Available online at http://scik.org
Adv. Fixed Point Theory, 2021, 11:5
https://doi.org/10.28919/afpt/5231
ISSN: 1927-6303

APPROXIMATION OF COMMON SOLUTIONS OF FIXED POINT PROBLEM FOR α-HEMICONTRACTIVE MAPPING, SPLIT EQUILIBRIUM AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. ln this paper, a Halpern-type algorithm for approximating a common fixed point of multivalued α-hemicontractive mappings and a set of solutions of split equilibrium and variational inequality problems is constructed. Strong convergence of the sequence generated by the algorithm is proved in the setting real Hilbert spaces. Our results improved and generalised the results of Meche and Zegeye [2] in particular and some recent results in Literature.

Keywords: split equilibrium; variational inequality; α-hemicontractive mapping; iterative approximation.

2010 AMS Subject Classification: 47H09, 47H10, 47J05, 65J15.

1. Introduction

Throughout this paper, $F(S)$ denotes the set of fixed point of the multivalued mapping S, $\mathbb{R}$ denotes the set of all real numbers, and $\mathbb{N}$ a set of positive integers. Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $CB(C)$ denotes the family of nonempty, closed and bounded subsets of C and $K(C)$ denotes a family of nonempty and compact subsets of C.

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Received November 18, 2020

The Hausdorff metric is defined by

$$D(A,B) = max\{(\sup d(x,B), x \in A), (\sup d(y,A), y \in B)\},$$

for all $A, B \in CB(C)$, where $d(x,B) = \inf\{\|x-b\| : b \in B\}$

Definition 1.1. *(see [2]) Let $S : C \longrightarrow CB(C)$ be a multivalued mapping. Then, S is said to be L-Lipshitizian if there exists $L > 0$ such that*

$$D(Sx,Sy) \leq L\|x-y\|, \forall x,y \in C \tag{1.1}$$

S is said to be nonexpansive if it is Lipschitz continous with $L = 1$ in (1.1). Note that the class of nonexpansive mapping is one of the initial classes of mappings for which fixed point results were obtained using the geometric structure of the underlying Banach space rather than the compactness property. An element $x \in C$ is called the fixed point of a multivalued mapping S if $x \in Sx$. A nonexpansive multivalued mapping S with a nonempty fixed point set is called quasi-nonexpansive multivalued mapping(i.e, a mapping $S : C \longrightarrow CB(C)$ such that $D(Sx,Sp) \leq |x-p\|, \forall (x,p) \in C \times F(S)$).

Definition 1.2. *(see [2]) Let $S : C \longrightarrow CB(C)$ be a multivalued mapping. Then, S is said to be demicontractive if $F(S) = \{x \in C : x \in Sx\} \neq \emptyset$ and for all $u \in S$ satisfying $\|u-p\| \leq D(Sx,Sp)$, there exists $k \in (0,1)$ such that*

$$D^2(Sx,Sp) \leq \|x-p\|^2 + k\|x-u\|^2, \forall x \in C \text{ and } \forall p \in F(S), \tag{1.2}$$

Note that if k in (1.2) is 1, then S is called hemicontractive multivalued mapping. Thus, the class of demicontractive multivalued napping is a proper subclass of the class of hemicontractive multivalued mapping.

ln 2015, Osilike and Onah [12] introduced a new class of mapping called α-hemicontractive mapping in a closed convex subset of a real Hilbert space. They showed that the class of α-demicontractive mapping introduced by Maruster and Maruster in [18] is a subclass of the class of α-hemicontractive mapping. Also, it was shown in [12] that the class of hemicontractive mapping and the class of α-hemicontractive mapping are independent(see [12] for details).

Definition 1.3. *Let $S : C \longrightarrow CB(C)$ be a multivalued mapping. Then, S is said to be α-hemicontractive multivalued mapping if $F(S) = \{x \in C : x \in Sx\} \neq \emptyset$ and for all $u \in S$ satisfying $\|u - p\| \leq D(Sx, Sp)$, we have*

$$D^2(Sx, S\alpha p) \leq \|x - \alpha p\|^2 + \|x - u\|^2, \forall x \in C \text{ and } \forall p \in F(S), \tag{1.3}$$

for some $\alpha \geq 1$. The class of mapping defined by (1.3) is a superclass of the class of α-demicontractive multivalued mapping(where a mapping $S : C \longrightarrow CB(C)$ is called α-demicontractive multivalued mapping if $F(S) = \{x \in C : x \in Sx\} \neq \emptyset$ and for all $u \in S$ satisfying $\|u - p\| \leq D(Sx, Sp)$, there exists $k \in (0, 1)$ such that $D^2(Sx, S\alpha p) \leq |x - \alpha p\|^2 + k\|x - u\|^2$, $\forall x \in C$, $\forall p \in F(S)$ and for some $\alpha \geq 1$).

Observe that (1.3) is equivalent to

$$\langle x - u, x - \alpha p \rangle \geq 0, \forall x \in C, \forall p \in F(S), \forall u \in S \text{ and for some } \alpha \geq 1. \tag{1.4}$$

Let $F : C \times C \longrightarrow R$ be a bifunction. The equilibrium problem for F is to find $z \in C$ such that

$$F(z, y) \geq 0, \forall y \in C \tag{1.5}$$

The set of all solutions of (1.5) is denoted by $EP(F)$, that is, $EP(F) = \{z \in C : F(z, y) \geq 0, \forall y \in C$.

Let $A : C \longrightarrow R$ be a nonlinear mapping. The classical variational inequality problem, which was developed as a useful tool in solving partial differential equation by Stampacchia(see [22] for details), is the problem of finding $x \in C$ such that

$$\langle u - x, Ax \rangle \geq 0, \forall u \in C \tag{1.6}$$

The set of all solutions of (1.6) is denoted by $VI(C,A)$.

Recently, Karmi and Rizvi [17] considered a problem which they called split equilibrium problem. Let H_1 and H_2 be two Hilbert spaces and C,K be two nonempty closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C\times C \longrightarrow R$ and $F_2 : K\times K \longrightarrow R$ be two bifunctions and $A : H_1 \longrightarrow H_2$ be a bounded linear operator.The split equilibrium problem is to find $x^\star \in C$ such that

$$F_1(x^\star,x) \geq 0, \forall x \in C \; and \; y^\star = Ax^\star \in K \; such \; that \; F_2(y\star,y) \geq 0, \forall y \in K \tag{1.7}$$

The set of solutions of split equilibrium problem is denoted by Ω, that is, $\Omega = \{z \in C : x \in EP(F_1), Ax \in EP(F_2)\}$.

Very recently, Meche and Zegeye [2] first introduced an iteration sequence (for finding common set of solutions of fixed point problem, split equilibrium and variational inequality problems) defined as follows:

Let $x_o, u \in C$ be arbitrary and let x_n be a sequence in C generated by

$$\left.\begin{aligned}
z_n &= T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n,\\
y_n &= J_t z_n,\\
u_n &= (1-\delta_n)y_n + \delta_n v_n,\\
x_{n+1} &= a_n u + b_n w_n + c_n y_n,
\end{aligned}\right\} \tag{1.8}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s,r,t > 0, \lambda \in (0,\frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a,b \in (0,1)$.They used (1.8) to prove the following theorems:

Theorem MZ1[2]: Let H_1 and H_2 be two Hilbert spaces and C,Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C\times C \longrightarrow R$ and $F_2 : Q\times Q \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B : Q \longrightarrow H_2$ be a bounded linear operator. Let $S : C \longrightarrow CB(C)$ be L-Lipschitz hemicontractive-type multivalued mapping. Assume that $\Theta = F(S)\cap\Omega\cap VI(CA)$ is nonempty and $Sp = p$ for all $p \in \Theta$. Let $x_0, u \in C$ be arbitrary and let x_n be a sequence in C generated by

$$\left.\begin{aligned} z_n &= T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n, \\ y_n &= J_t z_n, \\ u_n &= (1-\delta_n)y_n + \delta_n v_n, \\ x_{n+1} &= a_n u + b_n w_n + c_n y_n, \end{aligned}\right\} \tag{1.9}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a, b \in (0,1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$,

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$.

Then, the sequence $\{x_n\}$ is bounded.

Theorem MZ2[2]: Let H_1 and H_2 be two Hilbert spaces and C, Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C \times C \longrightarrow R$ and $F_2 : Q \times Q \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B : Q \longrightarrow H_2$ be a bounded linear operator. Let $S : C \longrightarrow CB(C)$ be L-Lipschitz hemicontractive-type multivalued mapping. Assume that $\Theta = F(S) \cap \Omega \cap VI(CA)$ is nonempty, $Sp = p$ for all $p \in \Theta$ and $(I - S)$ is demiclosed at zero. Let $x_0, u \in C$ be arbitrary and let x_n be a sequence in C generated by

$$\left.\begin{aligned} z_n &= T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n, \\ y_n &= J_t z_n, \\ u_n &= (1-\delta_n)y_n + \delta_n v_n, \\ x_{n+1} &= a_n u + b_n w_n + c_n y_n, \end{aligned}\right\} \tag{1.10}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a, b \in (0,1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$,

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$.

Then, the sequence $\{x_n\}$ converges strongly to αp, where $\alpha p = P_\Theta(u)$.

It is our purpose in this paper to first introduce a new iterative sequence and then prove strong convergence theorem of our new iterative sequence to the common solutions of fixed point problem for α-hemicontractive mapping (which is a more general operator than the one used by Meche and Zegeye), split equilibrium and variational inequality problems.

2. Preliminary

In this section, we collect some concepts and results that play a crucial role in the sequel. Let $S : C \longrightarrow C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Then, for every $x \in C$ and $y \in F(S)$, we obtain that

$$\langle x - Sx, y - Sx \rangle \leq \frac{1}{2}\|Sx - x\|^2 \tag{2.1}$$

(see e.g. [17]). Let H be a real Hilbert space, C a closed convex subset of H and $P_C : H \longrightarrow 2^C$ a metric projection of H onto C. Recall that for every $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

The mapping $P_C : H \longrightarrow 2^C$ is characterised by

$$z = P_C x \in C \text{ if and only if } \langle x - z, z - y \rangle \geq 0, \forall x \in H, y \in C. \tag{2.2}$$

In what follows, we shall use the following assumptions:

Assumption G : Let H be a Hilbert space and C a nonempty, closed and convex subset of H. Let $F : C \times C \longrightarrow R$ be a bifunction satisfying the following conditions:

$G_1 : F(x,x) = 0, \forall x \in H$

G_2 :F is a monotone, i.e, $F(x,y) + F(y,x) \leq 0, \forall x, y \in C$

$G_3 : \lim_{t \longrightarrow 0}(tz + (1-t)y) \leq F(x,y), \forall x, y, z \in C$

G_4 : for each $x \in C, y \rightarrow F(x,y)$ is convex and lower semicontinous.

ln the proof of our main results, we make use of the following lemmas:

Lemma 2.1. *Let $F_1 : C \times C \longrightarrow R$ be a bifunction satisfying assumption G. For $s > 0$ and for all $x \in H$, define the mapping $T_s^{F_1} : H_1 \longrightarrow C$ as follows:*

$$T_s^{F_1} x = \{x \in C : F_1(x,y) + \frac{1}{s}\langle y - z, x - y\rangle \geq 0, \forall y \in C\} \tag{2.3}$$

Then, we have the following:

(1) $T_s^{F_1}$ is nonempty and single valued;
(2) $T_s^{F_1}$ is firmly nonexpansive, i.e, $\|T_s^{F_1}x - T_s^{F_1}y\| \leq \langle T_s^{F_1}x - T_s^{F_1}y, x - y\rangle$;
(3) $F(T_s^{F_1}) = EP(T_s^{F_1})$;
(4) $EP(F_1)$ is closed and convex.

Furthermore, assume that $F_2 : Q \times Q \longrightarrow R$ is another bifunction that satisfies assumption G. For $r > 0$ and for all $x \in H$ define the mapping $T_s^{F_2} : H_1 \longrightarrow Q$ as follows:

$$T_s^{F_2} x = \{x \in Q : F_1(x,y) + \frac{1}{s}\langle y - z, x - y\rangle \geq 0, \forall y \in Q\} \tag{2.4}$$

Then, we have the following:

(1) $T_s^{F_2}$ is nonempty and single valued;
(2) $T_s^{F_2}$ is firmly nonexpansive, i.e, $\|T_s^{F_2}x - T_s^{F_2}y\| \leq \langle T_s^{F_2}x - T_s^{F_1}y, x - y\rangle$;
(3) $F(T_s^{F_2}) = EP(T_s^{F_2})$;
(4) $EP(F_2)$ is closed and convex.

Lemma 2.2. *Let H be a Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0,1]$, for $i = 1,2,3$, such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the following equality holds:*

$$\|\alpha_1 + \alpha_2 + \alpha_3\|^2 = \sum_{i=1}^{2} \alpha_i \|x\|^2 - \sum_{1 \leq i,j \leq 3} \alpha_i \alpha_j \|x_i - x_j\|.$$

Lemma 2.3. *Let H be a real Hilbert space. Then, for every $x, y \in H$, we have the following:*

i. $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$;
ii. $\|x + y\| = \|x\|^2 + 2\langle x, x + y\rangle$.

Lemma 2.4. *Let H be a real Hilbert space. Let $A,B \in CB(H)$ and $a \in A$. Then for any $\varepsilon > 0$, there exists a point $b \in B$such that $\|a-b\| \leq D(A,B)+\varepsilon$. In particular, for every $a \in A$, there exists an element $b \in B$ such that $\|a-b\| \leq 2D(A,B)+\varepsilon$.*

Lemma 2.5. *Let $\{b_n\}$ be a sequence of nonnegative real numbers such that*

$$b_{n+1} \leq (1-\alpha_n)b_n + \alpha_n\delta_n,$$

for $n \geq n_0$, where $\alpha_n \subset (0,1)$ and $\delta_n \in R$ satisfying the following restrictions $: \lim_{n\to\infty} = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} sup\delta_n \leq 0.\text{Then}, \lim_{n\to\infty} b_n = 0.$

Lemma 2.6. *Let H be a Hilbert space and C a closed convex subset of H. Let $A : C \longrightarrow H$ be a continous monotone mapping. Then, for $t > 0$ and for all $x \in H$, there exists $z \in C$ such that*

$$\langle Az, y-z\rangle + \frac{i}{t}\langle y-z, z-x\rangle \geq 0, \forall x, y \in C$$

Moreover, the mapping $J_t : H \longrightarrow C$ defined by

$$J_t = \{z \in C : \langle Az, zx\rangle + \frac{i}{t}\langle y-z, z-x\rangle \geq 0, \forall y \in C\},$$

satisfies the following:

(1) J_t is nonempty and single valued;

(2) J_t is firmly nonexpansive;

(3) $F(J_t) = VI(C,A)$;

(4) $VI(C,A)$ is closed and convex.

Lemma 2.7. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leq a_{n_{i+1}}$ for all $i \in N$. Then, there exists a nondecreasing sequence $m_k \in N$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in N : a_{m_k} < a_{m_{k+1}}$ and $a_k < a_{m_{k+1}}$. lnfact, $m_k = max\{j \leq k : a_j < a_{j+1}\}$*

3. Main Results

Theorem 3.1. *Let H_1 and H_2 be two Hilbert spaces and C, Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C \times C \longrightarrow R$ and $F_2 : Q \times Q \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B :$*

$Q \longrightarrow H_2$ be a bounded linear operator. Let $S : C \longrightarrow CB(C)$ be L-Lipschitz α-hemicontractive multivalued mapping. Assume that $\Theta = F(S) \cap \Omega \cap VI(CA)$ is nonempty and $S\alpha p = \alpha p$ for all $p \in \Theta$ and for some $\alpha \geq 1$.Let $x_0, u \in C$ be arbitrary and let $x_n$be a sequence in C generated by

$$
(3.1) \qquad \left.\begin{aligned}
z_n &= T_s^{F_1}(1 - \lambda B^\star(1 - T_r^{F_2})B)x_n,\\
y_n &= J_t z_n,\\
u_n &= (1 - \delta_n)y_n + \delta_n v_n,\\
x_{n+1} &= a_n u + b_n[(1 - \gamma_n)w_n + \gamma_n x_n] + c_n y_n,
\end{aligned}\right\}
$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0, 1)$ and $b_n, c_n \subset [a, b]$ for some $a, b \in (0, 1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$,
ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$.

Then, the sequence $\{x_n\}$ is bounded.

Proof. First, we show that $B^\star(1 - T_r^{F_2})B$ is a $\frac{1}{2d}$-inversely strongly monotone mapping. Since, $T_r^{F_2}$ is nonexpansive, we have that $(1 - T_r^{F_2})$ is $\frac{1}{2}$-inversely strongly monotone. Thus, $\forall x, y \in H_1$, we have

$$
\begin{aligned}
&\|B^\star(1 - T_r^{F_2})Bx - B^\star(1 - T_r^{F_2})By\|^2\\
&= \langle B^\star(1 - T_r^{F_2})Bx - B^\star(1 - T_r^{F_2})By, B^\star(1 - T_r^{F_2})Bx - B^\star(1 - T_r^{F_2})By\rangle\\
&= \langle((1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By), BB^\star((1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By)\rangle\\
&\leq BB^\star|\langle(1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By), ((1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By)\rangle|\\
&= d\|(1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By\|^2\\
&\leq 2d\langle x - y, B^\star[(1 - T_r^{F_2})Bx - (1 - T_r^{F_2})By\rangle]\\
&= 2d\langle x - y, B^\star(1 - T_r^{F_2})Bx - B^\star(1 - T_r^{F_2})By\rangle],
\end{aligned}
$$

which implies that $B^\star(1 - T_r^{F_2})B$ is $\frac{1}{2d}$-inversely strongly monotone. Next, we show that $(1 - \lambda B^\star(1 - T_r^{F_2})B)$ is nonexpansive. Now, since $\lambda \in (0, \frac{1}{d})$, we otain that $(1 - \lambda B^\star(1 - T_r^{F_2})B)$ is nobexpansive. Hence, from the nonexpansiveness of $T_s^{F_1}$, we obtain

$$\begin{aligned} & \|T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x - T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)y\| \\ \leq\ & \|(1-\lambda B^\star(1-T_r^{F_2})B)x - (1-\lambda B^\star(1-T_r^{F_2})B)y\| \\ \leq\ & \|x-y\|. \end{aligned} \tag{3.2}$$

Now, let $\alpha p \in \Theta$, for some $\alpha \geq 1$. Then, we have $S(\alpha p) = \alpha p, J_t(\alpha p) = \alpha p$ and $\alpha p \in \Theta$; hence $\alpha p = T_s^{F_1}(\alpha p)$ and $B(\alpha p) = T_s^{F_2}(\alpha p)$, which implies that $T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)(\alpha p) = \alpha p$. Thus, from (3.1) and (3.2), we have

$$\begin{aligned} \|z_n - \alpha p\| &= \|T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n - \alpha p\| \\ &\leq \|(1-\lambda B^\star(1-T_r^{F_2})B)x_n - \alpha p\| \\ &\leq \|x_n - \alpha p\|. \end{aligned} \tag{3.3}$$

Since J_t is npnexpansive, we get

$$\begin{aligned} \|y_n - \alpha p\| &= \|J_t z_n - \alpha p\| \\ &= \|J_t z_n - J_t(\alpha p)\| \\ &\leq \|z_n - \alpha p\|, \end{aligned} \tag{3.4}$$

which by (3.3) gives

$$\|y_n - \alpha p\| \leq \|x_n - \alpha p\|. \tag{3.5}$$

Since S is α-hemicontractive, for all $w_n \in Su_n$, we have

$$\begin{aligned} \|w_n - \alpha p\|^2 &\leq D^2(Su_n, Sy_n) \\ &\leq \|u_n - \alpha p\|^2 + \|u_n - w_n\|^2. \end{aligned} \tag{3.6}$$

Also,

$$\begin{aligned} \|u_n - \alpha p\|^2 &= \|(1-\delta_n)y_n + \delta_n v_n - \alpha p\|^2 \\ &= \|(1-\delta_n)(y_n - \alpha p) + \delta_n(v_n - \alpha p)\|^2, \end{aligned}$$

which, using Lemma 2.2, gives

$$\|u_n - \alpha p\|^2 = (1-\delta_n)\|y_n - \alpha p\|^2 + \delta_n\|v_n - \alpha p\|^2 - \delta_n(1-\delta_n)\|y_n - v_n\|^2,$$

Since S is α-hemicontractive, for all, $v_n \in Sy_n$, we have

$$\begin{aligned}
\|u_n - \alpha p\|^2 &= (1-\delta_n)\|y_n - \alpha p\|^2 + \delta_n D^2(v_n - \alpha p) - \delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&\leq (1-\delta_n)\|y_n - \alpha p\|^2 + \delta_n[\|y_n - \alpha p\|^2 + \|y_n - v_n\|^2] - \delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= (1-\delta_n)\|y_n - \alpha p\|^2 + \delta_n\|y_n - \alpha p\|^2 + \delta_n\|y_n - v_n\|^2 - \delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= \|y_n - \alpha p\|^2 + \delta_n\|y_n - v_n\|^2 - \delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= \|y_n - \alpha p\|^2 + \delta_n^2\|y_n - v_n\|^2. \qquad (3.7)
\end{aligned}$$

(3.4) and (3.7) imply that

$$\|u_n - \alpha p\|^2 \leq \|x_n - \alpha p\|^2 + \delta_n^2\|y_n - v_n\|^2. \qquad (3.8)$$

(3.6) and (3.8) imply that

$$\|w_n - \alpha p\|^2 \leq \|x_n - \alpha p\|^2 + \delta_n^2\|y_n - v_n\|^2 + \|u_n - w_n\|^2. \qquad (3.9)$$

Next, we estimate $\|u_n - w_n\|^2$: From (3.1), we get

$$\begin{aligned}
\|u_n - w_n\|^2 &= \|(1-\delta_n)y_n + \delta_n v_n - w_n\|^2 \\
&= \|(1-\delta_n)y_n + \delta_n v_n - \delta_n w_n + \delta_n w_n - w_n\|^2 \\
&= \|(1-\delta_n)y_n + \delta_n(v_n - w_n) - (1-\delta_n)w_n\|^2 \\
&= \|(1-\delta_n)(y_n - w_n) + \delta_n(v_n - w_n)\|^2,
\end{aligned}$$

which by Lemma 2.2 gives

$$\|u_n - w_n\|^2 = (1-\delta_n)\|y_n - w_n\|^2 + \delta_n\|v_n - w_n\|^2 - \delta_n(1-\delta_n)\|y_n - v_n\|^2.$$

Since, $v_n, w_n \in Su_n, Sy_n$ respectively implies that $\|v_n - w_n\| \leq 2D(Sy_n, Su_n)$, we get

$$\|u_n - w_n\|^2 = (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n D^2(Sv_n, Sy_n) - \delta_n(1-\delta_n)\|y_n - v_n\|^2,$$

and since S is L-Lipschitizian mapping, we get

$$
\begin{aligned}
\|u_n - w_n\|^2 &\leq (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n L^2\|y_n - u_n\|^2 - \delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n L^2\|y_n - [(1-\delta_n)y_n + \delta_n v_n]\|^2 \\
&\quad -\delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n L^2\|y_n - (1-\delta_n)y_n - \delta_n v_n\|^2 \\
&\quad -\delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n L^2\|\delta_n(y_n - v_n)\|^2 \\
&\quad -\delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n^3 L^2\|y_n - v_n\|^2 \\
&\quad -\delta_n(1-\delta_n)\|y_n - v_n\|^2. \qquad (3.10)
\end{aligned}
$$

Putting (3.10) into (3.9), we have

$$
\begin{aligned}
\|w_n - \alpha p\|^2 &\leq \|x_n - \alpha p\|^2 + \delta_n^2\|y_n - v_n\|^2 + (1-\delta_n)\|y_n - w_n\|^2 + 4\delta_n^3 L^2\|y_n - v_n\|^2 \\
&\quad -\delta_n(1-\delta_n)\|y_n - v_n\|^2 \\
&= \|x_n - \alpha p\|^2 + \delta_n^2\|y_n - v_n\|^2 + (1-\delta_n)\|y_n - w_n\|^2 \\
&\quad +\delta_n(4\delta_n^2 L^2 + \delta_n - 1)\|y_n - v_n\|^2 \\
&= \|x_n - \alpha p\|^2 + (1-\delta_n)\|y_n - w_n\|^2 \\
&\quad -\delta_n(1 - 4\delta_n^2 L^2 - 2\delta_n)\|y_n - v_n\|^2. \qquad (3.11)
\end{aligned}
$$

Now, we estimate $\|x_{n+1} - \alpha p\|^2$:

$$
\begin{aligned}
\|x_{n+1} - \alpha p\|^2 &= \|au + b_n[\gamma_n w_n + (1-\gamma_n)x_n] + c_n - \alpha p\|^2 \\
&= \|a_n(u - \alpha p) + (a_n + b_n + c_n)\alpha p + b_n[\gamma_n w_n + (1-\gamma_n)x_n - \alpha p] \\
&\quad +c_n(y_n - \alpha p\|^2 \\
&= \|a_n(u - \alpha p) + b_n[\gamma_n(w_n - \alpha p) + (1-\gamma_n)(x_n - \alpha p)] \\
&\quad +c_n(y_n - \alpha p)\|^2,
\end{aligned}
$$

which by Lemma 2.2 gives

$$
\begin{aligned}
\|x_{n+1}-\alpha p\|^2 &= a_n\|u-\alpha p\|^2+b_n\|\gamma_n(w_n-\alpha p)+(1-\gamma_n)(x_n-\alpha p)\|^2 \\
&\quad +c_n\|y_n-\alpha p\|^2-c_nb_n\|(1-\gamma_n)w_n+\gamma_n x_n-y_n\|^2 \\
&= a_n\|u-\alpha p\|^2+b_n\|\gamma_n(w_n-\alpha p)+(1-\gamma_n)(x_n-\alpha p)\|^2 \\
&\quad +c_n\|y_n-\alpha p\|^2-c_nb_n\|(1-\gamma_n)(w_n-y_n)+\gamma_n(x_n-y_n)\|^2 \\
&= a_n\|u-\alpha p\|^2+b_n[(1-\gamma_n)\|w_n-\alpha p\|^2+\gamma_n\|x_n-\alpha p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|w_n-x_n\|^2]+c_n\|y_n-\alpha p\|^2 \\
&\quad -c_nb_n[(1-\gamma_n)\|w_n-y_n\|^2+\gamma_n\|x_n-y_n\|^2-\gamma_n(1-\gamma_n)\|w_n-x_n\|^2] \\
&= a_n\|u-\alpha p\|^2+b_n(1-\gamma_n)\|w_n-\alpha p\|^2+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad -b_n\gamma_n(1-\gamma_n)\|w_n-x_n\|^2+c_n\|y_n-\alpha p\|^2-c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&\quad -c_nb_n\gamma_n\|x_n-y_n\|^2+c_nb_n\gamma_n(1-\gamma_n)\|w_n-x_n\|^2 \\
&\leq a_n\|u-\alpha p\|^2+b_n(1-\gamma_n)\|w_n-\alpha p\|^2+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad -c_nb_n\gamma_n\|w_n-y_n\|^2+c_n\|y_n-\alpha p\|^2-c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&\quad +[c_nb_n\gamma_n(1-\gamma_n)-b_n\gamma_n(1-\gamma_n)]\|w_n-x_n\|^2. \qquad (3.12)
\end{aligned}
$$

(3.11) and (3.12) imply that

$$
\begin{aligned}
\|x_{n+1}-\alpha p\|^2 &\leq a_n\|u-\alpha p\|^2+b_n(1-\gamma_n)[\|x_n-\alpha p\|^2+(1-\delta_n)\|y_n-w_n\|^2 \\
&\quad -\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2]+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad +b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2+c_n\|y_n-\alpha p\|^2 \\
&\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&= a_n\|u-\alpha p\|^2+b_n(1-\gamma_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-\delta_n)\|y_n-w_n\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2]+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad +b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2+c_n\|y_n-\alpha p\|^2 \\
&\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq a_n\|u-\alpha p\|^2 + b_n(1-\gamma_n)\|x_n-\alpha p\|^2 + b_n(1-\gamma_n)(1-\delta_n)\|y_n-w_n\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2] + b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad +b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 + c_n\|x_n-\alpha p\|^2 \\
&\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \quad \text{(by (3.5))} \\
&= a_n\|u-\alpha p\|^2 + [b_n(1-\gamma_n)+b_n\gamma_n+c_n]\|x_n-\alpha p\|^2 \\
&\quad +b_n(1-\gamma_n)(1-\delta_n)\|-(w_n-y_n)\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2 \\
&\quad +b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 - c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&= a_n\|u-\alpha p\|^2 + [b_n(1-\gamma_n)+b_n\gamma_n+c_n]\|x_n-\alpha p\|^2 \\
&\quad +b_n(1-\gamma_n)(1-\delta_n)\|w_n-y_n\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2 \\
&\quad +b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 - c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&\leq a_n\|u-\alpha p\|^2 + [b_n(1-\gamma_n)+b_n\gamma_n+c_n]\|x_n-\alpha p\|^2 \\
&\quad +b_n(1-\gamma_n)(1-\delta_n-c_n)\|w_n-y_n\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2 \\
&= a_n\|u-\alpha p\|^2 + (b_n+c_n)\|x_n-\alpha p\|^2 \\
&\quad +b_n(1-\gamma_n)(1-\delta_n-c_n)\|w_n-y_n\|^2 \\
&\quad -b_n(1-\gamma_n)\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2. \qquad (3.13)
\end{aligned}
$$

Using conditions (i) and (ii), we obtain

$$
\begin{aligned}
1-4\delta_n^2L^2-2\delta_n &\geq 1-4c^2L^2-2c>0; \\
b_n\gamma_n(1-c_n-\delta_n) &= b_n\gamma_n(a_n+b_n-\delta_n)\leq 0; \qquad (3.14) \\
a_n+b_n+c_n &= 1.
\end{aligned}
$$

Putting (3.14) into (3.13), we obtain

$$\begin{aligned}\|x_{n+1}-\alpha p\|^2 &\leq a_n\|u-\alpha p\|^2+(1-a_n)\|x_n-\alpha p\|^2 \\ &\leq \max\{\|u-\alpha p\|^2, \|x_n-\alpha p\|^2\}, \forall n \in (0,N).\end{aligned}$$

Using mathematical induction, we see that

$$\begin{aligned}\|x_{n+1}-\alpha p\|^2 &\leq \max\{\|u-\alpha p\|^2, \|x_n-\alpha p\|^2\} \\ &= \|x_0-\alpha p\|^2, \forall n \in \{0,N\}.\end{aligned} \tag{3.15}$$

Hence, $\{x_n\}$ is bounded. This completes the proof. □

Theorem 3.2. *Let H_1 and H_2 be two Hilbert spaces and C,Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C\times C \longrightarrow R$ and $F_2 : Q\times Q \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B : Q \longrightarrow H_2$ be a bounded linear operator. Let $S : C \longrightarrow CB(C)$ be L-Lipschitz α-hemicontractive multivalued mapping. Assume that $\Theta = F(S)\cap\Omega\cap VI(CA)$ is nonempty and $S\alpha p = \alpha p$ for all $p \in \Theta$ and for some $\alpha \geq 1$.Let $x_o,u \in C$ be arbitrary and let $x_n$be a sequence in C generated by*

$$\left.\begin{aligned} &z_n = T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n, \\ &y_n = J_t z_n, \\ &u_n = (1-\delta_n)y_n+\delta_n v_n, \\ &x_{n+1} = a_n u + b_n[(1-\gamma_n)w_n+\gamma_n x_n]+c_n y_n, \end{aligned}\right\} \tag{3.16}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s,r,t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a,b \in (0,1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$,

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$.

Then, the sequence $\{x_n\}$ converges strongly to $\alpha p \in \Theta$, where $\alpha p = P_\Theta(u)$

Proof. We note that P_Θ is well defined because Θ is nonempty, closed and convex subset of C, and from (3.1), it follows that the sequence $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$ and $\{z_n\}$. Now, let $\alpha p \in \Theta$. Then, since $T_s^{F_1}$ is nonexpansive, we have

$$\begin{aligned}
\|z_n-\alpha p\|^2 &= \|T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n-\alpha p\|^2\\
&= \|T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n-T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)\alpha p\|^2\\
&\leq \|(1-\lambda B^\star(1-T_r^{F_2})B)x_n-(1-\lambda B^\star(1-T_r^{F_2})B)\alpha p\|^2\\
&= \|x_n-\alpha p-\lambda(B^\star(1-T_r^{F_2})Bx_n-B^\star(1-T_r^{F_2})B\alpha p)\|^2,
\end{aligned}$$

which by Lemma 2.3(i) gives

$$\begin{aligned}
\|z_n-\alpha p\|^2 &\leq \|x_n-\alpha p\|^2+\lambda^2\|B^\star(1-T_r^{F_2})Bx_n-B^\star(1-T_r^{F_2})B\alpha p\|^2\\
&\quad -2\lambda\langle x_n-\alpha p, B^\star(1-T_r^{F_2})Bx_n-B^\star(1-T_r^{F_2})B\alpha p\rangle.
\end{aligned}$$

Since $B^\star(1-T_r^{F_2})B$ is $\frac{1}{2d}$-inversely strongly monotone, we have

$$\begin{aligned}
\|z_n-\alpha p\|^2 &\leq \|x_n-\alpha p\|^2+\lambda^2\|B^\star(1-T_r^{F_2})Bx_n-B^\star(1-T_r^{F_2})B\alpha p\|^2\\
&\quad -\frac{\lambda}{d}\|B^\star(1-T_r^{F_2})Bx_n-B^\star(1-T_r^{F_2})B\alpha p\|^2\\
&= \|x_n-\alpha p\|^2+\lambda^2\|B^\star(1-T_r^{F_2})Bx_n-B^\star(B\alpha p-T_r^{F_2}B\alpha p)\|^2\\
&\quad -\frac{\lambda}{d}\|B^\star(1-T_r^{F_2})Bx_n-B^\star(B\alpha p-T_r^{F_2}B\alpha p)\|^2. \qquad (3.17)
\end{aligned}$$

Since $B\alpha p=T_r^{F_2}B\alpha p$, (3.17) becomes

$$\begin{aligned}
\|z_n-\alpha p\|^2 &\leq \|x_n-\alpha p\|^2+\lambda^2\|B^\star(1-T_r^{F_2})Bx_n\|^2\\
&\quad -\frac{\lambda}{d}\|B^\star(1-T_r^{F_2})Bx_n\|^2\\
&= \|x_n-\alpha p\|^2+\lambda(\lambda-\frac{1}{d})\|B^\star(1-T_r^{F_2})Bx_n\|^2. \qquad (3.18)
\end{aligned}$$

Also, from (3.16), we get

$$\begin{aligned}
\|x_{n+1}-\alpha p\|^2 &= \|a_nu+b_n[1-\gamma_n)w_n+\gamma_nx_n]+c_ny_n-\alpha p\|^2\\
&= \|a_n(u-\alpha p)+b_n[(1-\gamma_n)w_n+\gamma_nx_n)-\alpha p]+c_n(y_n-\alpha p)\|^2\\
&= \|b_n[(1-\gamma_n)w_n+\gamma_nx_n)-\alpha p]+c_n(y_n-\alpha p)+a_n(u-\alpha p)\|^2,
\end{aligned}$$

which by Lemma 2.3(ii) gives

$$\begin{aligned}\|x_{n+1}-\alpha p\|^2 &\leq \|b_n[(1-\gamma_n)w_n+\gamma_n x_n)-\alpha p]+c_n(y_n-\alpha p)\|^2\\ &\quad +2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle,\end{aligned}$$

and by Lemma 2.2, we obtain

$$\begin{aligned}\|x_{n+1}-\alpha p\|^2 &\leq b_n\|(1-\gamma_n)w_n+\gamma_n x_n-\alpha p\|^2+c_n\|y_n-\alpha p\|^2\\ &\quad -c_nb_n\|(1-\gamma_n)w_n+\gamma_n x_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle\\ &= b_n\|(1-\gamma_n)w_n+\gamma_n x_n-\gamma_n(\alpha p)+\gamma_n(\alpha p)-\alpha p\|^2+c_n\|y_n-\alpha p\|^2\\ &\quad -c_nb_n\|(1-\gamma_n)w_n+\gamma_n x_n-\gamma_n y_n+\gamma_n y_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle\\ &= b_n\|(1-\gamma_n)(w_n-\alpha p)+\gamma_n(x_n-\alpha p)\|^2+c_n\|y_n-\alpha p\|^2\\ &\quad -c_nb_n\|(1-\gamma_n)(w_n-y_n)+\gamma_n(x_n-y_n)\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle\\ &= b_n[(1-\gamma_n)\|w_n-\alpha p\|^2+\gamma_n\|x_n-\alpha p\|^2-\gamma_n(1-\gamma_n)\|w_n-x_n\|^2]\\ &\quad +c_n\|y_n-\alpha p\|^2-c_nb_n[(1-\gamma_n)\|w_n-x_n\|^2+\gamma_n\|x_n-y_n\|^2\\ &\quad -\gamma_n(1-\gamma_n)\|w_n-x_n\|^2]+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle\\ &= b_n(1-\gamma_n)\|w_n-\alpha p\|^2+b_n\gamma_n\|x_n-\alpha p\|^2\\ &\quad +c_n\|y_n-\alpha p\|^2+b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2\\ &\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2-c_nb_n\gamma_n\|x_n-y_n\|^2\\ &\quad +2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \qquad (3.19)\end{aligned}$$

Using (3.4), (3.16) and (3.11) into (3.19), we obtain

$$
\begin{aligned}
\|x_{n+1}-\alpha p\|^2 &\leq b_n(1-\gamma_n)[\|x_n-\alpha p\|^2+(1-\delta_n)\|y_n-w_n\|^2-\delta_n(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2] \\
&\quad +b_n\gamma_n\|x_n-\alpha p\|^2+c_n\|z_n-\alpha p\|^2+b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 \\
&\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2-c_nb_n\gamma_n\|x_n-y_n\|^2+2a_n\langle u-\alpha p,x_{n+1}-\alpha p\rangle \\
&= b_n(1-\gamma_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-\delta_n)\|y_n-w_n\|^2 \\
&\quad -\delta_nb_n(1-\gamma_n)(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&\quad +c_n\|z_n-\alpha p\|^2+b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2-c_nb_n(1-\gamma_n)\|w_n-y_n\|^2 \\
&\quad -c_nb_n\gamma_n\|x_n-y_n\|^2+2a_n\langle u-\alpha p,x_{n+1}-\alpha p\rangle. \qquad (3.20)
\end{aligned}
$$

(3.18) and (3.20) imply that

$$
\begin{aligned}
&\|x_{n+1}-\alpha p\|^2 \leq b_n(1-\gamma_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-\delta_n)\|y_n-w_n\|^2 \\
&-\delta_nb_n(1-\gamma_n)(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2+b_n\gamma_n\|x_n-\alpha p\|^2 \\
&+c_n[\|x_n-\alpha p\|^2+\lambda(\lambda-\frac{1}{d})\|B^\star(I-T_r^{F_2})Bx_n\|^2]+b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 \\
&-c_nb_n(1-\gamma_n)\|w_n-y_n\|^2-c_nb_n\gamma_n\|x_n-y_n\|^2+2a_n\langle u-\alpha p,x_{n+1}-\alpha p\rangle \\
= \; &(b_n+c_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-c_n-\delta_n)\|w_n-y_n\|^2 \\
&-\delta_nb_n(1-\gamma_n)(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2 \\
&+c_n\lambda(\lambda-\frac{1}{d})\|B^\star(I-T_r^{F_2})Bx_n\|^2]+b_n\gamma_n(1-\gamma_n)(c_n-1)\|w_n-x_n\|^2 \\
&-c_nb_n\gamma_n\|x_n-y_n\|^2+2a_n\langle u-\alpha p,x_{n+1}-\alpha p\rangle \\
\leq \; &(b_n+c_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-c_n-\delta_n)\|w_n-y_n\|^2 \\
&-\delta_nb_n(1-\gamma_n)(1-4\delta_n^2L^2-2\delta_n)\|y_n-v_n\|^2 \\
&+c_n\lambda(\lambda-\frac{1}{d})\|B^\star(I-T_r^{F_2})Bx_n\|^2] \\
&-c_nb_n\gamma_n\|x_n-y_n\|^2+2a_n\langle u-\alpha p,x_{n+1}-\alpha p\rangle
\end{aligned}
$$

Using (3.14), condition (i) and (ii), and the fact that $\lambda \in (0, \frac{1}{d})$, we get

$$\begin{aligned} \|x_{n+1} - \alpha p\|^2 &\leq (1-a_n)\|x_n - \alpha p\|^2 + c_n \lambda (\lambda - \frac{1}{d})\|B^\star (I - T_r^{F_2})Bx_n\|^2] \\ &\quad +2a_n \langle u - \alpha p, x_{n+1} - \alpha p \rangle. \end{aligned} \tag{3.21}$$

Then, we complete the proof by the next two cases:

Case 1: Suppose that there exists a positive integer n_0 such that $\{\|x_n - \alpha p\|\}$ is decreasing for all $n \geq n_0$. Then, the sequence $\{\|x_n - \alpha p\|\}$ is convergent, and from (3.21), we have

$$\begin{aligned} c_n \lambda (\frac{1}{d} - \lambda)\|B^\star (I - T_r^{F_2})Bx_n\|^2 &\leq (1-a_n)\|x_n - \alpha p\|^2 - \|x_{n+1} - \alpha p\|^2 \\ &\quad +2a_n \langle u - \alpha p, x_{n+1} - \alpha p \rangle. \end{aligned}$$

Hence, assumption of $\{c_n\}$, convergence of $\{\|x_n - \alpha p\|\}$ and the fact that $a_n \to 0$ as $n \to \infty$ imply that

$$\lim_{n\to\infty} \|B^\star (I - T_r^{F_2})Bx_n\| = 0, \tag{3.22}$$

and hence

$$\lim_{n\to\infty} \|x_n - (x_n - \lambda B^\star (I - T_r^{F_2})Bx_n\| = 0. \tag{3.23}$$

And since $T_s^{F_1}$ is firmly nonexpansive and $(I - \lambda B^\star (I - T_r^{F_2})B)$ is nonexpansive, using (3.16) and Lemma 2.3(i), we obtain

$$\begin{aligned} \|z_n - \alpha p\|^2 &= \|T_s^{F_1}(I - \lambda B^\star (I - T_r^{F_2})B)x_n - T_s^{F_1}\alpha p\|^2 \\ &\leq \langle z_n - \alpha p, (I - \lambda B^\star (I - T_r^{F_2})Bx_n) - \alpha p \rangle \\ &= \|z_n - \alpha p\|\|(I - \lambda B^\star (I - T_r^{F_2})B)x_n - \alpha p\| \\ &\leq \frac{1}{2}\{\|z_n - \alpha p\|^2 + \|(I - \lambda B^\star (I - T_r^{F_2})B)x_n - \alpha p\|^2 \\ &\quad -\|(z_n - (I - \lambda B^\star (I - T_r^{F_2})B)x_n\|^2\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\{\|z_n-\alpha p\|^2+\|x_n-\alpha p\|^2+\lambda^2\|B^\star(I-T_r^{F_2})Bx_n\|^2 \\ &\quad -2\lambda\langle x_n-\alpha p, B^\star(I-T_r^{F_2})Bx_n\rangle-\|z_n-x_n\|^2-\lambda^2\|B^\star(I-T_r^{F_2})Bx_n\|^2 \\ &\quad +2\lambda\langle z_n-x_n, B^\star(I-T_r^{F_2})Bx_n\rangle\} \\ &\leq \frac{1}{2}\{\|z_n-\alpha p\|^2+\|x_n-\alpha p\|^2-\|z_n-x_n\|^2+2\lambda\langle z_n-x_n, B^\star(I-T_r^{F_2})Bx_n\rangle\}. \end{aligned}$$

Thus,

$$\|z_n-\alpha p\|^2 \leq \|x_n-\alpha p\|^2-\|z_n-x_n\|^2+2\lambda\langle z_n-x_n, B^\star(I-T_r^{F_2})Bx_n\rangle. \tag{3.24}$$

Now, from (3.4) and (3.20), we get

$$\begin{aligned} \|x_{n+1}-\alpha p\|^2 &\leq b_n(1-\gamma_n)\|w_n-\alpha p\|^2+\gamma_n b_n\|x_n-\alpha p\|^2+c_n\|z_n-\alpha p\|^2 \\ &\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \end{aligned} \tag{3.25}$$

Substituting (3.11) and (3.24) into (3.25), we obtain

$$\begin{aligned} \|x_{n+1}-\alpha p\|^2 &\leq b_n(1-\gamma_n)[\|x_n-\alpha p\|^2+(1-\delta_n)\|y_n-w_n\|^2 \\ &\quad -\delta_n(1-4\delta_nL^2-2\delta_n)\|y_n-v_n\|^2]+\gamma_nb_n\|x_n-\alpha p\|^2 \\ &\quad +c_n[\|x_n-\alpha p\|^2-\|z_n-x_n\|^2+2\lambda\langle z_n-x_n, B^\star(I-T_r^{F_2})Bx_n\rangle] \\ &\quad -c_nb_n(1-\gamma_n)\|w_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle \\ &= (b_n+c_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-c_n-\delta_n)\|w_n-y_n\|^2 \\ &\quad -b_n\delta_n(1-\gamma_n)(1-4\delta_nL^2-2\delta_n)\|y_n-v_n\|^2-c_n\|z_n-x_n\|^2 \\ &\quad +2c_n\lambda\langle z_n-x_n, B^\star(I-T_r^{F_2})Bx_n\rangle+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \end{aligned} \tag{3.26}$$

It follows from condition (i),(3.14) and (3.26) that

$$\begin{aligned} c_n\|z_n-x_n\|^2 &\leq (1-a_n)\|x_n-\alpha p\|^2-\|x_{n+1}-\alpha p\|^2+2c_n\lambda\|z_n-x_n\|\|B^\star(I-T_r^{F_2})Bx_n\| \\ &\quad +2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \end{aligned}$$

Hence, since $\{x_n\}$ and $\{z_n\}$ are bounded and $a_n \to 0$ as $n \to \infty$, from (3.22) and the assumption of c_n, we obtain that

$$\lim_{n\to\infty}\|z_n-x_n\| = 0 \tag{3.27}$$

On the other hand, since J_t is firmly nonexpansive, it follows from lemma 2.3(i) and (3.16) that

$$\begin{aligned} \|y_n-\alpha p\|^2 &= \|J_t z_n - J_t \alpha p\|^2 \\ &\leq \langle y_n-\alpha p, z_n-\alpha p\rangle \\ &= \frac{1}{2}\{\|y_n-\alpha p\|^2+\|z_n-\alpha p\|^2-\|z_n-y_n\|^2\}. \end{aligned} \tag{3.28}$$

(3.3) and (3.28) imply that

$$\begin{aligned} \|y_n-\alpha p\|^2 &\leq \|z_n-\alpha p\|^2-\|z_n-y_n\|^2 \\ &\leq \|x_n-\alpha p\|^2-\|z_n-y_n\|^2. \end{aligned} \tag{3.29}$$

Substituting (3.11) and (3.28) into (3.25), we get

$$\begin{aligned} \|x_{n+1}-\alpha p\|^2 &\leq b_n(1-\gamma_n)[\|x_n-\alpha p\|^2+(1-\delta_n)\|y_n-w_n\|^2 \\ &\quad -\delta_n(1-4\delta_n L^2-2\delta_n)\|y_n-v_n\|^2]+\gamma_n b_n\|x_n-\alpha p\|^2 \\ &\quad +c_n[\|x_n-\alpha p\|^2-\|z_n-y_n\|^2]-c_n b_n(1-\gamma_n)\|w_n-y_n\|^2 \\ &\quad +2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle \\ &= (b_n+c_n)\|x_n-\alpha p\|^2+b_n(1-\gamma_n)(1-c_n-\delta_n)\|w_n-y_n\|^2 \\ &\quad -b_n\delta_n(1-\gamma_n)(1-4\delta_n L^2-2\delta_n)\|y_n-v_n\|^2-c_n\|z_n-y_n\|^2 \\ &\quad +2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle, \end{aligned}$$

which from condition (i) and (3.14) yield

$$\begin{aligned} \|x_{n+1}-\alpha p\|^2 &\leq (1-a_n)\|x_n-\alpha p\|^2-b_n\delta_n(1-\gamma_n)(1-4\delta_n L^2-2\delta_n)\|y_n-v_n\|^2 \\ &\quad -c_n\|z_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle \end{aligned} \tag{3.30}$$

(3.30) implies that

$$\begin{aligned} b_n\delta_n(1-\gamma_n)(1-4\delta_n L^2-2\delta_n)\|y_n-v_n\|^2 &\leq (1-a_n)\|x_n-\alpha p\|^2-\|x_{n+1}-\alpha p\|^2 \\ &\quad -c_n\|z_n-y_n\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \end{aligned}$$

Hence, the assumption that $a_n \to 0$ as $n \to \infty$ and (3.14) imply that

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.31}$$

Since $v_n \in Sy_n$, using (3.31), we get

$$\lim_{n\to\infty} d(y_n, Sy_n) = 0. \tag{3.32}$$

In addition, from (3.30), we also have

$$c_n\|z_n - y_n\|^2 \leq (1-a_n)\|x_n - \alpha p\|^2 - \|x_{n+1} - \alpha p\|^2 + 2a_n\langle u - \alpha p, x_{n+1} - \alpha p\rangle,$$

and, because $a_n \to 0$ as $n \to \infty$, it follows from the assumption of $\{c_n\}$ that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.33}$$

From the Lipschitz condition of S and (3.30), we get

$$\begin{aligned}
\|y_n - w_n\| &\leq \|y_n - v_n\| + \|v_n - w_n\| \\
&\leq \|y_n - v_n\| + 2D(Sy_n, Su_n) \\
&\leq \|y_n - v_n\| + 2L\|y_n - u_n\| \\
&= (1 + 2L\delta_n)\|y_n - v_n\| \to 0 \text{ as } n \to \infty. && (3.34)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\
&= \|a_n u + b_n[(1-\gamma_n)w_n + \gamma_n x_n] + c_n y_n - y_n\| + \|y_n - x_n\| \\
&= \|a_n(u - y_n) - b_n(1-\gamma_n)(y_n - w_n) - \gamma_n b_n(y_n - x_n)\| + \|y_n - x_n\| \\
&\leq a_n\|u - y_n\| + b_n(1-\gamma_n)\|y_n - w_n\| + \gamma_n b_n\|y_n - x_n\| + \|y_n - x_n\| \\
&= a_n\|u - y_n\| + b_n(1-\gamma_n)\|y_n - w_n\| + (1 + \gamma_n b_n)\|y_n - x_n\|,
\end{aligned}$$

which, from (3.27), (3.33), (3.34) and the fact that $a_n \to 0$ as $n \to \infty$, gives

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq\ & a_n\|u - y_n\| + b_n(1-\gamma_n)\|y_n - w_n\| + (1 + \gamma_n b_n)\|z_n - y_n\| \\
& + (1 + \gamma_n b_n)\|z_n - x_n\| \to 0 \text{ as } n \to \infty. && (3.35)
\end{aligned}$$

Moreover, from (3.14) and (3.30), we get

$$\|x_{n+1}-\alpha p\|^2 \leq (1-a_n)\|x_n-\alpha p\|^2+2a_n\langle u-\alpha p, x_{n+1}-\alpha p\rangle. \tag{3.36}$$

Now, let $\alpha p = P_\Theta(u)$. Then, we show that $\limsup_{n\to\infty}\langle u-\alpha p, x_{n+1}-\alpha p\rangle \leq 0$.
Since the sequence $\{x_{n+1}\}$ is bounded in a real Hilbert space H_1, we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightharpoonup \omega$ as $n\to\infty$ and

$$\limsup_{n\to\infty}\langle u-\alpha p, x_{n+1}-\alpha p\rangle = \lim_{n\to\infty}\langle u-\alpha p, x_{n+1}-\alpha p\rangle.$$

Since C is closed and convex, C is weakly closed. So, we have ω in C and from (3.35), we find that $x_{n_i} \rightharpoonup \omega$ as $i\to\infty$, and thus it follows from (3.27) and (3.33) that $z_{n_i} \rightharpoonup \omega$ and $y_{n_i} \rightharpoonup \omega$ as $i\to\infty$

Next, we claim that $\omega\in\Theta$. From (3.32) and the hypothesis that $(I-S)$ is demiclosed at zero, we obtain that $\omega\in F$.

Since $(I-\lambda B^\star(I-T_r^{F_2})B)$ is nonexpansive, from (3.23) and demiclosedness principle for nonexpansive mapping, we have

$$(I-\lambda B^\star(I-T_r^{F_2})B)\omega=\omega,$$

which implies that

$$B^\star(I-T_r^{F_2})B\omega=0.$$

Thus, using (2.1), we obtain that $B\omega=T_r^{F_2}\omega$, hence $B\omega\in EP(F_2)$. In addition, from (3.27), we get

$$\lim_{n\to\infty}\|x_n-T_s^{F_1}(I-\lambda B^\star(I-T_r^{F_2})B)x_n\| = \lim_{n\to\infty}\|x_n-z_n\|=0.$$

Hence, since $T_s^{F_1}(I-\lambda B^\star(I-T_r^{F_2})B)$ is nonexpansive, from the demiclosedness of nonexpansive mapping, we obtain that

$$T_s^{F_1}(I-\lambda B^\star(I-T_r^{F_2})B)\omega=\omega.$$

This, with the fact that $B\omega = T_r^{F_2} B\omega$, gives $\omega = T_s^{F_1}\omega$ and hence $\omega \in EP(F_1)$. Therefore, $\omega \in \Omega$.

On the other hand, from (3.33) we have

$$\lim_{n\to\infty} \|z_{n_i} - J_t z_n\| = \lim_{n\to} \|z_n - y_n\| = 0$$

Since $z_{n_i} \rightharpoonup \omega$ and J_t is nonexpansive, then $(I - J_t)$ is demiclosed at zero and so, we get that $\omega = J_t\omega$ and hence $\omega \in VI(C,A)$. Therefore, $\omega \in \Theta$. Thus, since $\alpha q = P_\Theta(u)$ and $x_{n_i+1} \rightharpoonup \omega$, from the property of metric projection P_C given in (2.2), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle u - \alpha q, x_{n+1} - \alpha q\rangle &= \lim_{n\to\infty} \langle u - \alpha q, x_{n+1} - \alpha q\rangle \\ &= \langle u - \alpha q, x_{n+1} - \alpha q\rangle \leq 0 \end{aligned} \tag{3.37}$$

Furthermore, since αp was arbitrary, $\alpha q \in \Theta$, then from (3.36), (3.37) and lemma 2.5, we get that

$$\|x_n - \alpha q\| = 0 \text{ as } n \to \infty$$

Consequently $x_n \to \alpha q = P_\Theta(u)$.

Case 2. Suppose there exists a subsequence n_j of n such that

$$\|x_{n_j} - \alpha p\| \leq \|x_{n_j+1} - \alpha p\|,$$

for all $j \in N$. Then, by lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and

$$\|x_{m_k} - \alpha p\| \leq \|x_{m_k+1} - \alpha p\| \text{ and } \|x_k - \alpha p\| \leq \|x_{m_k+1} - \alpha p\|, \tag{3.38}$$

for all $k \in N$. Thus, from condition (i), (3.14),(3.26), (3.30),(3.38) and the hypothesis that $a_n \to 0$ as $n \to \infty$, we get

$$\|z_{m_k} - x_{m_k}\| \to o,\ \|y_{m_k} - v_{m_k}\| \to o \text{ and } \|y_{m_k} - z_{m_k}\| \to o \text{ as } k \to \infty.$$

Then, since $\alpha q = P_\Theta(u)$, following the same procedure as in Case 1, we get

$$\limsup_{k\to\infty} \langle u - \alpha q, x_{m_k+1} - \alpha q\rangle \leq 0 \tag{3.39}$$

Now, since $\alpha q \in \Theta$, from (3.30) and (3.33), we have that

$$\|x_{m_k+1}-\alpha q\|^2 \leq (1-a_{m_k})\|x_{m_k}-\alpha q\|^2 + 2a_{m_k}\langle u-\alpha q, x_{m_k+1}-\alpha q\rangle \tag{3.40}$$

and hence (3.38) and (3.40) imply that

$$\begin{aligned} a_{m_k}\|x_{m_k}-\alpha q\|^2 &\leq \|x_{m_k}-\alpha q\|^2 - \|x_{m_k+1}-\alpha q\|^2 + 2a_{m_k}\langle u-\alpha q, x_{m_k+1}-\alpha q\rangle \\ &\leq 2a_{m_k}\langle u-\alpha q, x_{m_k+1}-\alpha q\rangle. \end{aligned}$$

Hence, in view of the fact that $a_{m_k} > 0$, we have that

$$\|x_{m_k}-\alpha q\|^2 \leq 2\langle u-\alpha q, x_{m_k+1}-\alpha q\rangle.$$

Hence, using (3.39), we obtain that $\|x_{m_k}-\alpha q\| \to 0$ as $k\to\infty$. This together with (3.40) implies that$\|x_{m_k+1}-\alpha q\| \to 0$ as $k\to\infty$. Because $\|x_k-\alpha p\| \leq \|x_{m_k+1}-\alpha p\|$, for all $k\in N$,we have that $x_k \to \alpha q$. Therefore, from the above two Cases, we deduce that the sequence $\{x_n\}$ converges strongly to $\alpha q = P_\Theta(u)$. This completes the proof. □

If, in Theorem 3.2, we assume that S is a single-valued Lipschitz α-hemicontractive mapping, then we obtain the following results:

Corollary 3.3. *Let H_1 and H_2 be two Hilbert spaces and C,Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C\times C \longrightarrow R$ and $F_2 : K\times K \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B : Q \longrightarrow H_2$ be a bounded linear operator. Let $S : C \longrightarrow CB(C)$ be L-Lipschitz α-hemicontractive mapping. Assume that $\Theta = F(S)\cap\Omega\cap VI(CA)$ is nonempty and $S\alpha p = \alpha p$ for all $p\in\Theta$ and for some $\alpha \geq 1$.Let $x_o, u \in C$ be arbitrary and let $x_n$be a sequence in C generated by*

$$\left.\begin{aligned} z_n &= T_s^{F_1}(1-\lambda B^\star(1-T_r^{F_2})B)x_n \\ y_n &= J_t z_n \\ u_n &= (1-\delta_n)y_n + \delta_n v_n \\ x_{n+1} &= a_n u + b_n[(1-\gamma_n)w_n + \gamma_n x_n] + c_n y_n \end{aligned}\right\} \tag{3.41}$$

for all $n \geq 0$, *where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, *where* $B^\star$ *is the adjoint of* B, $a_n, \delta_n \subset (0,1)$ *and* $b_n, c_n \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following conditions:*

i. $a_n + b_n + c_n = 1$

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_n\}$ *converges strongly to* $\alpha p \in \Theta$, *where* $\alpha p = P_\Theta(u)$

If, in Theorem 3.2, we assume that $A \equiv 0$, then we find the following result on split equilibrium and fixed point problem for Lipschitz α-hemicontractive multivalued mapping:

Corollary 3.4. *Let* H_1 *and* H_2 *be two Hilbert spaces and* C, Q *be two nonempty, closed and convex subsets of* H_1 *and* H_2 *respectively. Let* $F_1 : C \times C \longrightarrow R$ *and* $F_2 : K \times K \longrightarrow R$ *be two bifunctions satisfying Assumption G. Let* $A : C \longrightarrow H_1$ *be a continous monotone mapping and* $B : Q \longrightarrow H_2$ *be a bounded linear operator. Let* $S : C \longrightarrow CB(C)$ *be L-Lipschitz* α*-hemicontractive mapping. Assume that* $\Theta = F(S) \cap \Omega$ *is nonempty and* $S\alpha p = \alpha p$ *for all* $p \in \Theta$ *and for some* $\alpha \geq 1$.*Let* $x_o, u \in C$ *be arbitrary and let* x_n*be a sequence in* C *generated by*

$$\left.\begin{aligned} z_n &= T_s^{F_1}(1 - \lambda B^\star(1 - T_r^{F_2})B)x_n \\ u_n &= (1 - \delta_n)y_n + \delta_n v_n \\ x_{n+1} &= a_n u + b_n[(1 - \gamma_n)w_n + \gamma_n x_n] + c_n y_n \end{aligned}\right\} \tag{3.42}$$

for all $n \geq 0$, *where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, *where* $B^\star$ *is the adjoint of* B, $a_n, \delta_n \subset (0,1)$ *and* $b_n, c_n \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following conditions:*

i. $a_n + b_n + c_n = 1$

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_n\}$ *converges strongly to* $\alpha p \in \Theta$, *where* $\alpha p = P_\Theta(u)$

If, in Theorem 3.2, we assume that $H_1 = H_2$, $C = Q, B \equiv 1$ and $F_2 \equiv 0$, then we obtain the following corollary:

Corollary 3.5. *Let H_1 a real Hilbert spaces and C be a nonempty, closed and convex subsets of H_1 . Let $F_1 : C \times C \longrightarrow R$ be a bifunctions satisfying Assumption G and let $A : C \longrightarrow H_1$ be a continous monotone mapping . Let $S : C \longrightarrow CB(C)$ be L-Lipschitz α-hemicontractive multivalued mapping. Assume that $\Theta = F(S) \cap EP(F_1) \cap VI(CA)$ is nonempty, closed and convex, $S\alpha p = \alpha p$ for all $p \in \Theta$ and for some $\alpha \geq 1$.Let $x_o, u \in C$ be arbitrary and let $x_n$be a sequence in C generated by*

$$
\left.
\begin{aligned}
&z_n = T_s^{F_1} x_n \\
&y_n = J_t z_n \\
&u_n = (1-\delta_n) y_n + \delta_n v_n \\
&x_{n+1} = a_n u + b_n[(1-\gamma_n) w_n + \gamma_n x_n] + c_n y_n
\end{aligned}
\right\} \tag{3.43}
$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0$, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a, b \in (0,1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$
ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_n\}$ converges strongly to $\alpha p \in \Theta$, where $\alpha p = P_\Theta(u)$

If, in Corollary 3.3, we assume that S is an identity mapping, then we get the following result on variational inequality and split equilibrium problems:

Corollary 3.6. *Let H_1 and H_2 be two Hilbert spaces and C, Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C \times C \longrightarrow R$ and $F_2 : K \times K \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $A : C \longrightarrow H_1$ be a continous monotone mapping and $B : Q \longrightarrow H_2$ be a bounded linear operator. . Assume that $\Theta = \Omega \cap VI(CA)$ is nonempty and $S\alpha p = \alpha p$ for all $p \in \Theta$ and for some $\alpha \geq 1$.Let $x_o, u \in C$ be arbitrary and let $x_n$be a sequence in C generated by*

$$
\left.
\begin{aligned}
z_n &= T_s^{F_1}(1-\lambda B^{\star}(1-T_r^{F_2})B)x_n \\
y_n &= J_t z_n \\
x_{n+1} &= a_n u + b_n \gamma_n x_n + c_n y_n
\end{aligned}
\right\} \tag{3.44}
$$

for all $n \geq 0$*, where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^{\star}$*, where* $B^{\star}$ *is the adjoint of B,* $a_n, \delta_n \subset (0,1)$ *and* $b_n, c_n \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following conditions:*

i. $a_n + b_n + c_n = 1$

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_{n\}}$ *converges strongly to* $\alpha p \in \Theta$*, where* $\alpha p = P_{\Theta}(u)$

If, in Corollary 3.5, we assume that $F_1 \equiv 0$, then we obtain the following corollary:

Corollary 3.7. *Let* H_1 *a real Hilbert spaces and C be a nonempty, closed and convex subsets of* H_1 *. let* $A : C \longrightarrow H_1$ *be a continous monotone mapping . Let* $S : C \longrightarrow CB(C)$ *be L-Lipschitz* α*-hemicontractive multivalued mapping. Assume that* $\Theta = F(S) \cap VI(CA)$ *is nonempty, closed and convex,* $S\alpha p = \alpha p$ *for all* $p \in \Theta$ *and for some* $\alpha \geq 1$*.Let* $x_o, u \in C$ *be arbitrary and let* x_n*be a sequence in C generated by*

$$
\left.
\begin{aligned}
y_n &= J_t x_n \\
u_n &= (1-\delta_n)y_n + \delta_n v_n \\
x_{n+1} &= a_n u + b_n[(1-\gamma_n)w_n + \gamma_n x_n] + c_n y_n
\end{aligned}
\right\} \tag{3.45}
$$

for all $n \geq 0$*, where* $v_n \in Sy_n$ *and* $w_n \in Su_n$ *are such that* $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0$*,* $a_n, \delta_n \subset (0,1)$ *and* $b_n, c_n \subset [a,b]$ *for some* $a, b \in (0,1)$ *satisfying the following conditions:*

i. $a_n + b_n + c_n = 1$

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_{n\}}$ *converges strongly to* $\alpha p \in \Theta$*, where* $\alpha p = P_{\Theta}(u)$

If, in Corollary 3.6, we assume that $A \equiv 0$, the we obtain the following corollary on split equilibrium problem:

Corollary 3.8. *Let H_1 and H_2 be two Hilbert spaces and C, Q be two nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C \times C \longrightarrow R$ and $F_2 : K \times K \longrightarrow R$ be two bifunctions satisfying Assumption G. Let $B : Q \longrightarrow H_2$ be a bounded linear operator. . Assume that Ω is nonempty and $S\alpha p = \alpha p$ for all $p \in \Theta$ and for some $\alpha \geq 1$.Let $x_o, u \in C$ be arbitrary and let $x_n$be a sequence in C generated by*

$$\left.\begin{aligned} & z_n = T_s^{F_1}(1 - \lambda B^\star(1 - T_r^{F_2})B)x_n \\ & y_n = J_t z_n \\ & x_{n+1} = a_n u + b_n \gamma_n x_n + c_n y_n \end{aligned}\right\} \tag{3.46}$$

for all $n \geq 0$, where $v_n \in Sy_n$ and $w_n \in Su_n$ are such that $\|v_n - w_n\| \leq 2D(Sy_n - Su_n), s, r, t > 0, \lambda \in (0, \frac{1}{d}), d = BB^\star$, where $B^\star$ is the adjoint of B, $a_n, \delta_n \subset (0,1)$ and $b_n, c_n \subset [a,b]$ for some $a, b \in (0,1)$ satisfying the following conditions:

i. $a_n + b_n + c_n = 1$

ii. $a_n + b_n \leq \delta_n \leq c < \frac{1}{\sqrt{1+4L^2}+1}$

Then, the sequence $\{x_n\}$ converges strongly to $\alpha p \in \Omega$, where $\alpha p = P_\Omega(u)$

We note that, since every α-demicontractive mappings are α-hemicontractive mappings, the results obtained in this paper for α-hemicontractive (single and multivalued) mapping also hold for α-demicontractive mappings provided that the indicated conditions are satisfied. Our results extend, improve and unify several recent results in the existing literature (e.g.,[1, 2, 3, 12, 17,18] etc) on approximation of common solution of fixed point problem for nonlinear mappings, classical variational inequality problem and split equilibrium problems. Theorem 3.2 extends the results of Meche and Zegeye [2] from Lipshitz hemicontractive-type mappings to the more general classs of Lipshitz α-hemicontractive mappings.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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