Available online at http://scik.org

Advances in Fixed Point Theory, 3 (2013), No. 1, 151-163

ISSN: 1927-6303

COMMON FIXED POINT THEOREMS IN GENERALIZED MENGER SPACES WITH T-NORM T OF HADŽIĆ-TYPE

Asha Rani[1,*], Sanjay Kumar[2]

[1]Department of Applied Sciences, B. M. I. E. T., Raipur, Sonepat, Haryana, India

[*]DCRUST, Murthal(Sonepat)-131039

[2]Department of Mathematics,DCRUST, Murthal (Sonepat), Haryana, India-131039

Abstract. In this paper we prove a common fixed point theorem for a class of generalized Kannan type mappings with continuous t-norm T of Hadžić-type (in short H-type) in generalized Menger spaces. In fact, our result generalizes the result of Choudhury et. al.[4] for weakly compatible maps. At the end, we give an example in support of our theorem.

Keywords: Generalized Menger spaces; Weakly compatible maps; t-norm T of Hadžić-type.

2000 AMS Classification: 47H10, 54H25.

1. Introduction

In 2000, Branciari[1] introduced the concept of generalized metric space by replacing triangular inequality with quadrangular inequality as follows:

Definition 1.1.[1] Let X be a nonempty set, $\mathbb{R}^+$ be the set of all positive real numbers and d: X$\times$ X $\rightarrow \mathbb{R}^+$ be a mapping such that for all x, y $\in$ X and for all points $\xi, \eta \in$ X, each of them different from x and y, satisfying:

(i) d(x, y) = 0 $\Leftrightarrow$ x = y,

(ii) d(x, y) = d(y, x) and

(iii) d(x, y) $\leq$ d(x, ξ) + d(ξ, η) + d(η, y).

[*]Corresponding author

Received July 3, 2012

Branciarinotedthat there are generalized metric spaces which are not metric spaces and further proved Banach contraction mapping theorem in setting of generalized metric spaces.

Definition 1.2. ([8],[9]) Let (X, d) be a metric space and f be a mapping on X. The mapping f is called a Kannan type mapping if there exists $0 \leq \alpha < \frac{1}{2}$ such that

d(f x, f y) $\leq \alpha$ [d(x, f x) + d(y, f y)] for all x, y $\in$ X. It is obvious that Kannan type mappings are not necessarily continuous. Kikkawa [11] verified that every metric space X is complete if and only if every Kannan type mapping has a fixed point.

First, we recall that a real valued function defined on the set of real numbers is known as a distribution functions if it is non-decreasing, left continuous with inf f(x) = 0 and sup f(x) = 1. In what follows, H(x) denotes the Heavy side function, a simple example of distribution function.

$$H(x) = \begin{cases} 0 \ if \ x \leq 0, \\ 1 \ if \ x > 0. \end{cases}$$

Definition 1.3. A probabilistic metric spaceis a pair (X, F) ,where X is a non empty set and F is a function defined on X $\times$X into L (the set of all distribution functions) satisfying the following properties:

(i) F(x, y, 0) = 0,

(ii) F(x, y, t) = H(t) iff x = y ,

(iii) F(x, y, t) = F(y, x, t) and

(iv) if F(x, y, s) = 1 and F(y, z, t) = 1, then F(x, z, s + t) = 1 f or all x, y, z $\in$ X and s, t $>$ 0.

For each x and y in X and for each real number t >0, F(x, y, t) is to be thought of as the probability that the distance between x and y is less than t. Of course, a metric space (X, d) induces a PM-space. Every metric space (X, d) can be realized as a probabilistic metric space by taking F: X×X $\rightarrow$ L defined by F(x, y, t) = H(t – d(x, y)), for all x, y in X.

Definition 1.4. A t-norm (in the sense of B. Schweizer and A. Sklar[18]) is a 2-place function, $\Delta : [0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following:

(i) $\Delta (0, 0) = 0$,

(ii) $\Delta (0, 1) = 1$,

(iii) $\Delta (a, b) = \Delta(b, a)$,

(iv) if $a \leq c$, $b \leq d$, then $\Delta (a, b) \leq \Delta (c, d)$ and

(v) $\Delta (\Delta (a, b),c) = \Delta (a, \Delta (b, c))$ for all a, b, c $\in$ [0, 1].

Basic examples of t-norm are the Lukasiewicz t-norm T_L, $T_L(a,b) = Max(a+b-1,0)$, t-norm T_P, $T_P(a,b) = ab$, and t-norm T_M, $T_M(a,b) = Min\{a,b\}$, $T_D(x,y) = \begin{cases} \min(x,y) \; if \; \max(x,y) = 1, \\ 0 \qquad otherwise. \end{cases}$

Definition 1.5. A Menger PM-space is a triplet (X, F, Δ) where (X, F) is a PM-space and Δ is a t-norm with the following condition:

$$F(x, z, s + t) \geq \Delta\, (F(x, y, s), F(y, z, t))$$

for all x, y, z ∈ X and s, t > 0.

This inequality is known as Menger's triangle inequality.

Definition 1.6. A sequence $\{x_n\}$ in (X, F, Δ) is said to

(i) converge to a point x ∈ X if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F(x_n, x, \epsilon) > 1 - \lambda$, for all $n \geq N(\epsilon, \lambda)$.

(ii) be a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F(x_n, x_m, \epsilon) > 1 - \lambda$, for all $n, m \geq N(\epsilon, \lambda)$.

A Menger space (X, F, Δ) is said to be complete if every Cauchy sequence in X converges to a point in X.

In [6], a special class of t-norms (called as a Hadžić- type t-norm) is introduced as follows:

Definition 1.7. Let T be a t-norm and let $T_n : [0, 1] \to [0, 1]$ ($n \in \mathbb{N}$) be defined in the following way :

$$T_1(x) = T(x, x),\ T_{n+1}(x) = T(T_n(x), x) \quad (n \in \mathbb{N},\ x \in [0, 1]).$$

We say that the t-norm T is of H-type if T is continuous and the family $\{T_n(x), n \in \mathbb{N}\}$ is equicontinuous at x = 1. The family $\{T_n(x), n \in \mathbb{N}\}$ is equicontinuous at x = 1, if for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that the following implication holds:

$$x > 1 - \delta(\lambda) \text{ implies } T_n(x) > 1 - \lambda \text{ for all } n \in \mathbb{N}.$$

A trivial example of t-norm of H-type is $T = T_M$.

Remark 1.8. Every t-norm T_M is of Hadžić-type but converse need not be true [7].

There is a nice characterization of continuous t-norm [15].

(i) If there exists a strictly increasing sequence $\{b_n\}_{n \in N}$ in [0,1] such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n,b_n) = b_n \forall\, n \in N$, then T is of Hadˇ zi´c-type.

(ii) If T is continuous and T is of Hadˇ zi´c-type, then there exists a sequence $\{b_n\}_{n \in N}$ as in (i).

Definition 1.9. [7] If T is a t-norm and $(x_1,x_2, \ldots ,x_n) \in [0,1]^n$ ($n \in N$), then $T^n_{i=1} x_i$ is defined recurrently by 1, if n = 0 and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n)$ for all $n \geq 1$. If $\{x_i\}_{i\in N}$ is a sequence of numbers from[0,1], then $T^\infty_{i=1} x_i$ is defined as $\lim_{n\to\infty} T^n_{i=1} x_i$ (this limit always exists) and $T^\infty_{i=n} x_i$ as $T^\infty_{i=1} x_{n+i}$. In particular, we are interested in sequences $\{x_n\} \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$and $\lim_{n\to\infty} T^\infty_{i=1} x_{n+i} = 1$.

Definition 1.10. (n-th order t-norm,[18]) A mapping $T:\prod_{i=1}^{n}[0,1] \to [0,1]$ is called an n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, \ldots, 0) = 0$, $T(a,1,1,\ldots,1) = a$ for all $a \in [0,1]$,

(ii) $T(a_1,a_2,a_3,\ldots,a_n) = T(a_2,a_1,a_3,\ldots,a_n) = T(a_2,a_3,a_1,\ldots,a_n) = \ldots\ldots = T(a_2,a_3,a_4,\ldots,a_n,a_1)$

(iii) $a_i \geq b_i$, $i = 1,2,3,\ldots\ldots,n$ implies $T(a_1,a_2,a_3,\ldots,a_n) \geq T(b_1,b_2,b_3,\ldots,b_n)$,

(iv) $T(T(a_1,a_2,a_3,\ldots,a_n),b_2,b_3,\ldots,b_n)$
$= T(a_1,T(a_2,a_3,\ldots,a_n,b_2),b_3,\ldots,b_n)$
$= T(a_1,a_2,T(a_3,a_4,\ldots,a_n,b_2,b_3) ,b_4,\ldots,b_n)$
$= T(a_1,a_2,a_3,\ldots,a_{n-1},T(a_n,b_2,b_3,\ldots,b_n))$.

For n = 2, the n-th order t-norm is known as binary t-norm and it is usually called as t-norm.

Definition 1.11.[4](Generalized Menger space). Let X be a non-empty set and T is a 3rd order t-norm. Then (X, F, T) is said to be a generalized Menger space if for all $x, y \in X$ and all distinct points $z, w \in X$ each of them different from x and y, the following conditions are satisfied:

(i) $F(x, y, 0) = 0$,

(ii) $F(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $F(x, y, t) = F(y, x, t)$ for all $t > 0$ and for all $x, y \in X$,

(iv) $F(x, y, t) \geqq T(F(x, z, t_1), F(z, w, t_2), F(w, y, t_3))$, where $t_1 + t_2 + t_3 = t$ and T is a 3rd order t-norm.

Recently, Choudhury[4] had shown that generalized metric space is a special case of generalized Menger space.

Definition 1.12.[4] Let (X, F, T) be a generalized Menger space. A sequence $\{x_n\} \subset X$ is said to

(i) converge to some point $x \in X$ if given $\epsilon > 0, \lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$, $F(x_n, x, \epsilon) > 1 - \lambda$.

(ii) be a Cauchy sequence in X if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that $F(x_n, x_m, \epsilon) > 1 - \lambda$ for all $m, n > N_{\epsilon,\lambda}$.

A generalized Menger space (X, F, T) is said to be complete if every Cauchy sequence is convergentin it.

Definition 1.13. Two **s**elf-maps f and g are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., iff p = gp for some p $\in$X then fgp = gfp.

In 1984, Khan et. al. [10] initiated the study of a new category of fixed point theorems by using a control function that alters the distance between two points in metric spaces. Many authors have proved fixed point theorems for mappings satisfying certain inequalities involving this altering distance function.

Recently, Choudhury et. al. [4] extended the idea of altering distance function in generalized Menger spaces using t – norm T_M.

Now we prove the result for a t – norm T of H – type.

Definition1.14 ($\boldsymbol{\theta}$- function [4]). A function φ: [0, 1] $\times$ [0, 1] $\rightarrow$ [0, 1] is said to be a $\boldsymbol{\theta}$ -function if

(i) φ is monotone increasing and continuous,

(ii) φ(x, x) > x for all 0 < x < 1,

(iii) φ(1, 1) = 1, φ(0, 0) = 0.

An example of $\boldsymbol{\theta}$- function:

$\varphi(x, y) = \frac{p\sqrt{x}+q\sqrt{y}}{p+q}$, p and q are positive numbers.

We will use the following type of functions:

Definition1.15(Φ-function). A function $\phi : R^+ \rightarrow R^+$ is said to be a **Φ**-function if it satisfies the following conditions:

(i) ϕ (t) = 0 if and only if t = 0,

(ii) ϕ (t) is increasing and ϕ (t) $\rightarrow\infty$ as t$\rightarrow\infty$,

(iii) ϕ is left continuous in (0, ∞),

(iv) ϕ is continuous at 0,

(v) ϕ is super-additive , i.e., $\phi (x + y) \geq \phi (x) + \phi (y)$, for all x, y $\geq$ 0.

An example of **Φ-function** as:

$$\phi(x) = \begin{cases} x \text{ } if \text{ } x > 0, \\ 0 \text{ } if \text{ } x = 0. \end{cases}$$

This idea of control function in fixed point theory has opened the possibility of proving new fixed point results. Some recent results on fixed point and coincidence point have been obtained in works like [3], [5], [13] and [14].

Definition1.16 (Ψ- function). A function $\psi : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a Ψ- function if it satisfies the following conditions:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(x, x) \geq x$ for all $0 < x < 1$,

(iii) $\psi(1, a) \geq a$, $\psi(0, 0) = 0$,

(iv) $\psi(1, 1) = 1$.

An example ofΨ- function:

$\psi(x, y) = \sqrt{xy}$.

The purpose of this paper is to prove a fixed point result for two mappings in generalized Menger space with continuous t-nornT of H-type. We will use **Φ-function** and **Ψ- function** as defined in definitions 1.15 and definition 1.16 respectively and also support our result by constructing an example.

Proposition 1.17[7]. Let $(x_n, n \in \mathbb{N})$ be a sequence of numbers in [0, 1] such that $\lim_{n\to\infty} x_n = 1$ and the t-norm T is of H-type, then

$\lim_{n\to\infty} T_{i=n}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1$.

2. Main Result

Recently, Choudhury et. al. [4] proved the following result:

Theorem 2.1. Let (X, F, T_M) be a complete generalized Menger space, where T_M is the 3rd order minimum t-norm given by $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$and the mapping $f : X \to X$ be a self mapping which satisfies the following condition:

$$F(fx, fy, t) \geq \varphi\left(F\left(x, fx, \left(\frac{t_1}{a}\right)\right), F\left(y, fy, \left(\frac{t_2}{b}\right)\right)\right), \quad (2.1)$$

for all $x, y \in X$ and for $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$ where φ is $\boldsymbol{\theta}$ - function. Then f has a unique common fixed point.

Now we prove our result under Hadˇzi´ t - norm T.

Theorem 2.2. Let (X, F, T) be a generalized Menger space, where T is the 3^{rd} order t-norm of Hadžić- type and the mapping f and $g : X \to X$ be self mappings which satisfy the following conditions:

(2.2) $f(X) \subseteq g(X)$,

(2.3) either $f(X)$ or $g(X)$ is complete,

(2.4) $F(fx, fy, \phi(t)) \geq \psi(F(fx, gx, \phi(\frac{t_1}{a})), F(fy, gy, \phi(\frac{t_2}{b})))$,

for all $x, y \in X$, and for $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$ where ϕ is **Φ**-function and ψ is a**Ψ- function**.

For any $x_0 \in X$, the sequence $\{y_n\}$ in X be constructed as follows : $y_n = fx_n = gx_{n+1}$, $n = 0,1,2,3,\ldots$,and for $\mu \in (c, 1)$ the following condition holds:

$lim_{n\to\infty} T_{i=n}^{\infty} F(y_0, y_1, \frac{1}{\mu^i}) = 1$.

Then f and g have a unique common fixed point provided f and g are weakly compatible on X.

Proof. Let $x_0 \in X$. then construct a sequence $\{y_n\}$ as $y_n = gx_{n+1} = fx_n$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Now we have for $t, t_1, t_2 > 0$, with $t = t_1 + t_2$ and in view of the properties of **Φ-function**, for $u > 0$ we can find a positive number t such that $u > \phi(t)$.

Then for $u > 0$, we have

$F(y_{n+1}, y_n, u) \geq F(y_{n+1}, y_n, \phi(t)) = F(fx_{n+1}, fx_n, \phi(t))$

$\geq \psi\{F(fx_{n+1}, gx_{n+1}, \phi(\frac{t_1}{a})), F(fx_n, gx_n, \phi(\frac{t_2}{b}))\}$

$= \psi\{F(y_{n+1}, y_n, \phi(\frac{t_1}{a})), F(y_n, y_{n-1}, \phi(\frac{t_2}{b}))\}$

Let $t_1 = \frac{at}{a+b}$, $t_2 = \frac{bt}{a+b}$ and $c = a + b$, then $0 < c < 1$, therefore, we have

(2.5) $F(y_{n+1}, y_n, \phi(t)) \geq \psi\{F(y_{n+1}, y_n, \phi(\frac{t}{c})), F(y_n, y_{n-1}, \phi(\frac{t}{c}))\}$.

We now claim that for all $t > 0$,

(2.6) $F(y_{n+1}, y_n, \phi(\frac{t}{c})) \geq F(y_n, y_{n-1}, \phi(\frac{t}{c}))$.

If possible, let for some $t > 0$, $F(y_{n+1}, y_n, \phi(\frac{t}{c})) < F(y_n, y_{n-1}, \phi(\frac{t}{c}))$, then we have

$F(y_{n+1}, y_n, \phi(t)) \geq \psi\{F(y_{n+1}, y_n, \phi(\frac{t}{c})), F(y_{n+1}, y_n, \phi(\frac{t}{c}))\}$

$\geq F(y_{n+1}, y_n, \phi(\frac{t}{c}))$

$\geq F(y_{n+1}, y_n, \phi(t))$, since $0 < c < 1$,which leads to a contradiction.

Therefore for all $t > 0$, $F(y_{n+1}, y_n, \phi(\frac{t}{c})) \geq F(y_n, y_{n-1}, \phi(\frac{t}{c}))$.

Therefore, in view of (2.6) and (2.5), we have

$F(y_{n+1},y_n,\phi(t)) \geq \psi\{F(y_{n+1}, y_n,\phi(\frac{t}{c})), F(y_n, y_{n-1}, \phi(\frac{t}{c}))\}$

$\geq \psi\{F(y_n, y_{n-1}, \phi(\frac{t}{c})), F(y_n, y_{n-1}, \phi(\frac{t}{c}))\}$

$\geq F(y_n, y_{n-1}, \phi(\frac{t}{c}))$

$\geq F(y_{n-1}, y_{n-2}, \phi(\frac{t}{c^2}))$

. . .

$\geq F(y_1, y_0, \phi(\frac{t}{c^n}))$.

Implies

$$F(y_{n+1},y_n,\phi(t)) \geq F(y_1, y_0, \phi(\frac{t}{c^n})).$$

Therefore, $lim_{n\to\infty} F(y_{n+1},y_n,\phi(t)) = 1$, for all $t > 0$,

implies $lim_{n\to\infty} F(y_{n+1},y_n, u) = 1$, for all $u > 0$.

Next we show that sequence $\{y_n\}$ is a Cauchy sequence.

Let $\sigma=\frac{c}{\mu}$, where $\mu \in (c, 1)$ and $c \in (0, 1)$, then $0 <\sigma< 1$, therefore the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $m_0 \in \mathbb{N}$ such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$. Now for every $m > m_0$ and for every $s \in \mathbb{N}$ and in view of **(Φ)**,

$$u > \phi(t) > \phi(t\sum_{i=m_0}^{\infty} \sigma^i) > \phi(t\sum_{i=m}^{m+s} \sigma^i),\text{which implies that}$$

$F(y_{m+s+1}, y_m, u) > F(y_{m+s+1}, y_m, \phi(t))$

$\geq F(y_{m+s+1}, y_m, \phi(t\sum_m^{m+s} \sigma^i))$

$\geq \underbrace{T(T \ldots T}_{s-times}(F(y_{m+s+1}, y_{m+s}, \phi(t\sigma^{m+s})), F(y_{m+s}, y_{m+s-1}, \phi(t\sigma^{m+s-1})),$

$\ldots, F(y_{m+1}, y_m \phi(t\sigma^m))))$

$\geq \underbrace{T(T \ldots T}_{s-times}(F(y_0, y_1, \phi(\frac{t\sigma^{m+s}}{c^{m+s}})), \ldots, F(y_0, y_1, \phi(\frac{t\sigma^m}{c^m}))))$

$\geq T_{i=m}^{m+s}$F(y_0, y_1, ϕ($\frac{t}{\mu^i}$))

$= T_{i=m}^{\infty}$F(y_0, y_1, ϕ($\frac{t}{\mu^i}$)) .

It is obvious that,

$lim_{n\to\infty}T_{i=n}^{\infty}$F($y_0$, y_1, $\frac{1}{\mu^i}$) = 1, implies that, $lim_{n\to\infty}T_{i=n}^{\infty}$F($y_0$, y_1, ϕ($\frac{1}{\mu^i}$)) = 1, and this implies that,

$lim_{n\to\infty}T_{i=n}^{\infty}$F($y_0$, y_1, ϕ($\frac{t}{\mu^i}$)) = 1, for every t> 0.

Now for every u > 0, there exists t> 0 such that u > ϕ(t) > 0, there exist m_1(ϕ(t), λ) such that

F(y_{m+s+1} , y_m, u) > 1$-\lambda$, for every m $\geq$ m_1(ϕ(t), λ) and every s $\in \mathbb{N}$.

This means that the sequence {y_n} is Cauchy sequence.Since either f(X) or g(X) is complete, for definiteness one can assume that g(X) is complete subspace of X then the subsequence of {y_n} must get a limit in g(X). Call it be w. Let p$\in$$g^{-1}$w. Then g p = w as {$y_n$} is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence {y_n} also convergent implying thereby the convergence of subsequence of the convergent sequence. Hence, we get

$lim_{n\to\infty}$$y_n$ = w = $lim_{n\to\infty}$gx_n = $lim_{n\to\infty}$ fx_n.

Let g(X) is complete, there exists a point p $\in$ X such that gp = w.Now we show that f p = w.

For some t > 0, since 0 < b < 1, we can chooseη_1,η_2,t_1,t_2> 0 such that

ϕ(t) = $\eta_1 + \eta_2$+ ϕ(t_1 + t_2) and$\frac{t_2}{b}$> t.

Then we have,

F(w, f p, ϕ(t)) = F(w, f p, $\eta_1 + \eta_2$+ ϕ(t_1 + t_2))

$\geq$ T {F(w, y_n, η_1), F(y_n, y_{n+1}, η_2), F(y_{n+1}, f p, ϕ(t_1 + t_2))}

$\geq$T {F(w, y_n, η_1), F(y_n, y_{n+1}, η_2), ψ (F(fx_{n+1}, gx_{n+1}, ϕ($\frac{t_1}{a}$)), F(fp, gp, ϕ($\frac{t_2}{b}$)))}

$\geq$T {F(w, y_n, η_1), F(y_n, y_{n+1}, η_2), ψ (F(fx_{n+1}, gx_{n+1}, ϕ($\frac{t_1}{a}$)), F(fp, gp, $\phi(t)$))}

Taking limit n$\to\infty$, we have

F(w, f p, ϕ(t)) $\geq$ T {1, 1,ψ (1, F(w, fp, ϕ(t)))} = ψ (1, F(w, fp, ϕ(t)))= F(w, fp, ϕ(t)).

This gives

F(w, fp, ϕ(t)) $\geq$F(w, fp, ϕ(t)), which is not possible.

Therefore, $F(u, fp, \phi(t)) = 1$ for all $t > 0$ and this implies that $w = fp = gp$. Since f and g are weakly compatible, it follows that $fg\,p = gf\,p$, that is, $fw = gw$.

Now, we show that w is a fixed point of f and g, from (2.4), we have,

$F(f\,w, f\,x_n, \phi(t)) \geq \psi\,(F(fw, gw, \phi(\frac{t_1}{a})), F(f\,x_n, g\,x_n, \phi(\frac{t_2}{b}))),$ (where $t = t_1 + t_2$)

i.e., $F(f\,w, y_n, \phi(t)) \geq \psi\,(F(fw, gw, \phi(\frac{t_1}{a})), F(y_n, y_{n-1}, \phi(\frac{t_2}{b}))).$

Taking $n \to \infty$, we have from the above inequality,

$F(f\,w, w, \phi(t)) \geq \psi\,(1,1) = 1$ gives $fw = w = gw$.

Thus w is a fixed point of f and g.

Uniqueness.

let z be another fixed point of f and g. Now for all $t > 0$ we have,

$F(z, w, \phi(t)) = F(fz, fw, \phi(t))$

$\geq \psi\,((F(fz, gz, \phi(\frac{t_1}{a})), F(fw, gw, \phi(\frac{t_2}{b})))$ (where $t_1, t_2 > 0$ and $t = t_1 + t_2$)

$= \psi\,(F(z, z, \phi(\frac{t_1}{a})), F(w, w, \phi(\frac{t_2}{b})))$

$= \psi(1, 1) = 1.$

Hence, we get $z = w$.

This completes the proof of the theorem.

Corollary 2.3. Let (X, F, T) be a complete generalized Menger space, where T is the 3rd order t-norm of Hadžić type and f be self mapping on X satisfy the following inequality:

(2.7) $F(fx, f\,y, \phi(t)) \geq \psi\,(F(fx, x, \phi(\frac{t_1}{a})), F(fy, y, \phi(\frac{t_2}{b}))),$

for all $x, y \in X$, and for $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$ where ϕ is **Φ-** function and ψ is a **Ψ- function**.

For any $x_0 \in X$, the sequence $\{x_n\}$ in X be constructed as follows: $x_n = f x_{n-1}$, n = 0,1, 2,3, .. . ,

such that for $\mu \in (c, 1)$ and $c = a + b$, the following condition holds:

$lim_{n\to\infty} T_{i=n}^{\infty} F\,(x_0, x_1, \frac{1}{\mu^i}) = 1,$

Then f has a unique common fixed point.

Proof. On putting $g(x) = I$ in Theorem 2.2 we get the required result.

Corollary 2.4. Let (X, F, T_M) be a complete generalized Menger space, where T_M is the 3rd order minimum t-norm given by T_M (α, β, γ) = min$\{\alpha, \beta, \gamma\}$ and the mapping f : X → X be a self mapping which satisfy the following inequality for all x, y ∈ X,

(2.8) $F(fx, fy, \phi(t)) \geq \psi(F(fx, x, \phi(\frac{t_1}{a})), F(fy, y, \phi(\frac{t_2}{b})))$,

where t_1, t_2, t > 0 with $t = t_1 + t_2$, a, b > 0 with 0 < a + b < 1 where ϕ is **Φ**-function and ψ is a **Ψ**-function. Then f has a unique common fixed point.

Proof: Since every t-norm T_M is of Hadžić-type. Therefore proof follows from Corollary 2.3.

Example 2.1. Let X = {1,2, 3, 4}, $T_M(\alpha, \beta, \gamma)$ = min$\{\alpha, \beta, \gamma\}$, i.e., T_M is the 3rd order minimum t-norm and F(x, y, t) be defined as

$$F(1, 2, t) = F(2, 1, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.60, & if\ 0 < t < 3, \\ 1, & if\ t \geq 3. \end{cases}$$

$$F(1, 3, t) = F(3, 1, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.80, & if\ 0 < t \leq 1.5, \\ 1, & if\ t > 1.5, \end{cases}$$

$$F(1, 4, t) = F(4, 1, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.70, & if\ 0 < t \leq 2, \\ 1, & if\ t > 2. \end{cases}$$

$$F(2, 3, t) = F(3, 2, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.85, & if\ 0 < t \leq 1.5, \\ 1, & if\ t > 1.5, \end{cases}$$

$$F(2, 4, t) = F(4, 2, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.70, & if\ 0 < t \leq 2, \\ 1, & if\ t > 2. \end{cases}$$

$$F(3, 4, t) = F(4, 3, t) = \begin{cases} 0, & if\ t \leq 0, \\ 0.60, & if\ 0 < t < 3, \\ 1, & if\ t \geq 3. \end{cases}$$

Then (X, F, T_M) is a complete generalized Menger space.

Let f : X → X be given by f (1) = f (2) = f (3) = f (4) = 1, and g(2) = g(3) = g(4) = 3, g(1)= 1. Also f(X) ⊆ g(X) and if we take $\phi(t) = t$, $\psi(x, y) = \sqrt{xy}$ and a = 0.2, b = 0.75, then f and g satisfy all the conditions of Theorem 2.2 and 1 is the unique fixed point of f and g.

In this example (X, F, T_M) is not a Menger space as can be seen from the fact that

$F(3, 4, 2) \ngtr T_M(F(3, 2, 1), F(2, 4, 1))$.

This shows that generalized Menger spaces are effective generalization of generalized metric spaces.

Example 2.2. Let $X = \{ \frac{1}{n} : n \in N \} \cup \{0\}$, $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$,i.e., T_M is the 3rd order minimum t-norm and F(x, y, t) be defined as $F(x, y, t) = \frac{t}{t+d(x,y)}$ and $d(x, y) = |x - y|$ with $t > 0$ and $x, y \in X$. Then (X, F, T_M) is a complete generalized Menger space.

Let $f(x) = 1$, $g(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$ on X.

Also $f(X) \subseteq g(X)$ and if we take $\phi(t) = t$, $\psi(x, y) = \sqrt{xy}$ and $a = 0.2$, $b = 0.75$, then f and g satisfy all the conditions of Theorem 2.2 and 1 is the unique fixed point of f and g.

Acknowledgement: Second author is highly thankful to UGC for providing financial help in form providing Major Research Project under Ref. No. 39-41/2010(SR).

REFERENCES

[1] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57 (2000), no. 1-2, 31-37.

[2] B.S. Choudhury and K.P. Das : A new contraction principle in Menger spaces, ActaMathematicaSinica, English Series, 24(8) (2008), 1379-1386.

[3] B.S. Choudhury and K.P. Das, A coincidence point result in Menger spaces using a control function, Chaos, Solitons and Fractals 42 (2009), 3058-3063.

[4] B. S. Choudhury and K. Das, Fixed points of generalized Kannan type mappings in generalized menger spaces, Commun. Korean Math. Soc. 24 (2009), No. 4, pp. 529-537. DOI 10.4134/CKMS.2009.24.4.529.

[5] P.N. Dutta, B.S. Choudhury and K.P. Das : Some fixed point results in Menger spaces using a control function, Surveys in Mathematics and its Applications, 4 (2009), 41-52.

[6] O. Hadzic, A fixed point theorem in Menger spaces, Publ. Inst. Math. Beograd, 20 (1979), 107-112.

[7] O. Had zic and E.Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, Dordrecht, 2001.

[8] R. Kannan, Some results on fixed point, Bull. Cal. Math. Soc. 60 (1968), 71-76.

[9] R. Kannan, Some results on fixed point II, Amer. Math. Monthly 76 (1969), 405-408.

[10] M.S. Khan, M. Swaleh and S. Sessa : Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984), 1-9.

[11] M. Kikkawa and T. Suzuki, Some similarity between contractions and Kannan mappings, Fixed Point Theory and Applications 2008 (2008), Article ID 649749.

[12] K. Menger, “Statistical metrices,” Proceedings of the National Academy of Sciences (USA), Vol. 28, 1942, pp. 535-537.

[13] D. Mihet, Altering distances in probabilistic Menger spaces, Nonlinear Analysis 71 (2009), 2734-2738.

[14] S.V.R. Naidu : Some fixed point theorems in Metric spaces by altering distances, Czechoslovak Mathematical Journal, 53(128) (2003),205-212.

[15] V. Radu, Lectures on Probabilistic Analysis, Surveys, Lecture Notes and Monographs, Series on Probability, Statistics and Applied Mathematics, vol. 2, Universitatea din Timisoara, Timisoara, 1994.

[16] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math., 10(1960), 313-334.

[17] V.M. Sehgal and A.T. Bharucha-Reid, “Fixed points of contraction mappings on probabilistic metric spaces,” Mathematical Systems Theory, Vol. 6, 1972, pp.97-102.

[18] Y. Shi, L. Ren and X. Wang, The extension of fixed point theorems for set valued mapping, J. Appl. Math. & Computing 13 (2003), no. 1-2, 277-286.