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ON QUASICONVEX METRIC SPACES

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Abstract. We introduce the notion of quasiconvexity in metric space in this paper. Some fixed point theorems in this newly introduced space are also formulated and proved. Our results are significant extension of some fixed point results of convex metric spaces and metric spaces in literature.

Keywords: quasiconvex metric spaces; fixed point; quasiconvexity; convex netric spaces.

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1. Introduction and Preliminaries

Metric space is a vital tool in functional analysis, topology, nonlinear analysis and many other branches of mathematics. Its topological formation has attracted the attention of many mathematicians partly because of its usefulness in the fixed point theory. In recent years, applications of fixed point theorems have made researchers to introduce different generalizations of metric spaces. These spaces include 2-metric spaces, D-metric spaces, D^*-metric spaces, G-metric spaces, b-metric spaces, quasimetric spaces, G_b-metric spaces, complex valued G_b-metric spaces, S-metric spaces, S_b-metric spaces, complex valued S_b-metric spaces, A-metric spaces, γ-generalized quasi metric spaces and , most recently, S_p-metric spaces (see [1-17]).

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In his investigation, Ponstein in 1969 was able to review that quasiconvex function was the weakest among the convex functions. The breakdown is as shown below:

Function	Region	SC	C	SPC	PC	SQC	QC	XC
$f_1(x) = 0$ $f_1(x) = -(x-1)^2$	$0 \leq x \leq 1$ $1 \leq x \leq 2$	N •	N	N	N •	N	Y	N
$f_2(x,y) = -x^2$	$0 \leq x,y \leq 1$	N	N	N	N	Y	Y	N
$f_3(x,y) = -x^2 - x$	$0 \leq x,y \leq 1$	N	N	N	Y	Y	Y	N
$f_4(x) = 0$	$0 \leq x \leq 1$	N	Y	N	Y	Y	Y	N
$f_5(x) = -x$	$0 \leq x \leq 1$	N	Y	Y	Y	Y	Y	Y
$f_6(x) = x^2$	$0 \leq x \leq 1$	Y	Y	Y	Y	Y	Y	Y
$f_7(x) = -x^2 - x$	$0 \leq x \leq 1$	N	N	Y	Y	Y	Y	Y
$f_8(x) = -x^2$	$0 \leq x \leq 1$	N	N	N	N	Y	Y	Y

Keys:

N=No

Y=Yes

SC=Strict convexity

C=convexity

SPC=Strict Pseudoconvexity

PC=Pseudoconvexity

SQC= Strict Quasiconvexity

QC=Quasiconvexity

XC=Unnamed convexity.

The notion of convexity in metric space was introduced by Takahashi in 1970 and he established that all normed spaces and their convex subsets are convex metric spaces. He also gave examples of convex metric spaces which are not imbedded in any Normed or Banach spaces.

Motivated by the work of Ponstein and Takahashi, we introduce the concept of quasiconvex metric spaces by replacing the notion of convex function with a more weaker function, quasiconvex function. We give examples in quasiconvex metric space which are not embedded in Banach or convex metric space. Our results generalize convex metric spaces in literature.

We give some basic definitions of concepts which are needed in this work.

The following are the definitions of convexity as defined by Aibinu and Mewomo 2018.

Definition 1.1. [6]

A subset K of E is said to be convex if for every $x,y\in K$, and $\lambda\in[0,1]$, we have $\lambda x+(1-\lambda)y\in K$.

Definition 1.2. [6]

A function $f:K\to R$ defined on a convex subset K of E is convex if for any $x,y\in K$ and $\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y).$$

If we have strict inequality for all $x\neq y$ in the above definition, the function is said to be strictly convex.

Definition 1.3. [6]

A function $f:K\to R$ is quasiconvex if $f(\lambda x+(1-\lambda)y)\leq\max\{f(x),f(y)\}$, $\forall x,y\in K$ and $\lambda\in[0,1]$. Clearly, every convex function is quasiconvex but the converse is not always true. If function $f:R\to R$ defined by:

$$f(x) = \begin{cases} x-1, & x\leq 1; \\ \\ \ln x, & if x>y. \end{cases}$$

Then f is quasiconvex but it is not convex.

Definition 1.4. [8]

Let (X,d) be a complete metric space and $E\subset X$. A mapping $T:E\to E$ is said to be an involution if $T^2(x)=x$.

Definition 1.5. [8]

A function $\Psi:R^+\to R^+$ is called a comparison function if:

(i) Ψ is monotone increasing, and

(ii) $\lim_{n\to\infty}\Psi^n(t)=0\ \forall t\in R^+$.

2. MAIN RESULTS

In this section, we introduce the notion of quasiconvex metric space and prove some fixed point theorems in this new space. Throughout this paper, we denote metric space as (X,d) or X.

Definition 2.1.

Let (X,d) be a metric space, A mapping $\gamma : X \times X \times [0,1] \to X$ is said to have quasiconvex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u,\gamma(x,y,\lambda)) \leq max\{d(u,x),d(u,y)\} \tag{1}$$

Definition 2.2

A metric space (X,d) having quasiconvex structure γ is called a quasiconvex metric space.

Remark 2.3

If $max\{d(u,x),d(u,y)\} = \lambda d(u,x)+(1-\lambda)d(u,y)$ in Definition 2.1 where $\lambda \in [0,1]$, we obtain convex structure in metric spaces as defined by Takahashi [12].

Example 2.4

Considering a linear space, V which is at the same time a metric space with metric, d. For all $x,y \in V$ and $\lambda \in [0,1]$ if:

(i) $d(x,y) = d(x-y,0)$, and

(ii) $d(\lambda x+(1-\lambda)y,0) = \max\{d(x,0),d(y,0)\}$

Then V is a quasiconvex metric space.

Example 2.5

Considering a linear space, V which is at the same time a metric space with metric, d defined by

$$d(x,y) \;=\; \begin{cases} 0, & if \quad x=y=0; \\ 1, & if \quad x,y \in N; \\ 0.5, & Otherwise. \end{cases}$$

For all $x, y, z \in V$ and $\lambda \in [0,1]$ if:

(i) $d(x, y+z) = d(x-y, z)$, and

(ii) $d(\lambda x + (1-\lambda)y, z) \leq \max\{d(x,z), d(y,z)\}$

Then V is a quasiconvex metric space but not convex metric space because if $x = 0, y = 2, z = 3$ and $\lambda = 0.5$, we obtain $d(1,3) = 1 > 0.5d(0,3) + 0.5d(2,3) = 0.75$.

Definition 2.6.

A subset C of a quasiconvex metric space X is said to be quasiconvex if $\gamma(x, y, \lambda) \in C$ and $\gamma(x, y, \lambda) \leq \max\{x, y\}$ for all $x.y \in C$ and $\lambda \in [0,1]$.

Definition 2.7.

Let (X, d, γ) be a complete quasiconvex metric space and E a nonempty closed convex subset of X. A mapping $T : E \to E$ is said to be (k, L)-Lipschitzian if there exists $k \in [1, \infty)$, $L \in [0,1)$ such that

$$d(Tx, Ty) \leq Ld(x, Tx) + kd(x, y), \forall x, y \in E. \tag{2}$$

The above mapping generalizes many known mappings in literature.

Definition 2.8.

Let (X, d, γ) be a quasiconvex metric space. An open ball $S(z, r)$ in (X, d, γ) is defined by

$$S(z, r) = \{(x, y) \in X^2 : d(z, \gamma(x, y, \lambda)) < r\}.$$

Definition 2.9.

Let (X, d, γ) be a quasiconvex metric space. A closed ball $\bar{S}(z, r)$ in (X, d, γ) is defined by

$$\bar{S}(z, r) = \{(x, y) \in X^2 : d(z, \gamma(x, y, \lambda)) \leq r\}.$$

The following propositions show that an open ball and a closed ball in quasiconvex metric space are respectively open and closed subset of the space.

Proposition 2.10.

Let X be a quasiconvex metric space. Open ball $S(x, r)$ and closed ball $\bar{S}(x, r)$ in X are quasiconvex subsets of X.

Proof

For $y,z \in S(x,r) \subset X$ and $\lambda \in [0.1]$, we have $\gamma(y,z,\lambda) \in X$. It is sufficient to show that $\gamma(y,z,\lambda) \in S(x,r)$. Since X is a quasiconvex metric space,

$$d(x,\gamma(y,z,\lambda)) \leq max\{d(x,y),d(x,z)\} < r.$$

Therefore, $\gamma(y,z,\lambda) \in S(x,r)$. Similarly, $\bar{S}(x,r)$ is a quasiconvex subset of X.

Proposition 2.11.

Let X be a quasiconvex metric space. For $y,z \in X$ and $\lambda \in [0.1]$,

$$d(x,y) \leq d(x,\gamma(x,y,\lambda)) + d(\gamma(x,y,\lambda),y).$$

These theorems extend the results of Beg[4] as well as result of Goebel[6].

Theorem 2.12

Let (X,d,γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a (k,L)-Lipschitzian mapping. Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in F$ there exists $q \in F$ with

$$d(Tq,q) \leq \psi(d(Tx,x)) \tag{3}$$

Then T has a fixed point in F.

Proof:

Suppose $x_0 \in F$ is an arbitrary point. By condition (3), we obtain

$$d(Tx_{n+1},x_{n+1}) \leq \psi(d(Tx_n,x_n)), n = 0,1,2,... \tag{4}$$

By induction in (4), we obtain

$$\begin{aligned} d(Tx_{n+1},x_{n+1}) &\leq \psi(d(Tx_n,x_n)) && (5)\\ &\leq \psi^2(d(Tx_{n-1},x_{n-1})) \leq ... \leq && (6)\\ &\leq \psi^{n+1}(d(Tx_0,x_0)) && (7) \end{aligned}$$

Using (7) along with (1), the following are obtained:

$$\begin{aligned} d(Tx_n,x_n) &\leq d(Tx_n,\gamma(Tx_n,x_n,\lambda))+d(\gamma(Tx_n,x_n,\lambda),x_n) \\ &\leq \max\{d(Tx_n,Tx_n),d(Tx_n,x_n)\}+\max\{d(Tx_n,x_n),d(x_n,x_n)\} \\ &\leq 2d(Tx_n,x_n) \\ &\leq 2\psi^n(d(Tx_0,x_0)). \end{aligned}$$

Since ψ is a comparison function, we get $d(Tx_n,x_n) \to 0$ as $n \to \infty$.

Hence, $\{x_n\}$ is a Cauchy sequence in F. By completeness, there exists $x^* \in F$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

By (2), (7) and triangle inequality, we have

$$\begin{aligned} d(Tx^*,x^*) &\leq d(Tx^*,Tx_n)+d(Tx_n,x_n)+d(x_n,x^*) && (8)\\ &\leq Ld(Tx_n,x_n)+kd(x_n,x^*)+d(Tx_n,x_n)+d(x_n,x^*) && (9)\\ &\leq (1+L)d(Tx_n,x_n)+(1+k)d(x_n,x^*) && (10)\\ &\leq (1+L)\psi^n(d(Tx_0,x_0))+(1+k)d(x_n,x^*) && (11) \end{aligned}$$

So, $d(Tx^*,x^*) \to 0$ as $n \to \infty$ which implies that $Tx^* = x^*$. Hence, x^* is a fixed point of T.

Theorem 2.12 can be extended to the next result.

Theorem 2.13

Let (X,d,γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a (k,L)-Lipschitzian mapping. Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in F$ there exists $q \in F$ with

(i) $d(Tq,q) \leq \psi(d(Tx,x))$;

(ii) $d(Tq,Tx) \leq cd(Tx,x)$, $c > 0$.

Then T has a fixed point in F.

Proof:

Suppose $x_0 \in F$ is an arbitrary point. By condition (i) and (ii), we obtain

$$d(Tx_{n+1},x_{n+1}) \leq \psi(d(Tx_n,x_n)), n = 0,1,2,... \qquad (12)$$

and

$$d(Tx_n, x_n) \leq cd(x_n, x_{n-1}), n = 0, 1, 2, ... \tag{13}$$

By induction in (13), we obtain

$$d(Tx_{n+1}, x_{n+1}) \leq \psi(d(Tx_n, x_n)) \tag{14}$$

$$\leq \psi^2(d(Tx_{n-1}, x_{n-1})) \leq ... \leq \tag{15}$$

$$\leq \psi^{n+1}(d(Tx_0, x_0)) \tag{16}$$

(16) and (13)along with (1) give

$$d(Tx_{n+1}, x_{n+1}) \leq 2cd(Tx_n, x_n) \leq 2c\psi^n(d(Tx_0, x_0)) \tag{17}$$

Since ψ is a comparison function, we get $d(Tx_n, x_n) \to 0$ as $n \to \infty$.

Hence, $\{x_n\}$ is a Cauchy sequence in F. By completeness, there exists $x^* \in F$ such that

$$\lim_{n \to \infty} x_n = x^*.$$

By (2), (16) and triangle inequality, we have

$$d(Tx^*, x^*) \leq d(Tx^*, Tx_n) + d(Tx_n, x_n) + d(x_n, x^*) \tag{18}$$

$$\leq Ld(Tx_n, x_n) + kd(x_n, x^*) + d(Tx_n, x_n) + d(x_n, x^*) \tag{19}$$

$$\leq (1 + L)d(Tx_n, x_n) + (1 + k)d(x_n, x^*) \tag{20}$$

$$\leq (1 + L)\psi^n(d(Tx_0, x_0)) + (1 + k)d(x_n, x^*) \tag{21}$$

So, $d(Tx^*, x^*) \to 0$ as $n \to \infty$ which implies that $Tx^* = x^*$. Hence, x^* is a fixed point of T.

Theorem 2.14

Let (X, d, γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a k-Lipschitzian involution with $k \in [0, 1)$. Then T has a fixed point in F.

Proof:

For any $x \in F$, let $u = \gamma(x, Tx, \alpha_n)$. Then,

$$
\begin{aligned}
d(u,x) &= d(\gamma(x,Tx,\alpha_n),x) && (22)\\
&\leq \max\{d(x,x),d(x,Tx)\} && (23)\\
&= d(Tx,x) && (24)
\end{aligned}
$$

Also

$$
\begin{aligned}
d(u,Tu) &= d(\gamma(x,Tx,\alpha_n),Tu) && (25)\\
&\leq \max\{d(x,Tu),d(Tx,Tu)\} && (26)\\
&\leq \max\{d(T^2x,Tu),d(Tx,Tu)\} && (27)\\
&\leq \max\{kd(Tx,u),kd(x,u)\} && (28)\\
&= kd(x,Tx) && (29)
\end{aligned}
$$

Using (29) repeatedly, we have

$$d(Tx_n,x_n) \leq k^n d(Tx_0,x_0) \tag{30}$$

By taking the limit, $d(Tx_n,x_n) \to 0$ as $n \to \infty$. So, $\{x_n\}$ is a Cauchy sequence in F. By completeness of F, there exists $x^* \in F$ such that

$$\lim_{n\to\infty} x_n = x^*$$

By using triangle inequality, we get

$$
\begin{aligned}
d(Tx^*,x^*) &\leq d(Tx^*,Tx_n)+d(Tx_n,x_n)+d(x_n,x^*) && (31)\\
&\leq kd(x^*,x_n)+d(Tx_n,x_n)+d(x_n,x^*) && (32)\\
&= d(Tx_n,x_n)+(1+k)d(x_n,x^*) && (33)\\
&\leq k^n d(Tx_0,x_0)+(1+k)d(x_n,x^*) && (34)
\end{aligned}
$$

So, $d(Tx^*,x^*) \to 0$ as $n \to \infty$ which implies that $Tx^* = x^*$. Hence, x^* is the fixed point of T.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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