APPROXIMATE FIXED POINT THEOREMS OF CYCLICAL CONTRACTION MAPPING ON G-METRIC SPACES

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Abstract. This paper introduce a new class of operators and contraction mapping for a cyclical map T on G-metric spaces and the approximately fixed point properties. Also, we prove two general lemmas regarding approximate fixed Point of cyclical contraction mapping on G-metric spaces. Using these results we prove several approximate fixed point theorems for a new class of operators on G-metric spaces (not necessarily complete). These results can be exploited to establish new approximate fixed point theorems for cyclical contraction maps. Further, there is a new class of cyclical operators and contraction mapping on G-metric space (not necessarily complete) which do not need to be continuous. Finally, examples are given to support the usability of our results.

Keywords: approximate fixed points; G-Mohseni-semi cyclical operator G-Mohseni cyclical operator; G-Mohsenialhosseini cyclical operator; diameter approximate fixed point.

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1. INTRODUCTION

Fixed point theory is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the study of existence and

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uniqueness of solutions for a class of nonlinear integral equations. Some recent work on fixed point theorems of integral type in G-metric spaces, stability of functional difference equation can be found in [19, 20] and the references therein.

In 1968, Kannan (see [7]) proved a fixed point theorem for operators which need not be continuous. Further, Chatterjea (see[6]), in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. In 1972, by combining the above three independent contraction conditions above, Zamfirescu (see [22]) obtained another fixed point result for operators which satisfy the following. In 2001, Rus (see [21]) defined $\alpha-$contraction. In [3], the author obtained a different contraction condition, also he formulated a corresponding fixed point theorem. In 2006, Berinde (see [4]) obtained some result on $\alpha-$contraction for approximate fixed point in metric space. Miandaragh et al. [9, 10] obtained some result on approximate fixed points in metric space.

On the other hand, in 2006, Mustafa and Sims [17, 18] introduced the notion of generalized metric spaces or simply G-metric spaces. Many researchers have obtained fixd point, coupled fixed point, coupled common fixed point results on G-metric spaces (see [2, 5, 19]).

In 2011, Mohsenalhosseini et al [11], introduced the approximate best proximity pairs and proved the approximate best proximity pairs property for it. Also, In 2012 , Mohsenalhosseini et al [12], introduced the approximate fixed point for completely norm space and map T_{α} and proved the approximate fixed point property for it. In 2014 , Mohsenalhosseini [13] introduced the Approximate best proximity pairs on metric space for contraction maps. Also, Mohsenalhosseini in [14] introduced the approximate fixed point in G-metric spaces for various types of operators. Recently, in 2017 Mohsenialhosseini [15] introduced the approximate fixed points of operators on G-metric spaces. The aim of this paper is to introduce the new classes of operators and contraction maps (not necessarily continuous) regarding approximate fixed point and diameter approximate fixed point for cyclical contraction mapping on G-metric spaces. Also, we give some illustrative example of our main results.

2. PRELIMINARIES

This section recalls the following notations and the ones that will be used in what follows. In 2003, kirk et al [8], obtained an extension of Banach's fixed point theorem by considering a cyclical operator.

Definition 1.1. [8] *Let $\{X_i\}_{i=1}^m$ be nonempty susets of a complete metric space X. A mapping $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ satisfies the following condition(where $X_{i+1} = X_1$)*

$$T(X_1) \subseteq X_2, ..., T(X_{m-1}) \subseteq X_m, T(X_m) \subseteq X_1,$$

is called a cyclical operator.

Definition 2.1. cite Mus Let X be a nonempty set and let $G : X \times X \times X \longrightarrow R^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ *if and only if* $x = y = z$;

(G2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (*symmetry in all three variables*);

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ (*rectangle inequality*).

Then, the function G is called generalized metric or, more specifically $G - metric$ on X, and the pair (X,G) is called a $G - metric$ space.

Proposition 2.2. [17] *Every G-metric (X,G) defines a metric space (X,d_G) by*

1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.

if (X,G) is a symmetric G- metric space. Then

2) $d_G(x,y) = 2G(x,y,y)$.

Definition 2.3. [12] *Let $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is an $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.*

Remark 2.4. [12] *In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :*

$$F_\varepsilon(T) = \{x \in X \mid x \text{ is an } \varepsilon - \text{ fixed point of } T\}.$$

Definition 2.5. [12] Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \varepsilon > 0, \ F_\varepsilon(T) \neq \varnothing.$$

Lemma 2.6. [16] *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X and $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ be a cyclical operator. Let $x_0 \in \cup_{i=1}^m X_i$ and $\varepsilon > 0$. If $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is asymptotically regular at each point $x_0 \in \cup_{i=1}^m X_i$, then T has an $\varepsilon-$ fixed point.*

Lemma 2.7. [16] *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X and $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ be a cyclical operator. Let $x_0 \in \cup_{i=1}^m X_i$ and $\varepsilon > 0$. If $d(T^n(x_0), T^{n+k}(x_0)) \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

Lemma 2.8. [16] *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X, $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ a cyclical operator and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(T) \neq \emptyset$;

(ii) $\forall \theta > 0, \ \exists \phi(\theta) > 0$ *such that*;

$$d(x,y) - d(Tx,Ty) \ \leq \ \theta \Rightarrow d(x,y) \leq \phi(\theta), \ \forall x, y \in F_\varepsilon(T) \neq \emptyset.$$

Then:

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon).$$

3. MAIN RESULTS

We begin with two lemmas which will be used in order to prove all the results given in third section. Let (X, G) be a $G-metric$ space.

Definition 3.1. Let $\{X_i\}_{i=1}^m$ be nonempty susets of a $G-metric$ space X and $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ be a cyclical operator. Let $\varepsilon > 0$ and $x_0 \in \cup_{i=1}^m X_i$. Then x_0 is an ε- fixed point of T if

$$[G(x_0, Tx_0, Tx_0) + G(Tx_0, x_0, x_0)] < \varepsilon.$$

Remark 3.2. In this paper we will denote the set of all ε-fixed points of T, for a given ε, by:

$$F_G^\varepsilon(T) = \{x \in \cup_{i=1}^m X_i \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Definition 3.3. Let $\{X_i\}_{i=1}^m$ be nonempty closed subsets of a $G-metric$ space X, $T : \cup_{i=1}^m X_i \rightarrow$ $\cup_{i=1}^m X_i$ be a cyclical operator and $\varepsilon > 0$. We define diameter of the set $F_G^{\varepsilon}(T)$, i.e.,

$$\delta(F_G^{\varepsilon}(T)) = \sup\{G(x,y,z) : x,y,z \in F_G^{\varepsilon}(T)\}.$$

Definition 3.4. Let $\{X_i\}_{i=1}^m$ are closed subsets of a $G-metric$ space X and $T : \cup_{i=1}^m X_i \rightarrow \cup_{i=1}^m X_i$ be a cyclical operator. Then T has the approximate fixed point property (a.f.p.p) if $\forall \varepsilon > 0$,

$$F_G^{\varepsilon}(T) \neq \emptyset.$$

Definition 3.5. Let $\{X_i\}_{i=1}^m$ are closed subsets of a $G-metric$ space X. A cyclical operator $T : \cup_{i=1}^m X_i \rightarrow \cup_{i=1}^m X_i$ is said to be asymptotically regular at a point $x \in \cup_{i=1}^m X_i$, if

$$\lim_{n \to \infty} \{G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)\} = 0,$$

where T^n denotes the *n*th iterate of T at x.

Lemma 3.6. *Let $\{X_i\}_{i=1}^m$ are closed subsets of a $G-metric$ space X. If $T : \cup_{i=1}^m X_i \rightarrow \cup_{i=1}^m X_i$ is asymptotically regular at a point $x \in \cup_{i=1}^m X_i$, Then T has an approximate fixed point.*

Proof. The proof of Lemma is the same as the proof of Lemma 2.6 for $x \in \cup_{i=1}^m X_i$.   □

Lemma 3.7. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ space X and $T : \cup_{i=1}^m X_i \rightarrow \cup_{i=1}^m X_i$ be a cyclical operator. Let $x_0 \in \cup_{i=1}^m X_i$ and $\varepsilon > 0$. If*

$$G(T^n(x_0), T^{n+k}(x_0), T^{n+k}(x_0)) + G(T^{n+k}(x_0), T^n(x_0), T^n(x_0)) \to 0$$

as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.

Proof. The proof of Lemma is the same as the proof of Lemma 2.7 for $x \in \cup_{i=1}^m X_i$.   □

Lemma 3.8. *Let $\{X_i\}_{i=1}^m$ are closed subsets of a $G-metric$ space X, $T : \cup_{i=1}^m X_i \rightarrow \cup_{i=1}^m X_i$ a cyclical operator and $\varepsilon > 0$. We assume that:*
a) $F_G^{\varepsilon}(T) \neq \emptyset$;
b) $\forall \xi > 0 \ \exists \psi(\xi) > 0$ such that

$$[G(x,y,y) + G(y,x,x)] - [G(Tx,Ty,Ty) + G(Ty,Tx,Tx)] < \xi \implies$$

$$G(x,y,y)+G(y,x,x) \leq \psi(\xi), \quad \forall x,y \in F_G^\varepsilon(T).$$

Then:

$$\delta(F_G^\varepsilon(T)) \leq \psi(2\varepsilon).$$

Proof. The proof of Lemma is the same as the proof of Lemma 2.8 for $x \in \cup_{i=1}^m X_i$.   □

4. APPROXIMATE FIXED POINT FOR SEVERAL OPERATOR ON $G-$ METRIC SPACES

In this section a series of qualitative and quantitative results will be obtained regarding the properties of approximate fixed point. Also, by using Proposition 2.2 and lemma 2.7 we prove approximate fixed point theorems and diameter approximate theorems for a new class of cyclical operators on G-metric spaces.

Definition 4.1. [14] Let (X,d) be a metric space. A mapping $T : X \to X$ is a Mohseni operator if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha[d(x,y)+d(Tx,Ty)].$$

Definition 4.2. [16] Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X, $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a $\alpha-$cyclical contraction if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha d(x,y) \quad \forall x \in X_i, y \in X_{i+1}.$$

Definition 4.3. Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ space X, $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a $G-\alpha-$cyclical contraction if there exists $\alpha \in (0,1)$ such that

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \alpha[G(x,y,y)+G(y,x,x)$$
$$+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \quad \forall x \in X_i, y \in X_{i+1}.$$

Theorem 4.4. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ metric space X and Suppose $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a $G-\alpha-$cyclical contraction. Then T has an $\varepsilon-$fixed point.*

Proof: Let $\varepsilon > 0$ and $x \in \cup_{i=1}^{m} X_i$.

$$
\begin{aligned}
G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x) &= G(T(T^{n-1} x), T(T^n x), T(T^n x)) \\
&+ G(T(T^n x), T(T^{n-1} x), T(T^{n-1} x)) \\
&\leq \alpha[G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)] \\
&\vdots \\
&\leq (\alpha)^n[G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}
$$

But $\alpha \in (0, \frac{1}{2})$. Hence

$$
lim_{n \to \infty}(G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)) = 0, \forall x \in \cup_{i=1}^{m} X_i.
$$

Hence by Lemma 3.6 it follows that $F_G^{\varepsilon}(T) \neq \emptyset, \forall \varepsilon > 0.$ $\square$

In 1972, Chatterjea (see [6]) considered another operator in which continuity is not imposed. Now, the appoximate fixed point theorems by using cyclical operators on *G*-metric spaces are obtained.

Definition 4.5. Let $\{X_i\}_{i=1}^{m}$ be nonempty subsets of a $G-metric$ metric space X, $T : \cup_{i=1}^{m} X_i \to \cup_{i=1}^{m} X_i$ is a G-Chatterjea cyclical operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$
\begin{aligned}
[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] &\leq \alpha[G(x, Ty, Ty) + G(Ty, x, x) \\
&+ G(y, Tx, Tx) + G(Tx, y, y)] \quad \forall x \in X_i, y \in X_{i+1}.
\end{aligned}
$$

Theorem 2.4. *Let* $\{X_i\}_{i=1}^{m}$ *be nonempty subsets of a* $G-metric$ *space* X. *Suppose that the mapping* $T : \cup_{i=1}^{m} X_i \to \cup_{i=1}^{m} X_i$ *is a G-Chatterjea cyclical operator. Then T has an* $\varepsilon-fixed$ *point.*

Proof: Let $\varepsilon > 0$ and $x \in \cup_{i=1}^{m} X_i$.

$$
\begin{aligned}
[G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] &= [G(T(T^{n-1} x), T(T^n x), T(T^n x)) \\
&+ \ G(T(T^n x), T(T^{n-1} x), T(T^{n-1} x))] \\
&\leq \ \alpha[G(T^{n-1} x, T(T^n x), T(T^n x)) \\
&+ \ G(T(T^n x), T^{n-1} x, T^{n-1} x) \\
&+ \ G(T^n x, T(T^{n-1} x), T(T^{n-1} x)) \\
&+ \ G(T(T^{n-1} x), T^n x, T^n x)] \\
&= \ \alpha[(G(T^{n-1} x, T^{n+1} x, T^{n+1} x) \\
&+ \ G(T^{n+1} x, T^{n-1} x, T^{n-1} x))].
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
G(T^{n-1} x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^{n-1} x, T^{n-1} x) &\leq \ [G(T^{n-1} x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1} x, T^{n-1} x) \\
&+ \ G(T^n x, T^{n+1} x, T^{n+1} x) \\
&+ \ G(T^{n+1} x, T^n x, T^n x)].
\end{aligned}
$$

Then

$$
\begin{aligned}
(1 - \alpha)(G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)) &\leq \ \alpha[G(T^{n-1} x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1} x, T^{n-1} x)],
\end{aligned}
$$

hence

$$
\begin{aligned}
(G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)) &\leq \ \frac{\alpha}{1 - \alpha}[(G(T^{n-1} x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1} x, T^{n-1} x))] \\
&\ \ \vdots \\
&\leq \ (\frac{\alpha}{1 - \alpha})^n [G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}
$$

But $\alpha \in (0, \frac{1}{2})$ hence $\frac{\alpha}{1-\alpha} \in (0,1)$. Therfore

$$lim_{n\to\infty}(G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)) = 0, \forall x \in \cup_{i=1}^m X_i.$$

Hence by Lemma 3.6 it follows that $F_G^\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0$. □

Definition 4.6. Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G - metric$ space X, $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a **G-Mohseni cyclical operator** if there exists $\alpha \in (0, \dfrac{1}{2})$ such that

$$[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \alpha[G(x, y, y) + G(y, x, x)$$
$$+ G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \quad \forall x \in X_i, y \in X_{i+1}.$$

Theorem 4.7. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G - metric$ space X and Suppose $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a G-Mohseni cyclical operator. Then T has an $\varepsilon-$fixed point.*

Proof: Let $\varepsilon > 0$ and $x \in \cup_{i=1}^m X_i$.

$$\begin{aligned}
[G(T^n x, T^{n+1} x, T^{n+1} x) &+ G(T^{n+1} x, T^n x, T^n x)] = [G(T(T^{n-1} x), T(T^n x), T(T^n x)) \\
&+ G(T(T^n x), T(T^{n-1} x), T(T^{n-1} x))] \\
&\leq \alpha[G(T^{n-1} x, T^n x, T^n x) + G(T^n x, T^{n-1} x, T^{n-1} x) \\
&+ G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha)[G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] &\leq \alpha[G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)].
\end{aligned}$$

So,

$$\begin{aligned}
[G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] &\leq \frac{\alpha}{(1-\alpha)}[G(T^{n-1} x, T^n x, T^n x) \\
&+ G(T^n x, T^{n-1} x, T^{n-1} x)] \\
&\vdots \\
&\leq (\frac{\alpha}{(1-\alpha)})^n[G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$, therefore $(\frac{\alpha}{1-\alpha}) \in (0,1)$. Hence

$$lim_{n\to\infty}(G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)) = 0, \forall x \in \cup_{i=1}^m X_i.$$

Hence by Lemma 3.6 it follows that $F_G^{\mathcal{E}}(T) \neq \emptyset, \forall \varepsilon > 0$. $\square$

Example 4.8. Consider the sets: $A_1 = \{\frac{1}{k}\}_{k=1}^\infty \cup \{\frac{-1}{2k}\}_{k=1}^\infty$ and $A_2 = \{\frac{-1}{k}\}_{k=1}^\infty \cup \{\frac{1}{2k-1}\}_{k=1}^\infty$. Define the map $T : \cup_{i=1}^2 A_i \to \cup_{i=1}^2 A_i$ as

$$Tx = \begin{cases} \frac{-x}{x+4} & if \quad x \in A_1 \\ \\ \frac{-x}{4} & if \quad x \in A_2 \end{cases}$$

It is easily to be checked that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. For any $x \in A_1$ and $y \in A_2$ and Proposition 2.2 we have the chain of inequalities

$$\begin{aligned}
[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] = d_G(Tx, Ty) &= |\frac{x}{x+4} - \frac{y}{4}| \\
&\leq \frac{1}{3}(|x| + |y|) \\
&\leq \frac{1}{3}(|x-y| + |\frac{x}{x+4} - \frac{y}{4}|) \\
&= \frac{1}{3}(d_G(x,y) + d_G(Tx, Ty)) \\
&= \frac{1}{3}[G(x,y,y) + G(y,x,x) \\
&+ G(Tx, Ty, Ty) + G(Ty, Tx, Tx)].
\end{aligned}$$

So T satisfies all the conditions of Theorem 4.7 and thus it has a approximate fixed point.

Example 4.9. Let X be a subset in R endowed with the usual metric. Suppose $A_1 =]0, 0.8]$ and $A_2 =]0, \frac{1}{2}]$. Define the map $T : \cup_{i=1}^2 A_i \to \cup_{i=1}^2 A_i$ as $Tx = \frac{x}{4}$ for all $x \in \cup_{i=1}^2 A_i$. It is easily to be checked that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. For any $x \in A_1$ and $y \in A_2$ and Proposition 2.2 we

have the chain of inequalities

$$
\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]=d_G(Tx,Ty) &= |\frac{x}{4}-\frac{y}{4}| \\
&\leq \frac{1}{3}(|x-y|+|\frac{x}{4}-\frac{y}{4}|) \\
&= \frac{1}{3}(d_G(x,y)+d_G(Tx,Ty)) \\
&= \frac{1}{3}[G(x,y,y)+G(y,x,x) \\
&+ G(Tx,Ty,Ty)+G(Ty,Tx,Tx)].
\end{aligned}
$$

So T satisfies all the conditions of Theorem 4.7 and thus for every $\varepsilon > 0$, $F_\varepsilon(T)) \neq \emptyset$. on the other hand take $0 < \varepsilon < \frac{1}{2}$ and select $x_0 \in \cup_{i=1}^2 A_i$ such that $x_0 < \frac{4}{3}\varepsilon$. Then

$$
d(Tx,x) = |\frac{x}{4}-x| \leq \varepsilon.
$$

Hence by by Proposition 2.2, $G(Tx,x,x)+G(x,Tx,Tx) \leq \varepsilon$. So T has an approximate fixed point which implies that $F_\varepsilon(T)) \neq \emptyset$. On the contrary, there is no fixed point of T in $\cup_{i=1}^2 A_i$.

By combining the three independent contraction conditions: $G-\alpha-$cyclical contraction, G-Mohseni cyclical, and G-Chatterjea cyclical operators we obtain another approximate fixed point result for operators which satisfy the following.

Definition 4.10. Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ space X, $T:\cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a **G-Mohsenialhosseini cyclical operator** if there exists $\alpha,\beta,\gamma \in R$, $\alpha \in [0,1[,\beta \in [0,\frac{1}{2}[,\gamma \in [0,\frac{1}{2}[$ such that for all $x \in_{i=1}^m X_i, y \in X_{i+1}$ at least one of the following is true:

$(i)G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \alpha[G(x,y,y)+G(y,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)];$

$(ii)[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \beta[G(x,y,y)+G(y,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)];$

$(iii)[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \gamma[G(x,Ty,Ty)+G(Ty,x,x)+G(y,Tx,Tx)+G(Tx,y,y)].$

Theorem 4.11. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ metric space X and Suppose $T:\cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a G-Mohsenialhosseini cyclical operator. Then T has an $\varepsilon-$fixed point.*

Proof: Let $x, y \in \cup_{i=1}^{m} X_i$. Supposing *ii*) holds, we have that:

$$
\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \;\leq\;& \beta[G(x,y,y)+G(y,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]; \\
\leq\;& \beta[G(x,Tx,Tx)+G(Tx,x,x)+G(Tx,y,y)+G(y,Tx,Tx) \\
+\;& G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
\leq\;& \beta[G(x,Tx,Tx)+G(Tx,x,x)+G(Tx,x,x)+G(x,Tx,Tx) \\
+\;& G(x,y,y)+G(y,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
=\;& 2\beta[G(x,Tx,Tx)+G(Tx,x,x)]+\beta[G(x,y,y) \\
+\;& G(y,x,x)]+\beta[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)].
\end{aligned}
$$

Thus

(4.1)
$$
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \frac{2\beta}{1-\beta}[G(x,Tx,Tx)+G(Tx,x,x)]+\frac{\beta}{1-\beta}[G(x,y,y)+G(y,x,x)].
$$

Supposing *iii*) holds, we have that:

$$
\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \;\leq\;& \gamma[G(x,Ty,Ty)+G(Ty,x,x)+[G(y,Tx,Tx)+G(Tx,y,y)] \\
\leq\;& \gamma[G(x,y,y)+G(y,x,x)+G(y,Ty,Ty)+G(Ty,y,y)] \\
+\;& \gamma[G(y,Ty,Ty)+G(Ty,y,y) \\
+\;& G(Ty,Tx,Tx)+G(Tx,Ty,Ty)] \\
=\;& \gamma[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]+2\gamma[G(y,Ty,Ty) \\
+\;& G(Ty,y,y)]+\gamma[G(x,y,y)+G(y,x,x)].
\end{aligned}
$$

Thus

(4.2)
$$
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \frac{2\gamma}{1-\gamma}[G(y,Ty,Ty)+G(Ty,y,y)]+\frac{\gamma}{1-\gamma}[G(x,y,y)+G(y,x,x)].
$$

Similarly:

$$
\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \gamma[G(x,Ty,Ty)+G(Ty,x,x)+G(y,Tx,Tx)+G(Tx,y,y)] \\
&\leq \gamma[G(x,Tx,Tx)+G(Tx,x,x) \\
&\quad + G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
&\quad + \gamma[[G(y,x,x)+G(x,y,y)+G(x,Tx,Tx)+G(Tx,x,x)] \\
&= \gamma[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
&\quad + 2\gamma[G(x,Tx,Tx)+G(Tx,x,x)]+\gamma[G(x,y,y)+G(y,x,x)].
\end{aligned}
$$

Then

(4.3)
$$
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \frac{2\gamma}{1-\gamma}[G(x,Tx,Tx)+G(Tx,x,x)] + \frac{\gamma}{1-\gamma}[G(x,y,y)+G(y,x,x)].
$$

Therefore for T satisfying at least one of the conditions (i), (ii), (iii) we have that

(4.4)
$$
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq 2\eta[G(x,Tx,Tx)+G(Tx,x,x)]+\eta[G(x,y,y)+G(y,x,x)],
$$

and

(4.5)
$$
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq 2\eta[G(y,Ty,Ty)+G(Ty,y,y)]+\eta[G(x,y,y)+G(y,x,x)],
$$

where $\eta := max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, hold. Using these conditions implied by (i) - (iii) and taking $x \in \cup_{i=1}^{m}X_i$, we have:

$$
\begin{aligned}
[G(T^n x, T^{n+1}x, T^{n+1}x)+G(T^{n+1}x, T^n x, T^n x)] &= [G(T(T^{n-1}x), T(T^n x), T(T^n x)) \\
&\quad + G(T(T^n x), T(T^{n-1}x), T(T^{n-1}x))] \\
&\leq^{(4.4)} 2\eta[G(T^{n-1}x, T(T^{n-1})x, T(T^{n-1})x)
\end{aligned}
$$

$$+ \quad G(T(T^{n-1})x, T^{n-1}x, T^{n-1}x)]$$

$$+ \quad \eta[G(T^{n-1}x, T^n x, T^n x)$$

$$+ G(T^n x, T^{n-1}x, T^{n-1}x)]$$

$$= \quad 3\eta[G(T^{n-1}x, T^n x, T^n x)$$

$$+ \quad G(T^n x, T^{n-1}x, T^{n-1}x)]$$

$$\vdots$$

$$\leq \quad (3\eta)^n [G(x, Tx, Tx) + G(Tx, x, x)].$$

Therfore

$$Lim_{n\to\infty}[G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] = 0, \ \forall x \in \cup_{i=1}^m X_i.$$

Now by Lemma 3.6 it follows that $F_G^\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0.$ $\square$

Example 4.12. Let $X = [0, \infty)$ and let d be usual metric on X. Suppose $A_1 = [0.1, 2]$ and $A_2 = [0.1, 1]$. Define the map $T : \cup_{i=1}^2 A_i \to \cup_{i=1}^2 A_i$ as

$$Tx = \begin{cases} 0 & x \in [0, 1-\beta) \\ \frac{x}{4} & x \in [1-\beta, 1) \\ \frac{1-\beta}{4} & x \in [1, 2] \end{cases}$$

It is easily to be checked that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. For any $x, y \in \cup_{i=1}^2 A_i$ there exists $\alpha \in (0, \frac{1}{2})$ such that holds at least one of the condition Theorem 4.11. Thus by Proposition 2.2 and Theorem 4.11 for every $\varepsilon > 0$, $F_G^\varepsilon(T) \neq \emptyset$.

Definition 4.13. Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a $G-metric$ space X, $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a **G-Mohseni-semi cyclical operator** if there exists $\alpha \in]0, \frac{1}{2}[$ such that

$$[G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \alpha([G(x, y, y) + G(y, x, x)] + [G(x, Tx, Tx) + G(Tx, x, x)]),$$

$\forall x \in X_i, y \in X_{i+1}.$

Theorem 4.14. *Let $\{X_i\}_{i=1}^{m}$ be nonempty subsets of a $G-$ metric space X and Suppose T :* $\cup_{i=1}^{m}X_i \to \cup_{i=1}^{m}X_i$ *is a G-Mohseni-semi cyclical operator. Then:*

$$\forall \varepsilon > 0, \ F_\varepsilon(T) \neq \emptyset.$$

Proof: Let $x \in \cup_{i=1}^{m}X_i$.

$$
\begin{aligned}
[G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] \ &= \ [G(T(T^{n-1}x), T(T^n x), T(T^n x)) \\
&+ \ G(T(T^n x), T(T^{n-1}x), T(T^{n-1}x))] \\
&\leq \ \alpha[G(T^{n-1}x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1}x, T^{n-1}x)] \\
&+ \ \alpha[G(T^{n-1}x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1}x, T^{n-1}x)] \\
&= \ 2\alpha[G(T^{n-1}x, T^n x, T^n x) \\
&+ \ G(T^n x, T^{n-1}x, T^{n-1}x)] \\
&\ \ \vdots \\
&\leq \ (2\alpha)^n[G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}
$$

But $\alpha \in]0, \frac{1}{2}[$. Therfore

$$Lim_{n \to \infty}[G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] = 0, \ \forall x \in \cup_{i=1}^{m}X_i.$$

Now by Lemma 3.6, it follows that $F_G^\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0$. $\square$

Example 4.15. Let X be a subset in R endowed with the usual metric. Suppose $A_1 = [0.01, 0.8]$ and $A_2 = [0.01, \frac{1}{2}]$. Suppose $X_1 = [0.01, 0.8]$ and $X_2 = [0.01, \frac{1}{2}]$. Define the map $T : \cup_{i=1}^{2}A_i \to \cup_{i=1}^{2}A_i$ as $Tx = \frac{x}{4}$ for all $x \in \cup_{i=1}^{2}A_i$. It is easily to be checked that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$.

For any $x \in A_1$ and $y \in A_2$ and Proposition 2.2 we have the chain of inequalities

$$
\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] = d_G(Tx,Ty) &= |\frac{x}{4}-\frac{y}{4}| \\
&\leq \frac{1}{3}(|x-y|+|x-\frac{x}{4}|) \\
&= \frac{1}{3}(d_G(x,y)+d_G(x,Tx)) \\
&= \frac{1}{3}[G(x,y,y)+G(y,x,x) \\
&+ [G(x,Tx,Tx)+G(Tx,x,x)].
\end{aligned}
$$

So T satisfies all the conditions of Theorem 4.14 and thus for every $\varepsilon > 0$, $F_G^\varepsilon(T) \neq \emptyset$.

5. DIAMETER APPROXIMATE FIXED POINT FOR SEVERAL OPERATOR ON $G-$ METRIC SPACES

In this section, using Lemma 3.8, quantitative results for new cyclical operators will be formulated and proved, and some results regarding diameter approximate fixed point of such operators on $G-$ metric spaces were given.

Theorem 5.1. *Let $\{X_i\}_{i=1}^{m}$ be nonempty subsets of a metric space X. Suppose that $T : \cup_{i=1}^{m} X_i \to \cup_{i=1}^{m} X_i$ is a $G-$Mohseni cyclical operator. Then for every $\varepsilon > 0$,*

$$
\delta(F_\varepsilon(T)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}.
$$

Proof: Let $\varepsilon > 0$. Condition i) in Lemma 3.8 is satisfied, as one can see in the proof of Theorem 4.7 we only verify that condition ii) in Lemma 3.8 holds. Let $\theta > 0$ and $x, y \in F_G^\varepsilon(T)$ and assume that Then:

$$
[G(x,y,y)+G(y,x,x)] - [G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] < \theta.
$$

Then:

$$
[G(x,y,y)+G(y,x,x)] \leq \alpha[G(x,y,y)+G(y,x,x)]+[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]+\theta.
$$

Therefore As $x, y \in F_G^\varepsilon(T)$, we know that

$$
G(x,Tx,Tx)+G(Tx,x,x) \leq \varepsilon, G(y,Ty,Ty)+G(Ty,y,y) \leq \varepsilon.
$$

Therfore, $G(x,y,y)+G(y,x,x) \le \frac{2\alpha\varepsilon+\theta}{1-2\alpha}$. So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha\varepsilon+\theta}{1-2\alpha} > 0$ such that

$$[G(x,y,y)+G(y,x,x)] - [G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] < \theta \Rightarrow G(x,y,y)+G(y,x,x)] \le \phi(\theta).$$

Now by Lemma 3.8, it follows that

$$\delta(F_G^\varepsilon(T)) \le \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_G^\varepsilon(T)) \le \frac{2\varepsilon(1+\alpha)}{1-2\alpha}. \ \square$$

Example 5.2. Let X be a subset in R endowed with the usual metric. Suppose $A_1 = [0.01, 0.8]$ and $A_2 = [0.01, \frac{1}{2}]$. Suppose $X_1 = [0.01, 0.8]$ and $X_2 = [0.01, \frac{1}{2}]$. Define the map $T : \cup_{i=1}^2 A_i \to \cup_{i=1}^2 A_i$ as $Tx = \frac{x}{4}$ for all $x \in \cup_{i=1}^2 A_i$.

By example 4.15 $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a Mohseni cyclical operator. So T satisfies all the conditions of Theorem 5.1 and thus for every $\varepsilon > 0$,

$$\delta(F_\varepsilon(T)) \le \frac{2\varepsilon(1+\alpha)}{1-2\alpha}.$$

Theorem 5.3. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X. Suppose that $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a G-Mohsenialhosseini cyclical operator. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T)) \le 2\varepsilon \frac{1+\eta}{1-\eta},$$

where $\eta := max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α, β, γ as in Definition 4.10.

Proof: In the proof of Theorem 4.11, we have already shown that if T satisfies at least one of the conditions $(i),(ii),(iii)$ from Definition 4.10, then

$$[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \le 2\eta[G(x,Tx,Tx)+G(Tx,x,x)] + \eta[G(x,y,y)+G(y,x,x)],$$

and

$$[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \le 2\eta[G(x,Ty,Ty)+G(Ty,y,y)] + \eta[G(x,y,y)+G(y,x,x)],$$

hold. Let $\varepsilon > 0$. We will only verify that condition (ii) in Lemma 3.8 is satisfied, as (i) holds, see the Proof of Theorem 4.11.

Let $\theta > 0$ and $x, y \in F_G^\varepsilon(T)$, and assume that

$$[G(x,y,y) + G(y,x,x)] - [G(Tx,Ty,Ty) + G(Ty,Tx,Tx)] \leq \theta.$$

Then

$$[G(x,y,y) + G(y,x,x)] \quad \leq \quad [G(Tx,Ty,Ty) + G(Ty,Tx,Tx)] + \theta \Rightarrow$$

$$[G(x,y,y) + G(y,x,x)] \quad \leq \quad 2\eta[G(x,Tx,Tx) + G(Tx,x,x)] + \eta[G(x,y,y) + G(y,x,x)] + \theta \Rightarrow$$

$$(1 - \eta)[G(x,y,y) + G(y,x,x)] \leq 2\eta\varepsilon + \theta$$

$$[G(x,y,y) + G(y,x,x)] \leq \frac{2\eta\varepsilon + \theta}{1 - \eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\eta\varepsilon+\theta}{1-\eta} > 0$ such that

$$[G(x,y,y) + G(y,x,x)] - [G(Tx,Ty,Ty) + G(Ty,Tx,Tx)] \leq \theta \Rightarrow [G(x,y,y) + G(y,x,x)] \leq \phi(\theta).$$

Now by Lemma 3.8, it follows that

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T)) \leq 2\varepsilon\frac{1 + \eta}{1 - \eta}, \ \forall \varepsilon > 0. \ \square$$

Example 5.4. Let $X = [0, \infty)$ and let d be usual metric on X. Suppose $A_1 = [0.1, 2]$ and $A_2 = [0.1, 1]$. Fix $\beta \in (0,1)$ and define $T : \cup_{i=1}^2 A_i \to \cup_{i=1}^2 A_i$ as

$$Tx = \begin{cases} 0 & x \in [0, 1 - \beta) \\ \frac{x}{4} & x \in [1 - \beta, 1) \\ \frac{1-\beta}{4} & x \in [1, 2] \end{cases}$$

By example 4.12 $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a G-Mohsenialhosseini cyclical operator. So T satisfies all the conditions of Theorem 5.3 and thus for every $\varepsilon > 0$, It is easy to check that

$T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. For any $x, y \in \cup_{i=1}^2 A_i$ there exists $\alpha \in (0, \frac{1}{2})$ such that holds at least one of the condition Theorem 4.11. Thus by Theorem 4.11 for every $\varepsilon > 0$, $F_G^{\varepsilon}(T) \neq \emptyset$.

Theorem 5.5. *Let $\{X_i\}_{i=1}^m$ be nonempty subsets of a metric space X. Suppose that $T : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ is a G-Mohseni-semi cyclical operator. Then for every $\varepsilon > 0$,*

$$\delta(F_G^{\varepsilon}(T)) \leq \varepsilon \frac{2 + \alpha}{1 - \alpha}.$$

Proof: Let $\varepsilon > 0$. We will only verify that condition 2) in Lemma 3.8 is satisfied. Let $\theta > 0$ and $x, y \in F_G^{\varepsilon}(T)$, and assume that $([G(x, y, y) + G(y, x, x)] - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)]) \leq \theta$. Then

$$[G(x, y, y) + G(y, x, x)] \leq [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] + \theta \Rightarrow$$

$$[G(x, y, y) + G(y, x, x)] \leq \alpha[[G(x, y, y) + G(y, x, x)] + [G(x, Tx, Tx) + G(Tx, x, x)]] + \theta \Rightarrow$$

$$(1 - \alpha)[G(x, y, y) + G(y, x, x)] \leq \alpha[G(x, Tx, Tx) + G(Tx, x, x)] + \theta$$

$$[G(x, y, y) + G(y, x, x)] \leq \frac{\alpha\varepsilon + \theta}{1 - \alpha}$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{\alpha\varepsilon + \theta}{1 - \alpha} > 0$ such that

$$[G(x, y, y) + G(y, x, x)] - [G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \theta \Rightarrow [G(x, y, y) + G(y, x, x)] \leq \phi(\theta)$$

Now by Lemma 3.8, it follows that

$$\delta(F_{\varepsilon}(T)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_{\varepsilon}(T)) \leq \varepsilon \frac{2 + \alpha}{1 - \alpha}, \forall \varepsilon > 0. \ \square$$

Remark 5.6. *Examples 4.9 and 4.12 holds in Theorem 5.5.*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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