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REALIZATION OF SYMPLECTIC CYCLIC ACTIONS ON ELLIPTIC SURFACES

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Abstract. In this paper, we study the existence of pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_p$ with order $7 < p \leq 40$ on elliptic surfaces $E(n)$. Especially, we construct the $\mathbf{Z}_{13}$ action on $E(n)$ and give the local representations of fixed points.

Keywords: symplectic cyclic action; elliptic surfaces; fixed points; local representation.

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1. Introduction

Let $E(n)(n \geq 2)$ be a minimal elliptic surface with rational base. $E(n)$ is a simply connected 4-manifold which is defined as the n-fold fiber sum of copies of $E(1)$, where $E(1) = \mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}$ being equipped with an elliptic fibration. Note that $\text{sign}(E(n)) = -8n$ and $\chi(E(n)) = 12n$. Thus $E(2) = E(1)\sharp_{\mathbb{T}^2} E(1)$ is the $K3$-surface. To see this just note that the Euler characteristic are additive under taking fiber connected sums over a torus. Hence $\text{sign}(E(2)) = -16$ and $\chi(E(2)) = 24$ which characterizes $K3$ surface [4]. Elliptic surface $E(n)$ is Kähler, hence can be equipped with a symplectic structure which is provided by the complex structure and the Kähler metric.

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Let $\mathbf{Z}_p$ be a symplectic cyclic action on 4-manifold $E(n)$ with odd prime order p and $\mathbf{Z}_p$ preserves the symplectic structure on $E(n)$. In the study of symplectic action of a finite group action on 4-manifolds, a central problem is describing the structure of fixed-point set and action around it. In this aspect, W. Chen and S. Kwasik [2] give a complete description of the fixed-point set structure of a symplectic cyclic action of prime order on a minimal symplectic 4-manifold with $c_1^2 = 0$. When $p \leq 7$, [6] study the pseudofree, homologically trivial, symplectic cyclic actions on $E(n)$ with order $2, 3, 5$ and 7. In this paper, we study the pseudofree, homologically trivial, symplectic cyclic actions on $E(n)$ with order $7 < p \leq 40$. In this case, the construction of the action is more complex. So we only construct the action $\mathbf{Z}_{13}$ on $E(n)$ and give the local representations of fixed points. While the other action $\mathbf{Z}_p$ with $p > 13$ can be constructed similarly.

2. Preliminaries

In this section, we review some theorems and notations such as the Lefschetz fixed point formula, the G-signature theorem, the realization theorem of Edmonds and Ewing [3] and some useful results of W. Chen and S. Kwasik [2].

Let $g : X \to X$ generate an action of G on a closed, simply connected 4-manifold X. we have the Lefschetz fixed point formula

$$\chi(F) = \Lambda(g) = 2 + \text{Trace}[g_* : H_2(X) \to H_2(X)].$$

We can refer to Allday and Puppe [1] for details.

Let X be a closed, oriented smooth 4-manifold. Let $G = \mathbf{Z}_p$ be an orientation-preserving cyclic group action with odd prime order on X. Then the fixed-point set of G on X, if nonempty, will be consist of a disjoint union of finitely many isolated points and 2-dimensional orientable submanifolds. On each isolated fixed point, G defines a local complex representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k, q \neq 0 \mod p$, where q is uniquely determined, and k is determined up to a sign and $\mu_p \equiv exp(\frac{2\pi i}{p})$.

Theorem 2.1.[5](G-Signature Theorem for Prime Order Cyclic Actions).

$$|G| \cdot \sigma(X/G) = \sigma(X) + \sum_{m \in F} def_m + \sum_{Y \in F} def_Y,$$

where m stands for an isolated fixed point, and Y stands for a 2-dimensional component of M^G. The terms def_m and def_Y are called signature defects and they are given by the following formulae:

$$def_m = \sum_{k=1}^{p-1} \frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}$$

if the local representation at m is given by $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$, and

$$def_Y = \frac{P^2 - 1}{3} \cdot (Y \cdot Y)$$

where $Y \cdot Y$ is the self-intersection of Y.

An action on a space is called pseudofree if it is free on the compliment of a discrete subset.

Theorem 2.2. [3](Realization Theorem for Pseudofree Prime Order Cyclic Actions) *Let G be the cyclic group of order p. Suppose that one is given a fixed point data*

$$D = \{(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})\},$$

where $a_i, b_i \in \mathbf{Z}_p \setminus \{0\}$, and a G-invariant symmetric unimodular form

$$\Phi \colon V \times V \to \mathbf{Z},$$

where V be a finitely generated $\mathbf{Z}$-free $\mathbf{Z}[G]$-module. Then the data D and the form (V, Φ) are realizable by a locally linear, pseudofree, G-action on a closed, simply-connected, topological 4-manifold if and only if they satisfy the following two conditions

1. *The condition REP: As a $\mathbf{Z}[G]$-module, V splits into $F \oplus T$, where F is free and T is a trivial $\mathbf{Z}[G]$-module with $rank_{\mathbf{Z}} T = n$.*
2. *The condition GSF: The G-Signature Formula is satisfied*

$$\sigma(g, (V, \Phi)) = \sum_{i=0}^{n+1} \frac{(\zeta^{a_i}+1)(\zeta^{b_i}+1)}{(\zeta^{a_i}-1)(\zeta^{b_i}-1)},$$

where $\zeta = exp(2\pi\sqrt{-1}/p)$.

Note that for homologically trivial action $\mathbf{Z}_p$(p being odd prime number), GSF is the only condition needed for realization of $\mathbf{Z}_p$.

Theorem 2.3. [2] *Let M be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrivial, pseudofree action of $G \equiv \mathbf{Z}_p$, where p is prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(M;Q)$ is trivial. Then the set of fixed points of G can be divided into groups each of which belongs to one of the following five possible types.*

(1) *One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0$ mod p, i.e., with local representation contained in $SL_2(\mathbf{C})$.*

(2) *Two fixed points with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ for some $k \neq 0$ mod p respectively. Fixed points of this type occur only when $p > 5$.*

(3) *Three fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ and the other two with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ for some $k \neq 0$ mod p. Fixed points of this type occur only when $p > 3$.*

(4) *Four fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ and the other three with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ for some $k \neq 0$ mod p. Fixed points of this type occur only when $p > 3$.*

(5) *Three fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ for some $k \neq 0$ mod p. Fixed points of this type occur only when $p = 3$.*

Theorem 2.4. [2] *Let M be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a homologically trivial (over $\mathbf{Q}$ coefficients), pseudofree, symplectic $\mathbf{Z}_p$-action for a prime $p > 1$. Then the following conclusions hold.*

(1) *The action is trivial if $p \neq 1$ mod 4, $p \neq 1$ mod 6, and the signature of M is nonzero. In particular, if the signature of M is nonzero, then for infinitely many primes p the manifold M does not admit any such nontrivial $\mathbf{Z}_p$-actions.*

(2) *The action is trivial as long as there is a fixed point of type (1) in Theorem 2.3.*

Theorem 2.5. [2] *Let $def(k)$ be the total signature defect contributed by one group of fixed points of type (k) in the Theorem 2.3, where $k = 1, 2, 3, 4$. Then we have*

(1) $def(1) = \frac{1}{3}(p-1)(p-2)$ *for all* $p > 1$.

(2) $def(2) = -8r$ *if* $p = 6r+1$, $4def(2) = 8r+8$ *if* $p = 6r+5$.

(3) $def(3) = -8r$ *if* $p = 4r+1$, $def(3) = 2$ *if* $p = 4r+3$.

(4) $def(4) = -8r$ *if* $p = 3r+1$, $def(4) = -4r$ *if* $p = 3r+2$.

3. Main results

In this section, we suppose there exists a pseudofree, homologically trivial, symplectic cyclic actions $G \equiv \mathbf{Z}_p$ on elliptic surfaces $E(n)$ with order $7 < p \leq 40$ being prime. Obviously, there are eight prime numbers in this scope, which is $11, 13, 17, 19, 23, 29, 31, 37$. When $p = 11, 23$, $\mathbf{Z}_p$ must be trivial by theorem 2.4. When $p = 13, 17, 19, 29, 31, 37$, $\mathbf{Z}_p$ could be nontrivial by theorem 2.4.

For the sake of convenience, we only construct a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$ and give the local representation of each fixed point.

Since the group action is pseudofree, the fixed point set is consist of isolated points. Hence the G-signature theorem becomes to

$$|G| \cdot \sigma(X/G) = \sigma(X) + \sum_{m \in F} def_m,$$

where F denotes the fixed point set. Since the induced action on $H^2(X; \mathbf{Q})$ is trivial, $\sigma(X/G) = \sigma(X)$ and $\chi(X/G) = \chi(X)$. Besides, the action is supposed to be nontrivial then from theorem 2.3, 2.4 the fixed-point set F may be composed of type (2), (3) and (4) by theorem 2.3. Let a_2, a_3 and a_4 be the numbers of groups of fixed points of type (2), (3) and (4) respectively. From the G-signature theorem and the Lefschetz fixed point formula we have

$$12n = 2a_2 + 3a_3 + 4a_4. \tag{1}$$

Next, we construct the action on $E(n)$ for $n = 2m$, $n = 3m$, $n = 4m$, $n = 4m-1$ and $n = 4m+1 (m \in \mathbf{Z})$ respectively. These five cases cover all possible value of n and would simplify our construction.

Case 1. Suppose there exists a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$, $(n = 2m, m \in \mathbf{Z})$.

We choose one solution $(6n, 0, 0)$ of equation (1). Since $n = 2m$, there are $24m$ fixed points all together. We divide them evenly into m group, and assign the points in each group with local representations

$$(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2),$$

evaluated at $k = 1, 2, 3, \cdots, 12$. Now we obtain a set of $24m$ ordered pairs of nonzero elements, we denote it D_1. In order to realize D_1 as the fixed point set of some pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on $E(2n)$, we need only to verify the GSF condition. In our study, the GSF condition is

$$\sigma(g, X) = m \cdot def_{(2)}.$$

Since $\sigma(g, X) = \sigma(X) = -8n = -16m$ for any $g \in \mathbf{Z}_{13}$ and $def_{(2)} = -16$, the GSF condition obviously exists.

Case 2. Suppose there exists a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$, $(n = 3m, m \in \mathbf{Z})$.

We choose one solution $(0, 4n, 0)$ of equation (1). Since $n = 3m$, there are $36m$ fixed points all together. We divide them evenly into m group, and assign the points in each group with local representations

$$(z_1, z_2) \mapsto (\mu_p^{k} z_1, \mu_p^{2k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. Now we obtain a set of $36m$ ordered pairs of nonzero elements, we denote it D_2. In order to realize D_2 as the fixed point set of some pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on $E(2n)$, we need only to verify the GSF condition. In our study, the GSF condition is

$$\sigma(g, X) = m \cdot def_{(3)}.$$

Since $\sigma(g, X) = \sigma(X) = -8n = -24m$ for any $g \in \mathbf{Z}_{13}$ and $def_{(3)} = -24$, the GSF condition obviously exists.

Case 3. Suppose there exists a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$, $(n = 4m, m \in \mathbf{Z})$.

We choose one solution $(0, 0, 3n)$ of equation (1). Since $n = 4m$, there are $48m$ fixed points all together. We divide them evenly into m group, and assign the points in each group with local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),$$

$$(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. Then there is a set of $48m$ ordered pairs of nonzero elements. We denote it D_3. In order to realize D_3 as the fixed point set of some pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on $E(2n)$, we need only to verify the GSF condition. In this case, the GSF condition is

$$\sigma(g, X) = m \cdot def_{(4)}.$$

Since $\sigma(g, X) = \sigma(X) = -8n = -32m$ for any $g \in \mathbf{Z}_{13}$ and $def_{(4)} = -32$, the GSF condition obviously exists.

Case 4. Suppose there exists a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$, $(n = 4m - 1, m \in \mathbf{Z})$.

Since $n = 4m - 1$, $(12, 0, 12(m - 1))$ is a solution of equation (1). Obviously, there are 36 fixed points of type (3) and $48(m - 1)$ fixed points of type (4). We divide them as follows. The fixed points of type (3) is in one group, and the local representations of each fixed point are

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. The fixed points of type (4) are evenly divided into $m - 1$ groups with local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),$$

$$(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. Now we obtain a set of $12(4m-1)$ ordered pairs of nonzero elements, we denote it D_4. In order to realize D_4 as the fixed point set of some pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on $E(2n)$, we need only to verify the GSF condition. In this case, the GSF condition is

$$\sigma(g, X) = def(3) + (m-1) \cdot def_{(4)}.$$

Since $\sigma(g, X) = \sigma(X) = -8n = -8(4m-1)$ for any $g \in \mathbf{Z}_{13}$ and $def_{(3)} = -24, def_{(4)} = -32$, the GSF condition exists and the action can be realized.

Case 5. Suppose there exists a pseudofree, homologically trivial but nontrivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$, $(n = 4m+1, m \in \mathbf{Z})$.

Since $n = 4m+1$, $(0, 36, 12(m-2))$ is a solution of equation (1). Obviously, there are 108 fixed points of type (3) and $48(m-2)$ fixed points of type (4). We divide them as below. The fixed points of type (3) is divided into 3 groups, and each group have local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. The fixed points of type (4) are evenly divided into $m-2$ groups, and each group have local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),$$

$$(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2),\ (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$$

evaluated at $k = 1, 2, 3, \cdots, 12$. Now we obtain a set of $12(4m+1)$ ordered pairs of nonzero elements, we denote it D_5. In order to realize D_5 as the fixed point set of some pseudofree, homologically trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on $E(2n)$, we need only to verify the GSF condition. In this case, the GSF condition is

$$\sigma(g, X) = 3def(3) + (m-2) \cdot def_{(4)}.$$

Since $\sigma(g, X) = \sigma(X) = -8n = -8(4m+1)$ for any $g \in \mathbf{Z}_{13}$ and $def_{(3)} = -24, def_{(4)} = -32$, the GSF condition obviously exists. Thus this action can be realized.

To sum up, for any n, we can construct a pseudofree, homologically trivial but non-trivial, symplectic cyclic actions $\mathbf{Z}_{13}$ on elliptic surfaces $E(n)$. For other $\mathbf{Z}_p$ action $(7 < p \leq 40)$ on elliptic surfaces $E(n)$, we can construct by the same way.

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