λ_2)bounded variation functions $(\Lambda_1,\Lambda_2)BV(I_a^b)$, then proof.

2. Preliminaries

Let $I_a^b=[a_1,b_1]\times[a_2,b_2]$ be the basic rectangle with $a=(a_1,a_2),b=(b_1,b_2)\in\mathbb{R}^2$ such that $a_1<a_2$ and $b_1<b_2$. With the sequence of positive numbers $\Lambda^{(1)}=\{\lambda_m^{(1)}\}_{m=1}^{\infty}$, $\Lambda^{(2)}=\{\lambda_n^{(2)}\}_{n=1}^{\infty}$ such that $\sum\frac{1}{\lambda_m^{(1)}}$ and $\sum\frac{1}{\lambda_n^{(2)}}$ are divergent separately. We denote $\Pi_1=\{t_i\}_{i=0}^{m}$, $\Pi_2=\{s_j\}_{j=0}^{n}$ be partitions of $[a_1,b_1]$ and $[a_2,b_2]$$(m,n\in\mathbb{N}$, $a_1=t_0<t_1<\cdots<t_{m-1}<t_m=b_1$ and $a_2=s_0<s_1<\cdots<s_{n-1}<s_n=b_2)$, for the two-dimensional function $f:I_a^b\to\mathbb{R}$. Referring to [10] and [12],

$$\Lambda^{(1)}V_1(f;I_a^b)=\sup_{s}\sup_{\Pi_1}\sum_{i=1}^{m}\frac{|f(t_i,s)-f(t_{i-1},s)|}{\lambda_i^{(1)}}$$

$$\Lambda^{(2)}V_2(f;I_a^b)=\sup_{t}\sup_{\Pi_2}\sum_{j=1}^{n}\frac{|f(t,s_j)-f(t,s_{j-1})|}{\lambda_j^{(2)}}$$

$$(\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(f;I_a^b)=\sup_{\Pi_1,\Pi_2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{|f(t_i,s_j)-f(t_{i-1},s_j)-f(t_i,s_{j-1})+f(t_{i-1},s_{j-1})|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

DEFINITION 1 *We define the $(\Lambda^{(1)},\Lambda^{(2)})$-variation of $f:I_a^b\to\mathbb{R}$ by*

$$(\Lambda^{(1)},\Lambda^{(2)})V(f;I_a^b)=\Lambda^{(1)}V_1(f;I_a^b)+\Lambda^{(2)}V_2(f;I_a^b)+(\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(f;I_a^b)$$

and the bounded $(\Lambda^{(1)},\Lambda^{(2)})$-variation on I_a^b space by

$$(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})=\{f:I_a^b\to\mathbb{R}|(\Lambda^{(1)},\Lambda^{(2)})V(f;I_a^b)<\infty\}.$$

with the norm

$$\|f\|_{(\Lambda_1,\Lambda_2)}=|f(a_1,a_2)|+(\Lambda^{(1)},\Lambda^{(2)})V(f;I_a^b)$$

for each $f\in(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$. For all $x\in I_a^b$, we have$(\Lambda^{(1)},\Lambda^{(2)})V(f;I_a^x)$ is a completely monotone function. According to Helly's selection principle in the 2-dimensional case[13][14], we can say the space in $definition1$ is a Banach space.

LEMMA 1 *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are seminorms on a given linear space X such that for each sequence $\{x_n\}$ of elements from X, we have $\|x_n\|_1\to 0$ implies $\|x_n\|_2\to 0(n\to 0)$, then there exists a constant M such that $\|x\|_2\le M\|x\|_1$ for each $x\in X$.*[8]

Proof. Assuming that for all $M \in \mathbb{N}$, there exist $x_k \in \{x_n\}$ such that $\|x_n\|_2 > M\|x_n\|_1$. Let's define $y_n = \frac{x_n}{\|x_n\|_2}$ while $\|y_n\|_2 = 1$. Then we have $0 \leq \|y_n\|_1 \leq \frac{1}{M}$ for all $M \in \mathbb{N}$. If $\|y_n\|_1 \to 0$ when $n \to 0$, according to $\|x_n\|_1 \to 0$ can imply $\|x_n\|_2 \to 0$, we can get $\|y_n\|_2 \to 0$. There's a contradiction. □

Applying Lemma 1 one can prove the following

LEMMA 2 *There exists a constant $\widetilde{c}$ such that*

$$\sup_{(t_1,t_2)\in I} |f(t_1,t_2)| \leq \widetilde{c}\|f\|_{(\Lambda_1,\Lambda_2)}$$

for any $f \in (\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$.

Proof. Let $\{f_i\}$ be a sequence of elements of the space $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$ such that we have $\|f_i\|_{(\Lambda_1,\Lambda_2)} \to 0$, $\|f_i\|_{(\Lambda_1,\Lambda_2)} = |f_i(a_1,a_2)| + (\Lambda^{(1)},\Lambda^{(2)})V(f_i;I_a^b)$, then $(\Lambda^{(1)},\Lambda^{(2)})V(f_i;I_a^b) \to 0$ and $|f_i(a_1,a_2)| = 0$. So we have $\sup\limits_{(t_1,t_2)\in I} |f_i(t_1,t_2)| \to 0$ as $n \to \infty$. By lemma 1, there exists a constant $\widetilde{c}$ such that

$$\sup_{(t_1,t_2)\in I} |f(t_1,t_2)| \leq \widetilde{c}\|f\|_{(\Lambda_1,\Lambda_2)}$$

for any $f \in (\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$. □

3. Main Results

Consider the Hammerstein integral equation (1.1) where "$\int$" stands for the Lebesgue integral. Assume the following hypothesis:

$H1$ $v : I_a^b \to \mathbb{R}$ is a $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$ function;

$H2$ $f : I_a^b \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function[15];

$H3$ $K : I_a^b \times I_a^b \to \mathbb{R}$ is a $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$ function such that

$$(\Lambda^{(1)},\Lambda^{(2)})V(K(\cdot,\alpha);I_a^b) \leq M(\alpha) \text{ for a.e.} \alpha = (\alpha_1,\alpha_2) \in I_a^b,$$

where $M : I_a^b \to \mathbb{R}_+$ is integrable in the Lebesgue sense(briefly: L-integrable) and $K(\beta,(\cdot,\cdot))$ is L-integrable for every $\beta = (\beta_1,\beta_2) \in I_a^b$.

THEOREM 3.1 *Under H, $H2$, $H3$, there exists a number $\rho > 0$ such that for every λ with $|\lambda| < \rho$, equation (1.1) has a unique $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$-solution, defined on I_a^b.*

Proof. Let $r > 0$ be such that $\|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} < r$ and denote by L_r the Lipschitz constant of f corresponding to the cube $I_a^b \times [-r,r]$. Choose a number $\rho > 0$ such that the following two inequalities true:

$$\|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + \rho \sup_{\overline{y}\in I_a^b} |f(\overline{y},u(\overline{y}))| \int_{I_a^b} (M(y) + |K(a,y)|)dy < r \tag{3.1}$$

which established a link between ρ and r, and

$$\widetilde{c}\rho L_r \int_{I_a^b} (M(y) + |K(a,y)|)ds < 1 \tag{3.2}$$

where $\widetilde{c}$ is the smallest number satisfying the inequality in LEMMA 2.

Denote by $\overline{B_r}$ the closed ball of center zero and radius r in the space $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$. Fix $|\lambda| < \rho$. Define

$$G(u)(x) = v(x) + \lambda F(u)(x),$$

where

$$F(u)(x) = \int_{I_a^b} K(x,y)f(y,u(y))dy, x = (x_1,x_2) \in \overline{B_r}.$$

Since the mapping $x \to f(x,u(x))$ is bounded and its discontinuities are at most denumerable, thus it is L-measurable. Therefore the mappings F and G are well defined.

Firstly, we verify that $G(\overline{B_r}) \subset \overline{B_r}$. Indeed, for any $x \in \overline{B_r}$, we have

$$(\Lambda^{(1)},\Lambda^{(2)})V(f\circ u;I_a^b) = \Lambda^{(1)}V_1(f\circ u;I_a^b) + \Lambda^{(2)}V_2(f\circ u;I_a^b) + (\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(f\circ u;I_a^b)$$
$$= \sup_s \sup_{\Pi_1} \sum_{i=1}^m \frac{|(f\circ u)(t_i,s) - (f\circ u)(t_{i-1},s)|}{\lambda_i^{(1)}}$$
$$+ \sup_t \sup_{\Pi_2} \sum_{j=1}^n \frac{|(f\circ u)(t,s_j) - (f\circ u)(t,s_{j-1})|}{\lambda_j^{(2)}}$$
$$+ \sup_{\Pi_1,\Pi_2} \sum_{i=1}^m \sum_{j=1}^n \frac{|(f\circ u)(t_i,s_j) - (f\circ u)(t_{i-1},s_j) - (f\circ u)(t_i,s_{j-1}) + (f\circ u)(t_{i-1},s_{j-1})|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$= \sup_{s} \sup_{\Pi_1} \sum_{i=1}^{m} \frac{|\int_{I_a^b}[K((t_i,s),y)-K((t_{i-1},s),y)]f(y,u(y))dy|}{\lambda_i^{(1)}}$$

$$+ \sup_{t} \sup_{\Pi_2} \sum_{j=1}^{n} \frac{|\int_{I_a^b}[K((t,s_j),y)-K((t,s_{j-1}),y)]f(y,u(y))dy|}{\lambda_j^{(2)}} + \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n}$$

$$\int_{I_a^b} \frac{|[K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)]f(y,u(y))dy|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$\leq \sup_{\bar{y}\in I_a^b} |f(\bar{y},u(\bar{y}))| \sup_{s} \sup_{\Pi_1} \sum_{i=1}^{m} \frac{|\int_{I_a^b} K((t_i,s),y)-K((t_{i-1},s),y)dy|}{\lambda_i^{(1)}}$$

$$+ \sup_{\bar{y}\in I_a^b} |f(\bar{y},u(\bar{y}))| \sup_{t} \sup_{\Pi_2} \sum_{j=1}^{n} \frac{|\int_{I_a^b} K((t,s_j),y)-K((t,s_{j-1}),y)dy|}{\lambda_j^{(2)}}$$

$$+ \sup_{\bar{y}\in I_a^b} |f(\bar{y},u(\bar{y}))| \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n}$$

$$\int_{I_a^b} \frac{|K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)dy|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$\leq \sup_{\bar{y}\in I_a^b} |f(\bar{y},u(\bar{y}))| \int_{I_a^b} M(y)dy < \infty$$

Hence $F(u) \in (\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$, we have

$$\begin{aligned}
\|G(u)\|_{(\Lambda^{(1)},\Lambda^{(2)})} &\leq \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + |\lambda|\|F(u)\|_{(\Lambda^{(1)},\Lambda^{(2)})} \\
&= \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + |\lambda|[|F(u)(a)| + (\Lambda^{(1)},\Lambda^{(2)})V(f\circ u;I_a^b)] \\
&\leq \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + |\lambda| \sup_{\bar{y}\in I_a^b} |f(\bar{y},u(\bar{y}))| \int\limits_{I_a^b} (|K(a,y)|+M(y))dy < r
\end{aligned}$$

so $G(\overline{B_r}) \subset \overline{B_r}$.

Then we show that G is a contraction. For any $u_1,u_2 \in \overline{B_r}$, we have

$$\|G(u_1 - G(u_2)\|_{(\Lambda^{(1)},\Lambda^{(2)})} = |G(u_1)(a) - G(u_2)(a)| + (\Lambda^{(1)},\Lambda^{(2)})V(f\circ u_1 - f\circ u_2; I_a^b)$$

$$= |\lambda|[F(u_1)(a) - F(u_2)(a)]| + \Lambda^{(1)}V_1(f\circ u_1 - f\circ u_2;I_a^b) + \Lambda^{(2)}V_2(f\circ u_1 - f\circ u_2;I_a^b)$$

$$+ (\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(f\circ u_1 - f\circ u_2;I_a^b)$$

$$\leq |\lambda \int_{I_a^b} K(a,y)[f(y,u_1(y)) - f(y,u_2(y))]dy|$$

$$+|\lambda|\sup_{s}\sup_{\Pi_1}\sum_{i=1}^{m}\int_{I_a^b}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_i^{(1)}}|K((t_i,s),y)-K((t_{i-1},s),y)|dy$$

$$+|\lambda|\sup_{t}\sup_{\Pi_2}\sum_{j=1}^{n}\int_{I_a^b}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_j^{(2)}}|K((t,s_j),y)-K(t,s_{j-1}),y)|dy$$

$$+|\lambda|\sup_{\Pi_1,\Pi_2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{I_a^b}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$|K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)|dy$$

$$\leq|\lambda|\sup_{\bar{y}\in I_a^b}|f(\bar{y},u_1(\bar{y}))-f(\bar{y},u_2(\bar{y}))|\{\int_{I_a^b}|K(a,y)|dy$$

$$+\sup_{s}\sup_{\Pi_1}\int_{I_a^b}\sum_{i=1}^{m}\frac{|K((t_i,s),y)-K((t_{i-1},s),y)|dy}{\lambda_i^{(1)}}$$

$$+\sup_{t}\sup_{\Pi_2}\int_{I_a^b}\sum_{j=1}^{n}\frac{|K((t,s_j),y)-K((t,s_{j-1}),y)|dy}{\lambda_j^{(2)}}$$

$$+\sup_{\Pi_1,\Pi_2}\int_{I_a^b}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{|K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)|dy}{\lambda_i^{(1)}\lambda_j^{(2)}}\}$$

$$\leq|\lambda|L_r|u_1(y)-u_2(y)|\int\limits_{I_a^b}(|K(a,y)|+M(y))dy$$

$$\leq|\lambda|L_r\widetilde{c}\|u_1-u_2\|_{(\Lambda_1,\Lambda_2)}\int\limits_{I_a^b}(|K(a,y)|+M(y))dy$$

$$\leq\|u_1-u_2\|_{(\Lambda_1,\Lambda_2)}$$

In view of the fixed point principle we infer that G has a unique fixed point in $\overline{B_r}$. It is clear that equation (1) has a unique $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$-solution, defined on I_a^b. □

We shall consider the nonlinear integral equation(1.2). We assume the additional conditions $H4$ $T=\{(t',s'):t'\in I_0^a,s'\in I_0^{t'}\}$ and $K:T\to\mathbb{R}$ is a function such that

$$|K(s',s')|+(\Lambda^{(1)},\Lambda^{(2)})V(K(\cdot,s'))<m(s')$$

for a.e. $s'\in I_a^b$, where $m:I_a^b\to\mathbb{R}_+$ is L-integrable and $K(t',\cdot)$ is L-integrable on $I_0^{t'}$ for every $t'\in I_a^b$.

THEOREM 4.1 *Under H1, H2, H4, there exists a rectangle $\mathscr{R} \subset I_a^b$ such that equation (1.2) has an unique $(\Lambda^{(1)}, \Lambda^{(2)})BV(I_a^b)$-solution, defined on $\mathscr{R}$.*

Proof. Let r, L_r denote the numbers defined in the proof of Theorem 3.1. Withoutloss of generality, we assume that $a = (a_1, a_2) = (0,0)$. Choose a number $d = (d_1, d_2)$ with $d_1, d_2 > 0$ in such a way that

$$\|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + \sup_{\bar{y} \in I_0^d} |f(\bar{y}, u(\bar{y}))| \int\limits_{I_0^d} m(s')ds' < r \tag{4.1}$$

and

$$\widetilde{c}L_r \int\limits_{I_0^d} m(s')ds' < 1. \tag{4.2}$$

Put

$$\widetilde{K}(t', s') = \begin{cases} K(t', s'), & s' \in I_0^{t'} \\ 0, & s' \in I_0^{t'}. \end{cases}$$

and $\mathscr{R} = [0, d_1] \times [0, d_2]$. Define

$$G(u)(t') = v(t') + F(u)(t'),$$

where

$$F(u)(t') = \int_{I_0^{t'}} K(t', s') f(s', u(s')) ds', u \in \overline{B_r}, t' \in \mathscr{R},$$

and $\overline{B_r}$ denotes the closed ball of center zero and radius r in the space $(\Lambda^{(1)}, \Lambda^{(2)})BV(\mathscr{R})$. Now, we verify that G maps $\overline{B_r}$ into itself. Indeed, we have

$$\begin{aligned}
&\|G(u)\|_{(\Lambda^{(1)},\Lambda^{(2)})} \leq \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + \|F(u)\|_{(\Lambda^{(1)},\Lambda^{(2)})} \\
&= \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + |F(u(0))| + (\Lambda^{(1)}, \Lambda^{(2)})V(f \circ u) \\
&= \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + \Lambda^{(1)}V_1(f; I_a^b) + \Lambda^{(2)}V_2(f; I_a^b) + (\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(f; I_a^b) \\
&= \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})} + \sup_s \sup_{\Pi_1} \sum_{i=1}^{m} \frac{|(F \circ u)(t_i, s) - (F \circ u)(t_{i-1}, s)|}{\lambda_i^{(1)}}
\end{aligned}$$

$$+\sup_t \sup_{\Pi_2} \sum_{j=1}^{n} \frac{|(F\circ u)(t,s_j)-(F\circ u)(t,s_{j-1})|}{\lambda_j^{(2)}}$$

$$+\sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{|(F\circ u)(t_i,s_j)-(F\circ u)(t_{i-1},s_j)-(F\circ u)(t_i,s_{j-1})+(F\circ u)(t_{i-1},s_{j-1})|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$=\|v\|_{(\Lambda^{(1)},\Lambda^{(2)})}+\sup_s \sup_{\Pi_1} \sum_{i=1}^{m} \frac{|\int_{I_0^{t'}}[K((t_i,s),y)-K((t_{i-1},s),y)]f(y,u(y))dy|}{\lambda_i^{(1)}}$$

$$+\sup_t \sup_{\Pi_2} \sum_{j=1}^{n} \frac{|\int_{I_0^{t'}}[K((t,s_j),y)-K((t,s_{j-1}),y)]f(y,u(y))dy|}{\lambda_j^{(2)}}+\sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n}$$

$$\int_{I_0^{t'}} \frac{|[K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)]f(y,u(y))dy|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$=\|v\|_{(\Lambda^{(1)},\Lambda^{(2)})}+\sup_s \sup_{\Pi_1} \sum_{i=1}^{m} \frac{|\int_{I_0^{d}}[\widetilde{K}((t_i,s),y)-\widetilde{K}((t_{i-1},s),y)]f(y,u(y))dy|}{\lambda_i^{(1)}}$$

$$+\sup_t \sup_{\Pi_2} \sum_{j=1}^{n} \frac{|\int_{I_0^{d}}[\widetilde{K}((t,s_j),y)-\widetilde{K}((t,s_{j-1}),y)]f(y,u(y))dy|}{\lambda_j^{(2)}}+\sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n}$$

$$\int_{I_0^{d}} \frac{|[\widetilde{K}((t_i,s_j),y)-\widetilde{K}((t_{i-1},s_j),y)-\widetilde{K}((t_i,s_{j-1}),y)+\widetilde{K}((t_{i-1},s_{j-1}),y)]f(y,u(y))dy|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$\leq \|v\|_{(\Lambda^{(1)},\Lambda^{(2)})}+\sup_{\overline{y}\in I_0^d}|f(\overline{y},u(\overline{y}))|\int_{I_0^d} m(s)ds<r$$

for $u_1,u_2\in\overline{B_r}$. Thus $G(\overline{B_r})\subset\overline{B_r}$, for any $u_1,u_2\in\overline{B_r}$, we obtain

$$\|G(u_1)-G(u_2)\|_{(\Lambda^{(1)},\Lambda^{(2)})}=(\Lambda^{(1)},\Lambda^{(2)})V(F(u_1)-F(u_2))$$

$$=\Lambda^{(1)}V_1(F(u_1)-F(u_2);I_a^b)+\Lambda^{(2)}V_2(F(u_1)-F(u_2);I_a^b)$$

$$+(\Lambda^{(1)}\Lambda^{(2)})V_{1,2}(F(u_1)-F(u_2);I_a^b)$$

$$\leq \sup_s \sup_{\Pi_1}\int_{I_0^{t'}}\sum_{i=1}^{m}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_i^{(1)}}|K((t_i,s),y)-K((t_{i-1},s),y)|dy$$

$$+\sup_t \sup_{\Pi_2}\int_{I_0^{t'}}\sum_{j=1}^{n}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_j^{(2)}}|K((t,s_j),y)-K((t,s_{j-1}),y)|dy$$

$$+\sup_{\Pi_1,\Pi_2}\int_{I_0^{\prime\prime}}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{|f(y,u_1(y))-f(y,u_2(y))|}{\lambda_i^{(1)}\lambda_j^{(2)}}$$

$$|K((t_i,s_j),y)-K((t_{i-1},s_j),y)-K((t_i,s_{j-1}),y)+K((t_{i-1},s_{j-1}),y)|dy$$

$$\leq \sup_{\bar{y}\in\mathscr{R}}|f(\bar{y},u_1(\bar{y}))-f(\bar{y},u_2(\bar{y}))|\{\sup_{s}\sup_{\Pi_1}\int_{I_0^d}\sum_{i=1}^{m}\frac{|\widetilde{K}((t_i,s),y)-\widetilde{K}((t_{i-1},s),y)|dy}{\lambda_i^{(1)}}$$

$$+\sup_{t}\sup_{\Pi_2}\int_{I_0^d}\sum_{j=1}^{n}\frac{|\widetilde{K}((t,s_j),y)-\widetilde{K}((t,s_{j-1}),y)|dy}{\lambda_j^{(2)}}$$

$$+\sup_{\Pi_1,\Pi_2}\int_{I_0^d}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{|\widetilde{K}((t_i,s_j),y)-\widetilde{K}((t_{i-1},s_j),y)-\widetilde{K}((t_i,s_{j-1}),y)+\widetilde{K}((t_{i-1},s_{j-1}),y)|dy}{\lambda_i^{(1)}\lambda_j^{(2)}}\}$$

$$\leq L_r|u_1-u_2|\int_{I_a^b}(m(y))dy$$

$$\leq L_r\widetilde{c}\|u_1-u_2\|_{(\Lambda_1,\Lambda_2)}\int_{I_a^b}(m(y))dy$$

$$\leq \|u_1-u_2\|_{(\Lambda_1,\Lambda_2)}$$

In view of the Banach fixed point principle we infer that G has a unique fixed point in $\overline{B_r}$, which is obviously a $(\Lambda_1,\Lambda_2)BV(I_a^b)$-solution of (1.2), defined on $\mathscr{R}$. The proof is complete. □

We shall study the existence of solutions of Hammerstein and the Volterra Hammerstein integral equations in the space of two-variables continuous $(\Lambda_1,\Lambda_2)BV$ functions. We denote the Banach space of all continuous functions $f: I_a^b\to\mathbb{R}$ by $C(I_a^b,\mathbb{R})$ with the norm

$$\|f\|_C=\sup_{(t,s)\in I_a^b}\|f(t,s)\|.$$

We assume the following hypothesis:

$H5$ $v: I_a^b\to\mathbb{R}$ is a continuous $(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})$ function.

$H6$ For each $\varepsilon>0$ there exists $\delta>0$ such that for all $t,s,\tau\in I_a^b: t=(t',t'')$, $s=(s',s'')$ and $\tau=(\tau',\tau'')$,

$$\|\tau-t\|<\delta\Rightarrow|K(\tau,s)-K(t,s)<\varepsilon.$$

THEOREM 5.1 *Under the assumptions H2, H3, H5, H6, there exists a number $\rho > 0$ such that for every λ with $|\lambda| < \rho$, equation (1.1) has a unique continuous $(\Lambda^{(1)}, \Lambda^{(2)})BV(I_a^b;\mathbb{R})$-solution, defined on I_a^b.*

Proof. Consider the space $(\Lambda^{(1)}, \Lambda^{(2)})BV_C(I_a^b;\mathbb{R}) = (\Lambda^{(1)}, \Lambda^{(2)})BV(I_a^b;\mathbb{R}) \cap C(I_a^b;\mathbb{R})$ with the norm $\|\cdot\|_{(\Lambda^{(1)},\Lambda^{(2)})BV_C(I_a^b;\mathbb{R})}$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $(\Lambda^{(1)}, \Lambda^{(2)})BV_C(I_a^b;\mathbb{R})$ such that $\|u_n - u\|_{(\Lambda^{(1)},\Lambda^{(2)})BV(I_a^b;\mathbb{R})} \to 0$, for some $u \in (\Lambda^{(1)}, \Lambda^{(2)})BV(I_a^b;\mathbb{R})$. Then it's clear that $\|u_n - u\|_{C(I_a^b;\mathbb{R})} \to 0$, so $u \in C(I_a^b;\mathbb{R})$. Hence $(\Lambda^{(1)}, \Lambda^{(2)})BV_C(I_a^b;\mathbb{R})$ is clear a Banach space. Denote the closed ball of center zero and radius r in the space $(\Lambda^{(1)}, \Lambda^{(2)})BV_C(I_a^b;\mathbb{R})$ by $\overline{B_r}$. Define the mappings F and G the same as in the proof of Theroem 3.1. Since

$$|G(u)(t) - G(u)(\tau)| \leq |v(t) - v(s)| + |\lambda| \sup_{s\in I_a^b} |f(u(s))| \int_{I_a^b} |K(t,s) - K(\tau,s)| ds$$

for $u \in \overline{B_r}$, $t, \tau \in I_a^b$. By $H5$ and $H6$, we infer that $G(u)$ is a continuous function. The next process of proof is similar to Theorem 3.1. $\square$

Further more,

$H7$ For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, \tau \in I_a^b$ and every $s \in I_0^t \cap I_0^\tau$,

$$\|\tau - t\| < \delta \Rightarrow |K(\tau,s) - K(t,s)| < \varepsilon.$$

THEOREM 5.2 *Under the assumptions H2, H4, H5, H7, there exists a rectangle$\mathscr{R} \subset I_a^b$ such that the equation (1.2) has a unique continuous $(\Lambda^{(1)}, \Lambda^{(2)})BV(I_a^b;\mathbb{R})$-solution, defined on I_a^b.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] C. Jordan, Sur la series de Fourier, C.R. Acad. Sci. Paris, 92 (1881), 228-230.

[2] J. Appell and P.P. Zabrejko, Nonlinear Superposition Operators, Cambridge Univer-sity Press, 1990.

[3] M. Josephy, Composing functions of bounded variation Proc. Amer. Math. Soc. 83(1981), 354-356.

[4] A. G. Ljamin, On the acting problem for the Nemytskij operator in the space of functions of bounded variation (Russian), 11th School Oper. Fcn. Spaces, Chel'jabinsk (1986), pp. 63-64.

[5] D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, Studia Math. 44 (1972), 107-117.

[6] A. A. Sahakian, On the convergence of double Fourier series of functions of bounded harmonic variation, Izv. Nats. Akad. Nauk Armenii Mat. 21 (1986), 517-529 (Russian).

[7] D. Bugajewski, On BV-Soludtions of some Nonlinear Integral Equations, Integral Equations Oper. Theory, 46 (2003), 387-398.

[8] D. Bugajewska, D.O'Regan, On nonlinear integral equations and Λ-bounded variation, Acta Math. Hungar, 107 (4) (2005), 295-306.

[9] D. Bugajewska, D. Bugajewski, H. Hudzik, BV φ-solutions of nonlinear integral equations. J. Math. Anal. Appl. 287 (1) (2003), 265-278.

[10] W. Aziz, H. Leiva, N. Merentes, Solutions of Hammerstein equations in the space $\mathrm{BV}(I_a^b)$. Quaest. Math. 37 (3) (2014), 359?370.

[11] J. Appell, C. Chen, How to solve Hammerstein equations, J. Integral Equations Appl. 18(3) (2006), 287-296.

[12] U. Goginava, Uniform summability of double Walsh-Fourier series of functions of bounded partial ?-variation, Math. Slovaca 64 (2014), 1451-1474.

[13] S. Bochner, Lectures on Fourier Integrals with an Author's Supplement on Monotonic Functions, Stieltjes Integrals and Harmonic Analysis, Princeton University Press, Princeton, N.J., 1959.

[14] A.S. Leonov, On the total variation for functions of several variables and a multidimensional analog of Helly?s selection principle. Math. Notes 63 (1996), 61-71.

[15] J.R. Giles, Sciffer Scott. Continuity characterisations of differentiability of locally Lipschitz functions. Bull. Aust. Math. Soc. 41 (3) (1990), 371-380.