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COMMON COUPLED FIXED POINTS OF SOME GENERALISED T-CONTRACTIONS IN RECTANGULAR B-METRIC SPACE AND APPLICATION

RENY GEORGE[1,2,*], K.P. RESHMA[3]

[1]Department of Mathematics, College of Science and Humanities at Alkharj, Prince Sattam bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia

[2]Permanent Affiliation: Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India

[3]Department of Mathematics, Rungta College of Engineering and Technology, Bhilai, C.G, India

Abstract. Common coupled fixed point theorems for a pair of generalised T-contraction mappings are proved in a rectangular b-metric space which generalize and improve some recent results due to Ramesh and Pitchamani [13] and Gu [2] and some references there in. We have given an application of our main result in establishing the existence and convergence of solution of a system of non linear integral equations under some weaker conditions, which has been properly verified using suitable example.

Keywords: coupled fixed points; rectangular b-metric space; T-contraction; weakly compatible mappings.

2010 AMS Subject Classification: Primary 47H10, Secondary 54H25.

1. Introduction and Preliminaries

In 2015 George et al [14] introduced rectangular b-metric space (in short *RbMS*) as a generalization of usual metric space, b-metric space and rectangular metric space. In recent years many

*Corresponding author

E-mail address: renygeorge02@yahoo.com

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fixed point theorems and their applications have been proved in b-metric space, *RbMS* and other similar generalised metric spaces (see [1], [4],,[5],[6], [7], [8],[9],[10],[11],[12],[15],,[16],[17], [18],[19],[20], [21],,[22], [23]). Some very recent results on common coupled fixed points can be seen in Gu [2] and Ramesh and Ptchamani [13]. In [2] the author has discussed coupled fixed point theorems for mappings defined on a set with two rectangular b-metrics r_{b1} and r_{b2} where $r_{b2} \leq r_{b1}$. Moreover in the proof of Theorem 2.1 in [2], the author shows that $r_{b1}(gx_n, gx_{n+p}) + r_{b1}(gy_n, gy_{n+p}) \leq \frac{sk^n(1+k)}{1-sk^2}.\delta_0 + s^{m-1}k^{n+2m-2}max\{\delta_0, \delta_0^*\}, 1-sk^2 \neq 0$ and on the basis of this the author claims that sequences $< gx_n >$ and $< gy_n >$ are Cauchy sequences. Note that here $p = 2m$ or $2m+1$ and hence the author's claim does not seems to be proper. In the present note we have given coupled fixed point results for a pair of generalised Reich type T contraction mappings in a *RbMS*. From our main theorem, we deduce a corrected and improved version of Theorem 2.1 of Gu [2]. At the same time we have also obtained an improved and generalised version of the results of Ramesh and Pitchamani [13]. In recent years fixed point theory has been successfully applied in establishing the existence of solution of non linear integral equations (see [13], [3]). We have applied our result in establishing convergence criteria for a unique solution of a system of non linear integral equations. We have used some weaker conditions as compared to those existing in literature.

Definition 1.1. [14] *Let M be a non empty set. Suppose that the mapping $d_r : M \times M \to R$ satisfies:*

(RbM1) $d_r(x,y) \geq 0$ *and* $d_r(x,y) = 0$ *if and only if* $x = y$

(RbM2) $d_r(x,y) = d_r(y,x)$

(RbM3) $d_r(x,y) \leq s[d_r(x,u) + d_r(u,v) + d_r(v,y)]$ *for some* $s \geq 1$, *all* $x,y,\in M$ *and all distinct points* $u,v \in M - \{x,y\}$

Then (M,d_r) is a rectangular b-metric space with coefficient s (in short RbMS(s)).

Definition 1.2. [14] *In the RbMS (M,d_r) the sequence $< x_n >$*

(a) converges to $x \in M$ *if and only if* $d_r(x_n,x) \to 0$ *as* $n \to \infty$.

(b) is a Cauchy sequence if and only if $d_r(x_n,x_{n+p}) \to 0$ *as* $n \to \infty$ *for all* $p > 0$.

Remark 1.3. *From Example 2.5 in* [14] *the following facts are easily observed:*

i) In a RbMS open balls may not be an open set.

ii)In a RbMS convergent sequences may not be a Cauchy sequence.

iii) RbMS is not necessarily Hausdorff.

iv) Rectangular b-mtric d is not necessarily continuous.

2. Main Results

Our main theorems are as follows :

Theorem 2.1. *Let (X,d_r) be a RbMS(s) , $T: X \to X$ be a one to one mapping, $S: X\times X \to X$ and $g: X \to X$ be mappings such that $S(X\times X) \subset g(X)$, $Tg(X)$ is complete. If there exist real numbers λ,μ,ν with $0 \le \lambda < 1$, $0 \le \mu,\nu \le 1$, minimum $\{\lambda\mu,\lambda\nu\} < \frac{1}{s}$ such that for all $u,v,w,z \in X$*

$$\begin{aligned} d_r(TS(u,v),TS(w,z)) &\le \lambda max\{d_r(Tgu,Tgw),d_r(Tgv,Tgz),\mu d_r(Tgu,TS(u,v)),\\ &\quad \mu d_r(Tgv,TS(v,u),\nu d_r(Tgw,TS(w,z)),\nu d_r(Tgz,TS(z,w))\} \end{aligned} \tag{2.1}$$

then

(i) S and g has a coupled coincident point.

(ii) A unique common coupled fixed point for S and g will exist provided S and g are weakly compatible.

(iii) If in addition T is sequentially continuous and convergent, then for some arbitrary $(u_0,v_0) \in X\times X$, the iterative sequences $< gu_n >, < gv_n >$ defined by $gu_n = S(u_{n-1},v_{n-1})$ and $gv_n = S(v_{n-1},u_{n-1})$ converges to the unique common coupled fixed point of S and g.

Proof: (i) We shall start the proof by showing that the sequences $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences, where $< gu_n >$ and $< gv_n >$ are as mentioned in the hypothesis.

By (2.1), we have

$$\begin{aligned} & d_r(Tgu_n, Tgu_{n+1}) = d_r(TS(u_{n-1}, v_{n-1}), TS(u_n, v_n)) \\ \leq\ & \lambda max\{d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_{n-1}, Tgv_n), \mu d_r(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})), \\ & \mu d_r(Tgv_{n-1}, TS(v_{n-1}, u_{n-1})), \nu d_r(Tgu_n, TS(u_n, v_n)), \nu d_r(Tgv_n, TS(v_n, u_n))\} \\ \leq\ & \lambda max\{d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n), \\ & d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_n, Tgu_{n+1}), d_r(Tgv_n, Tgv_{n+1})\} \end{aligned} \tag{2.2}$$

Similarly we get

$$\begin{aligned} d_r(Tgv_n, Tgv_{n+1}) \leq\ & \lambda max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_{n-1}, Tgv_n), \\ & d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_n, Tgv_{n+1}), d_r(Tgu_n, Tgu_{n+1})\} \end{aligned} \tag{2.3}$$

Let $K_n = max\{d_r(Tgu_n, Tgu_{n+1}), d_r(Tgv_n, Tgv_{n+1})\}$. By (2.2) and (2.3), we get

$$K_n \leq \lambda max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_n, Tgv_{n+1}), d_r(Tgu_n, Tgu_{n+1})\} \tag{2.4}$$

If

$$\begin{aligned} & max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_n, Tgv_{n+1}), d_r(Tgu_n, Tgu_{n+1})\} \\ & = d_r(Tgv_n, Tgv_{n+1}) \text{ or } d_r(Tgu_n, Tgu_{n+1}), \end{aligned}$$

then (2.4) will yield a contradiction. Thus we have

$$\begin{aligned} & max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_n, Tgv_{n+1}), d_r(Tgu_n, Tgu_{n+1})\} \\ & = max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n)\}, \end{aligned}$$

and then (2.4) gives

$$K_n \leq \lambda max\{d_r(Tgv_{n-1}, Tgv_n), d_r(Tgu_{n-1}, Tgu_n)\} = \lambda K_{n-1} \preceq \lambda^2 K_{n-2} \preceq \cdots \preceq \lambda^n K_0 \tag{2.5}$$

For any $m,n \in N$, we have

$$\begin{aligned}
& d_r(Tgu_m, Tgu_n) = d_r(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1}) \\
\leq\ & \lambda.Max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), \mu d_r(Tgu_{m-1}, TS(u_{m-1}, v_{m-1})), \\
& \mu d_r(Tgv_{m-1}, TS(v_{m-1}, u_{m-1})), \nu d_r(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})), \nu d_r(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))\} \\
\leq\ & \lambda max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), d_r(Tgu_{m-1}, Tgu_m), \\
& d_r(Tgv_{m-1}, Tgv_m), d_r(Tgu_{n-1}, Tgu_n), d_r(Tgv_{n-1}, Tgv_n)\}
\end{aligned}$$

Then by using 2.5 we get

$$d_r(Tgu_m, Tgu_n) \leq \lambda max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} \tag{2.6}$$

Similarly we have

$$d_r(Tgv_m, Tgv_n) \leq \lambda max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} \tag{2.7}$$

Let $K_{m,n} = max\{d_r(Tgu_m, Tgu_n), d_r(Tgv_m, Tgv_n)\}$. By (2.6) and (2.7), we get

$$K_{m,n} \leq \lambda max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} \tag{2.8}$$

If,

$$max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} = (\lambda^{m-1} + \lambda^{n-1})K_0\}$$

then (2.8) gives

$$max\{d_r(Tgu_m, Tgu_n), d_r(Tgv_m, Tgv_n)\} \leq (\lambda^m + \lambda^n)K_0$$

and since $0 < \lambda < 1$, we conclude that $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. Now if

$$max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} \neq (\lambda^{m-1} + \lambda^{n-1})K_0\}$$

then (2.8) gives

$$K_{m,n} \leq \lambda max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1})\} \tag{2.9}$$

$$\leq \lambda K_{m-1,n-1} \leq \lambda^2 K_{m-2,n-2} \leq \cdots \lambda^r K_{m-r,n-r} \tag{2.10}$$

for any positive integer $r \leq min\{m,n\}$. Since $0 < \lambda < 1$, we can find a positive integer q_0, such that $0 < \lambda^{q_0} < \frac{1}{s}$. Now from 2.9 we have

$$K_{m,m+q_0} \leq \lambda^m K_{0,q_0} \tag{2.11}$$

$$K_{n+q_0,n} \leq \lambda^n K_{q_0,0} \tag{2.12}$$

$$K_{m+q_0,n+q_0} \leq \lambda^{q_0} K_{m,n} \tag{2.13}$$

Using condition $(RbM3)$ of a rectangular b-metric and the above inequalities 2.11, 2.12 and 2.13, we have

$$\begin{aligned} K_{m,n} &\leq s[K_{m,m+q_0} + K_{m+q_0,n+q_0} + K_{n+q_0,n}] \\ &\leq \frac{s(\lambda^m + \lambda^n)}{1 - s\lambda^{q_0}} K_{0,q_0} \end{aligned}$$

Since $0 < \lambda < 1$, again we conclude that $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. Since $(Tg(X),d)$ is complete, we can find $w_{x_0}, w_{y_0} \in X$ such that

$$\lim_{n\to\infty} Tgu_n = Tgw_{x_0} and \lim_{n\to\infty} Tgv_n = Tgw_{y_0}. \tag{2.14}$$

Therefore

$$\begin{aligned} & d_r(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) \\ \leq\ & s[d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n) + d_r(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) + d_r(TS(u_{n+1}, v_{n+1}), Tgw_{x_0}) \\ \leq\ & s[\lambda max\{d_r(Tgw_{x_0}, Tgu_n), d_r(Tgw_{y_0}, Tgv_n), \mu d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \\ & \mu d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}), \nu d_r(Tgu_n, TS(u_n, v_n)), \nu d_r(Tgv_n, TS(v_n, u_n))\} \\ & +\lambda max\{d_r(Tgu_n, Tgu_{n+1}), d_r(Tgv_n, Tgv_{n+1}), \mu d_r(Tgu_n, TS(u_n, v_n)), \\ & \mu d_r(Tgv_n, TS(v_n, u_n), \nu d_r(Tgu_{n+1}, TS(u_{n+1}, v_{n+1})), \nu d_r(Tgv_{n+1}, TS(v_{n+1}, u_{n+1}))\} \\ & +d_r(Tgu_{n+2}, Tgw_{x_0}) \end{aligned}$$

$$\begin{aligned} &\leq s[\lambda max\{d_r(Tgw_{x_0},Tgu_n),d_r(Tgw_{y_0},Tgv_n),\mu d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),\\ &\mu d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0}),\nu d_r(Tgu_n,Tgu_{n+1}),\nu d_r(Tgv_n,Tgv_{n+1})\}\\ &+\lambda max\{d_r(Tgu_n,Tgu_{n+1}),d_r(Tgv_n,Tgv_{n+1}),\mu d_r(Tgu_n,Tgu_{n+1}),\\ &\mu d_r(Tgv_n,Tgv_{n+1}),\nu d_r(Tgu_{n+1},Tg(u_{n+2}),\nu d_r(Tgv_{n+1},Tgv_{n+2})\}\\ &+d_r(Tgu_{n+2},Tgw_{x_0}). \end{aligned} \tag{2.15}$$

Note that since as $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences, by definition $d_r(Tgu_n,Tgu_{n+1}) \to 0$, $d_r(Tgv_n,Tgv_{n+1}) \to 0$ as $n \to \infty$. Thus from 2.15, as $n \to \infty$ we get

$$d_r(TS(w_{x_0},w_{y_0}),Tgw_{x_0}) \leq s\lambda max\{\mu d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),\mu d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0})\}$$

Similarly, we get

$$d_r(TS(w_{y_0},w_{x_0}),Tgw_{y_0}) \leq s\lambda max\{\mu d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),\mu d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0})\}$$

Thus we have

$$\begin{aligned} &max\{d_r(TS(w_{x_0},w_{y_0}),Tgw_{x_0}),d_r(TS(w_{y_0},w_{x_0}),Tgw_{y_0})\}\\ &\leq s\lambda\mu Max\{d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0})\} \end{aligned} \tag{2.16}$$

Proceeding on the same lines as above we also have

$$\begin{aligned} &max\{d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0}))\}\\ &\leq s\lambda\nu Max\{d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0})),d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0})\} \end{aligned} \tag{2.17}$$

Using 2.16 and 2.17 along with the condition minimum $\{\lambda\mu,\lambda\nu\} < \frac{1}{s}$ we get $TS(w_{x_0},w_{y_0}) = Tgw_{x_0}$ and $TS(w_{y_0},w_{x_0}) = Tgw_{y_0}$. As T is one to one, we have $S(w_{x_0},w_{y_0}) = gw_{x_0}$ and $S(w_{y_0},w_{x_0}) = gw_{y_0}$. Therefore, (w_{x_0},w_{y_0}) is a coupled coincident point of S and g.

(ii) Suppose S and g are weakly compatible. First we will show that if $(w^*_{x_0},w^*_{y_0})$ is another coupled coincident point of S and g then $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, or in other words the point of coupled coincidence of S and g is unique. By 2.2 we have

$$\begin{aligned}
&d_r(Tgw^*_{x_0}, Tgw_{x_0}) = d_r(TS(w^*_{x_0}, w^*_{y_0}), TS(w_{x_0}, w_{y_0}))\\
&\leq \lambda max\{d_r(Tgw^*_{x_0}, Tgw_{x_0}), d_r(Tgw^*_{y_0}, Tgw_{y_0}), \mu d_r(Tgw^*_{x_0}, TS(w^*_{x_0}, w^*_{y_0})),\\
&\mu d_r(Tgw^*_{y_0}, TS(w^*_{y_0}, w^*_{x_0}), \nu d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \nu d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\}\\
&\leq \lambda max\{d_r(Tgw^*_{x_0}, Tgw_{x_0}), d_r(Tgw^*_{y_0}, Tgw_{y_0})\}
\end{aligned}$$

Similarly we have

$$d_r(Tgw^*_{y_0}, Tgw_{y_0}) \leq \lambda max\{d_r(Tgw^*_{x_0}, Tgw_{x_0}), d_r(Tgw^*_{y_0}, Tgw_{y_0})\}$$

Thus from the above two inequalities, we get

$$max\{d_r(Tgw^*_{x_0}, Tgw_{x_0}), d_r(Tgw^*_{y_0}, Tgw_{y_0}) \leq \lambda max\{d_r(Tgw^*_{x_0}, Tgw_{x_0}), d_r(Tgw^*_{y_0}, Tgw_{y_0})\}$$

which implies that, $Tgw^*_{x_0} = Tgw_{x_0}$ and $Tgw^*_{y_0} = Tgw_{y_0}$. Since T is one to one we get $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, that is the point of coupled coincidence of S and g is unique. Since S and g are weakly compatible and since (w_{x_0}, w_{y_0}) is a coupled coincident point of S and g, we have

$$ggw_{x_0} = gS(w_{x_0}, w_{y_0}) = S(gw_{x_0}, gw_{y_0})$$

and

$$ggw_{y_0} = gS(w_{y_0}, w_{x_0}) = S(gw_{y_0}, gw_{x_0})$$

which shows that (gw_{x_0}, gw_{y_0}) is a coupled coincident point of S and g. By the uniqueness of the point of coupled coincidence we get $ggw_{x_0} = gw_{x_0}$ and $ggw_{y_0} = gw_{y_0}$ and thus (gw_{x_0}, gw_{y_0}) is a common coupled fixed point of S and g. Uniqueness of the coupled fixed point follows easily from 2.2.

(iii) Now suppose T is sequentially convergent and continuous. Then since $\lim_{n\to\infty} Tgu_n = Tgw_{x_0}$ and $\lim_{n\to\infty} Tgv_n = Tgw_{y_0}$, using sequential convergence of T, we see that $< gu_n >$ and $< gv_n >$ are convergent and thus there exist u_0 and v_0 in X such that $\lim_{n\to\infty} gu_n = u_0$ and $\lim_{n\to\infty} gv_n = v_0$. Now since T is sequentially continuous we get $\lim_{n\to\infty} Tgu_n = Tu_0$ and $\lim_{n\to\infty} Tgv_n = Tv_0$. Therefore $Tgw_{x_0} = Tu_0$ and $Tgw_{y_0} = Tv_0$. Since

T is one to one, we get $gw_{x_0} = u_0$ and $gw_{y_0} = v_0$, that is $(< gu_n >, < gv_n >)$ converges to (gw_{x_0}, gw_{y_0}) which is the common coupled fixed point of S and g.

Theorem 2.2. *Theorem 2.1 with condition 2.1 replaced with the following:*

$$d_r(TS(u,v),TS(w,z)+d_r(TS(v,u),TS(z,w)\leq \lambda max\{d_r(Tgu,Tgw)+d_r(Tgv,Tgz),$$

$$\mu(d_r(Tgu,TS(u,v))+d_r(Tgv,TS(v,u)),\nu(d_r(Tgw,TS(w,z))+d_r(Tgz,TS(z,w)))\} \tag{2.18}$$

Proof: Putting $K_n^{'} = d_r(Tgu_n, Tgu_{n+1}) + d_r(Tgv_n, Tgv_{n+1})$ and $K_{m,n}^{'} = d_r(Tgu_m, Tgu_n) + d_r(Tgv_m, Tgv_n)$, and then proceeding exactly as in the proof of Theorem 2.1, we can show that

$$\begin{aligned} K_{m,n}^{'} &\leq s[K_{m,m+q_0}^{'} + K_{m+q_0,n+q_0}^{'} + mK_{n+q_0,n}^{'}] \\ &\leq \frac{s(\lambda^m+\lambda^n)}{1-s\lambda^{q_0}}K_{0,q_0}^{'} \end{aligned}$$

and so $d_r(Tgu_m, Tgu_n) \leq \frac{s(\lambda^m+\lambda^n)}{1-s\lambda^{q_0}}K_{0,q_0}^{'}$ and $d_r(Tgv_m, Tgv_n) \leq \frac{s(\lambda^m+\lambda^n)}{1-s\lambda^{q_0}}K_{0,q_0}^{'}$. Thus $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. Again proceeding as in the proof of Theorem 2.1 and taking into cosideration the fact that $d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) \leq d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) + d_r(TS(w_{y_0}, w_{x_0}), TS(u_n, v_n))$ and $d_r(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) \leq d_r(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) + d_r(TS(v_n, u_n), TS(v_{n+1}, u_{n+1}))$ we get

$$d_r(TS(w_{x_0},w_{y_0}),Tgw_{x_0})+d_r(TS(w_{y_0},w_{x_0}),Tgw_{y_0})\leq s[d_r(TS(w_{x_0},w_{y_0}),TS(u_n,v_n)$$

$$+d_r(TS(w_{y_0},w_{x_0}),TS(v_n,u_n)$$

$$+d_r(TS(u_n,v_n),TS(u_{n+1},v_{n+1}))+d_r(TS(v_n,u_n),TS(v_{n+1},u_{n+1}))$$

$$+d_r(TS(u_{n+1},v_{n+1}),Tgw_{x_0})+d_r(TS(v_{n+1},u_{n+1}),Tgw_{y_0})$$

$$\leq s[\lambda max\{d_r(Tgw_{x_0},Tgu_n)+d_r(Tgw_{y_0},Tgv_n),\mu(d_r(Tgw_{x_0},TS(w_{x_0},w_{y_0}))+$$

$$d_r(Tgw_{y_0},TS(w_{y_0},w_{x_0})),\nu(d_r(Tgu_n,TS(u_n,v_n))+d_r(Tgv_n,TS(v_n,u_n)))\}$$

$$+\lambda max\{d_r(Tgu_n,Tgu_{n+1})+d_r(Tgv_n,Tgv_{n+1}),\mu(d_r(Tgu_n,TS(u_n,v_n))+$$

$$d_r(Tgv_n,TS(v_n,u_n),\nu(d_r(Tgu_{n+1},TS(u_{n+1},v_{n+1}))+d_r(Tgv_{n+1},TS(v_{n+1},u_{n+1}))\}$$

$$+d_r(Tgu_{n+2},Tgw_{x_0})+d_r(Tgv_{n+2},Tgw_{y_0})]$$

$$\leq s[\lambda max\{d_r(Tgw_{x_0}, Tgu_n) + d_r(Tgw_{y_0}, Tgv_n), \mu(d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) +$$

$$d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})), \nu(d_r(Tgu_n, Tgu_{n+1}) + d_r(Tgv_n, Tgv_{n+1}))\}$$

$$+\lambda max\{d_r(Tgu_n, Tgu_{n+1}) + d_r(Tgv_n, Tgv_{n+1}), \mu(d_r(Tgu_n, Tgu_{n+1}) +$$

$$d_r(Tgv_n, Tgv_{n+1}), \nu(d_r(Tgu_{n+1}, Tgu_{n+2}) + d_r(Tgv_{n+1}, Tgv_{n+2})\}$$

$$+d_r(Tgu_{n+2}, Tgw_{x_0}) + d_r(Tgv_{n+2}, Tgw_{y_0})]$$

as $n \to \infty$, we get

$$\begin{aligned} & d_r(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) + d_r(TS(w_{y_0}, w_{x_0}), Tgw_{y_0}) \\ \leq \;& s\lambda\mu\{(d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\} \end{aligned}$$

Similarly we can show that

$$\begin{aligned} & d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})) \\ \leq \;& s\lambda\nu\{d_r(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_r(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})\} \end{aligned}$$

Rest of the proof follows on the same lines as in Theorem 2.1.

Our next result is a corrected and improved version of Theorem 2.1 of Gu [2].

Theorem 2.3. *Theorem 2.1 with condition 2.1 replaced with the following :*
There exist $\beta_1, \beta_2, \beta_3$ in the interval [0,1), such that $\beta_1 + \beta_2 + \beta_3 < 1$, minimum$\{\beta_2, \beta_3\} < \frac{1}{s}$ and for all $u, v, w, z \in X$

$$d_r(TS(u,v), TS(w,z) + d_r(TS(v,u), TS(z,w) \leq \beta_1(d_r(Tgu, Tgw) + d_r(Tgv, Tgz)) +$$

$$\beta_2(d_r(Tgu, TS(u,v)) + d_r(Tgv, TS(v,u)) + \beta_3(d_r(Tgw, TS(w,z)) + d_r(Tgz, TS(z,w))) \tag{2.19}$$

Proof: Proceeding on the same line and with the same notations as in the proof of Theorems 2.1 and 2.2, we can show that

$$K_n^{'} \leq \lambda^{'} K_{n-1}^{'} \leq \lambda^{'2} K_{n-2}^{'} \leq \cdots \leq \lambda^{'n} K_0^{'} \tag{2.20}$$

where $\lambda^{'} = \frac{\alpha_1+\alpha_2}{\alpha_3} < 1$. Now

$$\begin{aligned}
&d_r(Tgu_m, Tgu_n) + d_r(Tgv_m, Tgv_n) \\
&= d_r(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1}) + d_r(TS(v_{m-1}, u_{m-1}), TS(v_{n-1}, u_{n-1}) \\
&\leq \beta_1(d_r(Tgu_{m-1}, Tgu_{n-1}) + d_r(Tgv_{m-1}, Tgv_{n-1})) + \beta_2(d_r(Tgu_{m-1}, TS(u_{m-1}, v_{m-1})) \\
&+ d_r(Tgv_{m-1}, TS(v_{m-1}, u_{m-1})) + \beta_3(d_r(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})) + d_r(Tgv_{n-1}, TS(v_{n-1}, u_{n-1})) \\
&\leq \beta_1(d_r(Tgu_{m-1}, Tgu_{n-1}) + d_r(Tgv_{m-1}, Tgv_{n-1})) + \beta_2 K^{'}_{m-1} + \beta_3 K^{'}_{n-1} \\
&\leq \beta_1(d_r(Tgu_{m-1}, Tgu_{n-1}) + d_r(Tgv_{m-1}, Tgv_{n-1})) + \beta_2 \lambda^{'m-1} K^{'}_0 + \beta_3 \lambda^{'n-1} K^{'}_0
\end{aligned}$$

Thus we have

$$\begin{aligned}
K^{'}_{m,n} &\leq \beta K^{'}_{m-1,n-1} + \beta^{m-1} K^{'}_0 + \beta^{n-1} K^{'}_0 \\
&\leq \beta^r K^{'}_{m-r,n-r} + r(\beta^{m-1} + \beta^{n-1}) K^{'}_0
\end{aligned}$$

where β=Max$\{\beta_1, \beta_2, \beta_3, \lambda^{'}\}$. Note that $lim_{n\to\infty}\beta^n \to 0$ and so we can find natural number q_0 satisfying $0 < \beta^{q_0} < \frac{1}{s}$. Then we have

$$K^{'}_{m,m+q_0} \leq \beta^m K^{'}_{0,q_0} + m(\beta^m + \beta^{m+q_0}) K^{'}_0 \tag{2.21}$$

$$K^{'}_{n+q_0,n} \leq \beta^n K^{'}_{q_0,0} + n(\beta^{n+q_0} + \beta^n) K^{'}_0 \tag{2.22}$$

$$K^{'}_{m+q_0,n+q_0} \leq \beta^{q_0} K^{'}_{m,n} + q_0(\beta^{m+q_0} + \beta^{n+q_0}) K^{'}_0 \tag{2.23}$$

Now using 2.21,2.22 and 2.23 we get

$$\begin{aligned}
K^{'}_{m,n} &\leq s[K^{'}_{m,m+q_0} + K^{'}_{m+q_0,n+q_0} + K^{'}_{n+q_0,n}] \\
&\leq s\frac{(\beta^m + \beta^n) K^{'}_{0,q_0}}{1 - s\beta^{q_0}} \\
&+ s\frac{[\beta^m(m + (m+q_0)\beta^{q_0}) + \beta^n(n + (n+q_0)\beta^{q_0})] K^{'}_0}{1 - s\beta^{q_0}}
\end{aligned}$$

As $m,n \to \infty$, $K^{'}_{m,n} \to 0$ and so $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. Rest of the proof follows on the same line as in proof of Theorem 2.2, by taking into consideration the fact that minimum$\{\beta_2, \beta_3\} < \frac{1}{s}$

The next result can be proved in a similar way as in Theorem 2.1 and 2.3 and so we omit the proof.

Theorem 2.4. *Theorem 2.1 with condition 2.1 replaced with the following : There exist $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ in the interval [0,1), such that $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 < 1$, minimum$\{\beta_3 + \beta_4, beta_5 + \beta_6\} < \frac{1}{s}$ and for all $u, v, w, z \in X$*

$$d_r(TS(u,v), TS(w,z)) \leq \beta_1 d_r(Tgu, Tgw) + \beta_2 d_r(Tgv, Tgz) +$$

$$\beta_3 d_r(Tgu, TS(u,v)) + \beta_4 d_r(Tgv, TS(v,u) + \beta_5 d_r(Tgw, TS(w,z)) + \beta_6 d_r(Tgz, TS(z,w)) \tag{2.24}$$

Taking T to be the identity mapping in Theorems 2.1,2.2,2.3 and 2.4 we have the following respective corollaries:

Corollary 2.5. *Let (X,d) be a RbMS(s) , $S\colon X \times X \to X$ and $g\colon X \to X$ be mappings such that $S(X \times X) \subset g(X)$ and $g(X)$ is complete. Suppose there exist real numbers λ, μ, ν with $0 < \lambda < 1$, $0 \leq \mu, \nu \leq 1$, minimum $\{\lambda\mu, \lambda\nu\} < \frac{1}{s}$ such that for all $u, v, w, z \in X$ the following holds :*

$$\begin{aligned} d_r(S(u,v), S(w,z) \quad \leq \quad & \lambda max\{d_r(gu,gw), d_r(gv,gz), \mu d_r(gu, S(u,v)), \mu d_r(gv, S(v,u)), \\ & \nu d_r(gw, S(w,z)), \nu d_r(gz, S(z,w))\} \end{aligned} \tag{2.25}$$

Then S and g has a coupled coincident point. Further if S and g are weakly compatible then there exist a unique common coupled fixed point for S and g. Moreover for some arbitrary $(u_0, v_0) \in X \times X$, the iterative sequences $(< gu_n >, < gv_n >)$ defined by $gu_n = S(u_{n-1}, v_{n-1})$ and $gv_n = S(v_{n-1}, u_{n-1})$ converges to the unique common coupled fixed point.

Corollary 2.6. *Corollary 2.5 with condition 2.25 replaced with the following :*

$$\begin{aligned} & d_r(TS(u,v), TS(w,z) + d_r(TS(v,u), TS(z,w) \leq \lambda max\{d_r(gu,gw) + d_r(gv,gz), \\ & \mu(d_r(gu, S(u,v)) + d_r(gv, S(v,u)), \nu(d_r(gw, TS(w,z)) + d_r(gz, S(z,w)))\} \end{aligned} \tag{2.26}$$

Corollary 2.7. *Corollary 2.5 with condition 2.25 replaced with the following : There exist $\beta_1, \beta_2, \beta_3$ in the interval [0,1), such that $\beta_1 + \beta_2 + \beta_3 < 1$, minimum$\{\beta_2, \beta_3\} < \frac{1}{s}$ and for all*

$u,v,w,z \in X$

$$d_r(S(u,v),S(w,z)+d_r(S(v,u),S(z,w) \leq \beta_1(d_r(gu,gw)+d_r(gv,gz))+$$

$$\beta_2(d_r(gu,S(u,v))+d_r(gv,S(v,u))+\beta_3(d_r(gw,S(w,z))+d_r(gz,S(z,w))) \tag{2.27}$$

Corollary 2.8. *Corollary 2.5 with condition 2.25 replaced with the following : There exist $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6$ in the interval [0,1), such that $\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6 < 1$, minimum$\{\beta_3+\beta_4, beta_5+\beta_6\} < \frac{1}{s}$ and for all $u,v,w,z \in X$*

$$d_r(S(u,v),S(w,z)) \leq \beta_1 d_r(gu,gw)+\beta_2 d_r(gv,gz)+$$

$$\beta_3 d_r(gu,S(u,v))+\beta_4 d_r(gv,S(v,u)+\beta_5 d_r(gw,S(w,z))+\beta_6 d_r(gz,S(z,w)) \tag{2.28}$$

Remark 2.9. *Since every b-metric space is a rectangular b-metric space, we note that Theorem 2.1 is a substantial generalisation of Theorem 2.2 of Ramesh and Pitchamani* [13]. *Infact we donot require continuity and sub sequential convergence of the function T.*

Remark 2.10. *Note that condition 2.1 of Gu* [2] *implies 2.27 and hence Corollary 2.7 gives an improved version of Theorem 2.1 of Gu* [2].

Example 2.11. *Let $X=[0,1]$, $d(x,y)=|x-y|$.*

$$Tx = \begin{cases} x^2, & \text{if } x \in [0,\frac{1}{2}] \\ \frac{x^2}{2}, & \text{if } x \in (\frac{1}{2},1] \end{cases}$$

$$gx = \begin{cases} \frac{x^2}{2}, & \text{if } x \in [0,\frac{1}{2}] \\ x^2, & \text{if } x \in (\frac{1}{2},1] \end{cases}$$

$$S(x,y) = \sqrt{\frac{x^{16}+y^{16}}{8}}.$$

Then T, S and g satisfies conditions of Theorem 2.1 and $(0,0)$ is the unique common coupled fixed point of S and g. Note that T is not continuous.

3. An Application to Integral Equation

In this section, we apply Theorem 2.1 to study the existence and uniqueness of solutions of a system of nonlinear integral equations.

Let $X = C[0,A]$ be the space of all continuous real valued functions defined on $[0,A]$,$A > 0$. We consider the following system of nonlinear integral equations, for $t \in [0,A]$

$$x(t) = \int_0^A G(t,r)f(t,x(r),y(r))dr + K(t)$$
$$y(t) = \int_0^A G(t,r)f(t,y(r),x(r))dr + K(t) \tag{3.1}$$

where $f : [0,A] \times R \times R \to R$ and $G : [0,A] \times [0,A] \to R$ and $K \in C([0,A]$. Now suppose F : $X \times X \to X$ be given by

$$F(x(t),y(t)) = \int_0^A G(t,r)f(t,x(r),y(r))dr + K(t).$$

$$F(y(t),x(t)) = \int_0^A G(t,r)f(t,y(r),x(r))dr + K(t).$$

Then the system of nonlinear integral equations 3.1 is equivalent to the coupled fixed point problem $F(x,y) = x$, $F(y,x) = y$.

Theorem 3.1. *Suppose that the following hold:*

(i) $G : [0,A] \times [0,A] \to R$ and $f : [0,A] \times R \times R \to R$ are continuous functions.

(ii) $K \in C([0,A]$.

(iii) For all $x,y,u,v \in X$ and $t \in [0,A]$, we can find a function $g : X \to X$ and real numbers $s \geq 1$, λ,μ,ν with $0 \leq \lambda < 1$, $0 \leq \mu,\nu \leq 1$, minimum $\{\lambda\mu,\lambda\nu\} < \frac{1}{3^{s-1}}$ satisfying

(iiia):

$$\begin{aligned} & \mid f(t,x(r),y(r))) - f(t,u(r),v(r))) \mid^s \\ \leq\ & \lambda max\{\mid g(x(r)) - g(u(r)) \mid^s, \mid g(y(r)) - g(v(r)) \mid^s, \\ & \mu \mid g(x(r)) - F(x(r),y(r)) \mid^s, \mu \mid g(y(r)) - F(y(r),x(r)) \mid^s, \\ & \nu \mid g(u(r)) - F(u(r),v(r)) \mid^s, \nu \mid g(v(r)) - F(v(r),u(r)) \mid^s\}. \end{aligned}$$

and

(iiib): $F(g(x(t)),g(y(t))) = g(F(x(t),y(t)))$ *whenever* $F(x(t),y(t)) = g(x(t))$ *and* $F(y(t),x(t)) = g(y(t))$.

(iv) $sup_{t\in[0,A]} \int_0^A \mid G(t,r) \mid^s dr \leq \frac{1}{\lambda^{s-1}}$

Then 3.1 has a unique solution in $C[0,A]$*. Moreover, for some arbitrary* $x_0(t), y_0(t)$ *in X, the sequence* $\{< gx_n(t) >, < gy_n(t) >)$ *defined by*

$$gx_n(t) = \int_0^A G(t,r)f(t,x_{n-1}(r),y_{n-1}(r))dr + K(t)$$

$$gy_n(t) = \int_0^A G(t,r)f(t,y_{n-1}(r),x_{n-1}(r))dr + K(t) \tag{3.2}$$

converges to the unique solution .

Proof : Define $d_r \colon X\times X \to R$ such that for all $x,y \in X$,

$$d_r(x,y) = sup_{t\in[0,A]} \mid x(t) - y(t) \mid^s \tag{3.3}$$

Clearly d_r is a $RbMS(3^{s-1})$.

For some $r \in [0,A]$, we have

$$\begin{aligned}
& d_r(F(x,y),F(u,v)) = \mid F(x,y)(t) - F(u,v)(t) \mid^s \\
= \ & \mid [\int_0^A G(t,r)f(t,x(r),y(r))dr + g(t)] - [\int_0^A G(t,r)f(t,u(r),v(r))dr + g(t)] \mid^s \\
\leq \ & \int_0^A \mid G(t,r) \mid^s \mid f(t,x(r),y(r)) - f(t,u(r),v(r)) \mid^s dr \\
\leq \ & (\int_0^A \mid G(t,r) \mid^s dr)\lambda^s [max\{\mid g(x(r)) - g(u(r)) \mid^s, \mid g(y(r)) - g(v(r)) \mid^s, \\
& \mu \mid g(x(r)) - F(x(r),y(r)) \mid^s, \mu \mid g(y(r)) - F(y(r),x(r)) \mid^s, \\
& \nu \mid g(u(r)) - F(u(r),v(r)) \mid^s, \nu \mid g(v(r)) - F(v(r),u(r)) \mid^s\}. \\
\leq \ & (\int_0^A \mid G(t,r) \mid^s dr)\lambda^s [max\{d_r(x,u), d_r(y,v), \mu d_r(g(x),F(x,y)), \mu d_r(g(y),F(y,x)), \\
& \nu d_r(g(u),F(u,v)), \nu d_r(g(v),F(v,u))\} \\
\leq \ & \lambda [max\{d_r(x,u), d_r(y,v), \mu d_r(g(x),F(x,y)), \mu d_r(g(y),F(y,x)), \\
& \nu d_r(g(u),F(u,v)), \nu d_r(g(v),F(v,u))\}
\end{aligned}$$

Thus we have

$$\begin{aligned} d_r(F(x,y),F(u,v)) &= sup_{t\in[0,A]} \mid F(x,y)(t)-F(u,v)(t) \mid^s \\ &\leq \lambda[max\{d_r(x,u),d_r(y,v),\mu d_r(g(x),F(x,y)),\mu d_r(g(y),F(y,x)), \\ &\quad \nu d_r(g(u),F(u,v)),\nu d_r(g(v),F(v,u))\} \end{aligned}$$

This shows that contractive condition of Theorem 2.1 holds. Therefore, by Theorem 2.1 F has a unique coupled fixed point $(x',y') \in C([0,A] \times C([0,A]$ which is the unique solution of 3.1 and the sequence $\{< gx_n(t) >, < gy_n(t) >)$ defined by 3.2 converges to the unique solution of the system of integral equations 3.1.

Remark 3.2. *Condition (iv) of Theorem 3.1 above is weaker than the corresponding conditions used in similar theorems of* [13] *and* [3].

Example 3.3. *Let $X = C[0,1]$ be the space of all continuous real valued functions defined on $[0,1]$ and define $d_3\colon X\times X \to R$ such that for all $x,y \in X$,*

$$d_3(x,y) = sup_{t\in[0,1]} \mid x(t)-y(t) \mid^2 \tag{3.4}$$

Clearly d_3 is a rectangular b-metric with coefficient 3. Now consider the functions $f:[0,1]\times R\times R \to R$ given by $f(t,x,y) = t^2+\frac{9}{20}x+\frac{8}{20}y$, $G:[0,1]\times[0,1] \to R$ given by $G(t,r) = \frac{\sqrt{45}(t+r)}{10}$, $K \in C([0,1]$ given by $K(t) = t$. Then the system of non linear integral equations 3.1 becomes

$$\begin{aligned} x(t) &= t+\int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2+\frac{9}{20}x(r)+\frac{8}{20}y(r))dr \\ y(t) &= t+\int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2+\frac{9}{20}y(r)+\frac{8}{20}x(r))dr \end{aligned} \tag{3.5}$$

Then

$$\begin{aligned} \mid f(t,x,y)-f(t,u,v) \mid^2 &= \mid \frac{9}{20}(x-u)+\frac{8}{20}(y-v) \mid^2 \\ &\leq \mid Max\{\frac{9}{10}(x-u),\frac{8}{10}(y-v)\} \mid^2 \\ &\leq \frac{81}{100}Max\{\mid x-u \mid^2, \mid y-v) \mid^2\} \end{aligned}$$

Also

$$sup_{t\in[0,1]}\int_0^1 \mid G(t,r)\mid^2 dr = \int_0^1 \frac{45}{100}(t+r)^2 dr = 1.125$$

We see that all conditions of Theorem 3.1 are satisfied, with $\lambda = \frac{81}{100}, \mu = 0, \nu = 0, s = 2$ *and* $g = I_X$*(Identity mapping). Hence Theorem 3.1 ensures a unique solution of the system of non linear integral equations 3.5. Now for* $x_0(t) = 1$ *and* $y_0(t) = 0$*, we construct the sequence* $\{< x_n(t) >, < y_n(t) >\}$*, given by*

$$x_n(t) = t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}x_{n-1}(r) + \frac{8}{20}y_{n-1}(r))dr$$

$$y_n(t) = t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}y_{n-1}(r) + \frac{8}{20}x_{n-1}(r))dr \tag{3.6}$$

Using MATLAB we see that above sequence converges to $\{0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677, 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677\}$ *and this is the unique solution of the system of non linear integral equations 3.5. The convergence table is as given below.*

n	$x_n(t)=t+\int_0^1\frac{\sqrt{45}(t+r)}{10}(t^2+\frac{9}{20}x_{n-1}(r)+\frac{8}{20}y_{n-1}(r))dr$	$y_n(t)=t+\int_0^1\frac{\sqrt{45}(t+r)}{10}(t^2+\frac{9}{20}y_{n-1}(r)+\frac{8}{20}x_{n-1}(r))dr$
1	$x_1(t)=t+.0167(2t+1)(20t^2+9))$	$y_1(t)=t+.0671(2t+1)(5t^2+2))$
2	$x_2(t)=0.6708t^3+0.3354t^2+1.3t+0.5007$	$y_2(t)=0.6708t^3+0.3354t^2+1.29t+0.5115$
3	$x_3(t)=0.6708t^3+0.3354t^2+1.8210t+0.5174$	$y_3(t)=0.6708t^3+0.3354t^2+1.8208t+0.5171$
4	$x_4(t)=0.6708t^3+0.3354t^2+1.9734t+0.6179$	$y_4(t)=0.6708t^3+0.3354t^2+1.9734t+0.6178$
5	$x_5(t)=0.6708t^3+0.3354t^2+2.0743t+0.6755$	$y_5(t)=0.6708t^3+0.3354t^2+2.0743t+0.6755$
6	$x_6(t)=0.6708t^3+0.3354t^2+2.1359t+0.7111$	$y_6(t)=0.6708t^3+0.3354t^2+2.1359t+0.7111$
7	$x_7(t)=0.6708t^3+0.3354t^2+2.1737t+0.73298$	$y_7(t)=0.6708t^3+0.3354t^2+2.1737t+0.73298$
8	$x_8(t)=0.6708t^3+0.3354t^2+2.19699t+0.7464$	$y_8(t)=0.6708t^3+0.3354t^2+2.19699t+0.7464$
9	$x_9(t)=0.6708t^3+0.3354t^2+2.2113t+0.7547$	$y_9(t)=0.6708t^3+0.3354t^2+2.2113t+0.7547$
10	$x_{10}(t)=0.6708t^3+0.3354t^2+2.2200t+0.7597$	$y_{10}(t)=0.6708t^3+0.3354t^2+2.2200t+0.7597$
11	$x_{11}(t)=0.6708t^3+0.3354t^2+2.2254t+0.7628$	$y_{11}(t)=0.6708t^3+0.3354t^2+2.2254t+0.7628$
12	$x_{12}(t)=0.6708t^3+0.3354t^2+2.2287t+0.7647$	$y_{12}(t)=0.6708t^3+0.3354t^2+2.2287t+0.7647$
13	$x_{13}(t)=0.6708t^3+0.3354t^2+2.2308t+0.7658$	$y_{13}(t)=0.6708t^3+0.3354t^2+2.2308t+0.7658$
14	$x_{14}(t)=0.6708t^3+0.3354t^2+2.23199t+0.7666$	$y_{14}(t)=0.6708t^3+0.3354t^2+2.23199t+0.7666$
15	$x_{15}(t)=0.6708t^3+0.3354t^2+2.2328t+0.7671$	$y_{15}(t)=0.6708t^3+0.3354t^2+2.2328t+0.7671$
16	$x_{16}(t)=0.6708t^3+0.3354t^2+2.2333t+0.7674$	$y_{16}(t)=0.6708t^3+0.3354t^2+2.2333t+0.7674$
17	$x_{17}(t)=0.6708t^3+0.3354t^2+2.2336t+0.7675$	$y_{17}(t)=0.6708t^3+0.3354t^2+2.2336t+0.7675$
18	$x_{18}(t)=0.6708t^3+0.3354t^2+2.2338t+0.7676$	$y_{18}(t)=0.6708t^3+0.3354t^2+2.2338t+0.7676$
19	$x_{19}(t)=0.6708t^3+0.3354t^2+2.2339t+0.7677$	$y_{19}(t)=0.6708t^3+0.3354t^2+2.2339t+0.7677$
20	$x_{20}(t)=0.6708t^3+0.3354t^2+2.2339t+0.7677$	$y_{20}(t)=0.6708t^3+0.3354t^2+2.2339t+0.7677$

Remark 3.4. *In example 3.3 above we see that* $sup_{t\in[0,1]}\int_0^1 \mid G(t,r)\mid^2 dr=\int_0^1\frac{45}{100}(t+r)^2dr=1.125>1$ *and thus condition (v) of Theorem 3.1 of* [13] *and condition (30) of Theorem 3.1 of* [3] *is not satisfied.*

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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