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# A NOTE ON THE PHASE RETRIEVAL

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**Abstract.** The main conclusion of this paper is to give a special form of frame satisfying phase recovery condition. The object of discussion is separable Hilbert space, which is constructed by the basis of inner product space and the definition of frame, and the final result is obtained.

**Keywords:** phase retrieval; inner product; Hilbert space space.

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## 1. Introduction

Recovering a signal from the magnitude of its linear samples, commonly known as phase retrieval or phaseless reconstruction, has gained considerable attention in recent years. It has important application in $X-$ray imaging, crystallography, electron microscopy, coherence theory and other applications. In many applications the signals to be reconstructed are sparse. Thus it is natural to extend compressive sensing to the phase retrieval problem.

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Given a separable Hilbert space $H$, phase retrieval deals with the problem of recovering an unknown $f \in H$ from a set of intensity measurements $(|\langle f, \varphi_n \rangle|_{n\in I})$ for some countable collection $\Phi = \{\varphi_n\}_{n\in I} \subset H$. Note that if $f = \alpha g$ with $|\alpha| = 1$, then $|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|$ for every $n \in I$ regardless of our choice of $\Phi$; we say $\Phi$ does phase retrieval if the converse of this statement is true, i.e., if the equalities $|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|$ for every $n$ imply that there is a unimodular scalar $\alpha$ so that $f = \alpha g$. We call the operator $T_\Phi : H \to \ell^2(I)$ given by $T_\Phi(f) = (\langle f, \varphi_n \rangle)_{n\in I}$ the analysis operator of $\Phi$. We denote by $A_\Phi : H \to \ell^2(I)$ the nonlinear mapping given by $A_\Phi(f) = (|\langle f, \varphi_n \rangle|)_{n\in I}$ , so that $\Phi$ does phase retrieval if and only if $A_\Phi$ is injective on $H/\sim$, where $f \sim g$ if $f = \alpha g$ with $|\alpha| = 1$. We consider the intensity measurement process defined by $A^2_\Phi(f) = (|\langle f, \varphi_n \rangle|^2)_{n\in I}$ .

Frames are redundant systems of vectors in a Hilbert spaces. They satisfy the well-known property of perfect reconstruction, in that any vector of the Hilbert space can be synthesized back from its inner products with the frame vectors. More precisely, the linear transformation from the initial Hilbert space to the space of coefficients obtained by taking the inner product of a vector with the frame vectors is injective and hence admits a left inverse.

This property has been successfully used in a broad spectrum of applications, including Internet coding, multiple antenna coding, optics, quantum information theory, signal/image processing, and much more. The purpose of this paper is to study what kind of reconstruction is possible if we only have knowledge of the absolute values of the frame coefficients.

We will generally assume that $\varphi$ forms a frame for $H$, i.e., there are positive constants $0 < A < B < \infty$ so that

$$A\|f\|^2 \le \sum_{n\in I} |\langle f, \varphi_n \rangle|^2 \le B\|f\|^2 \tag{1}$$

for every $f \in H$.

Consider a Hilbert space $H$ with scalar product $\langle f, \varphi_n \rangle$. A finite or countable set of vectors $F = \{f_i : i \in I\}$ of $H$ is called a frame if there are two positive constants $A, B > 0$ such that for every vector $x \in H$,

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i \rangle|^2 \le B\|x\|^2$$

The frame is tight when the constants can be chosen equal to one another, $A = B$. For $A = B = 1$, $F$ is called a Parseval frame. The numbers $\langle x, f_i \rangle$ are called frame coefficients.

To a frame $F$ we associate the analysis and synthesis operators defined by

$$T : H \to \ell^2(I), T(x) = \{\langle x, f_i \rangle\}_{i \in I},$$

$$T^* : \ell^2(I) \to H, T^*(c) = \sum_{i \in I} c_i f_i,$$

which are well defined due to 1, and are adjoint to one another. The range of $T$ in $\ell^2(I)$ is called the range of coefficients. The frame operator defined by $S = T^*T : H \to H$ is invertible by 1 and provides the perfect reconstruction formula:

$$x = \sum_{i \in I} \langle x, f_i \rangle S^{-1} f_i.$$

**Definition 1.1.** We say a frame $\Phi = \{\varphi_n\}_{n \in I}$ for a Hilbert space $H$ has the complement property if for every subset $S \subset I$ we have $span\{\varphi_n\}_{n \in S} = H$ or $span\{\varphi_n\}_{n \notin S} = H$.

**Theorem 1.2.** (a) Let $H$ be a separable Hilbert space and let $\Phi$ be a frame for $H$. If $\Phi$ does phase retrieval, then $\Phi$ has the complement property.

(b) Let $H$ be a separable Hilbert space over the real numbers and let $\Phi$ be a frame for $H$. If $\Phi$ has the complement, then $\Phi$ does phase retrieval.

This theorem was originally proved in [2] where it was only stated in the finite dimensional case, but the proof still holds in infinite dimensions without any modifications. The original proof of part $(a)$ presented in [2] did not give the correct conclusion in the case where $H$ is a Hilbert space over the complex numbers. This was observed by the authors of [3] where they presented a much more complicated proof for this case. It turns out that the proof presented in [2] does hold in this case with only minor modifications, which is the proof presented above.

Note that the complement property is necessary but not sufficient for injectivity. To see this, consider measurement vectors $(1,0)$,$(0,1)$ and $(1,1)$. These certainly satisfy the complement property, but $A((1,i)) = (1,1,2) = A((1,-i))$, despite the fact that $(1,i) \neq (1,-i)$ mod $\mathbb{T}$; in general, real measurement vectors fail to yield injective intensity measurements in the complex setting since they do not distinguish complex conjugates.

Indeed, we have yet to find a "good" sufficient condition for injectivity in the complex case. As an analogy for what we really want, consider the notion of full spark: An ensemble $\{\varphi_n\}_{n=1}^N \subset \mathbb{R}^N$ is said to be full spark if every sub collection of $M$ vectors spans $\mathbb{R}^N$. It is easy to see that full spark ensembles with $N \geq 2M-1$ necessarily satisfy the complement property (thereby implying injectivity in the real case), and furthermore, the notion of full spark is simple enough to admit deterministic constructions. Deterministic measurement ensembles are particularly desirable for the complex case, and so finding a good sufficient condition for injectivity is an important problem that remains open.

One way to quantify the robustness of the phase retrieval process for a given frame $\Phi$ is in terms of the lower Lipschitz bound of the map $A_\Phi$ with respect to some metric on the space $H/\sim$. A natural choice of metric is the quotient metric induced by the metric on $H$ given by

$$d(\widetilde{f},\widetilde{g}) = \inf_{|\alpha|=1} \|f-\alpha g\|.$$

We would like to find a positive constant $C$ (depending only on $\Phi$) so that for every $f,g \in H$,

$$\inf_{|\alpha|=1} \|f-\alpha g\| \leq C\|A_\Phi(f)-A_\Phi(g)\| \tag{2}$$

## 2. Main Results

Let $H$ be a separable Hilbert space and let $\Phi = \{\varphi_n : n \in \mathbb{N}_N\}$ be a frame for $H$, where the $\mathbb{N}_N$ will stand for the set $\{1,2,...,N\}$ if $N < \infty$, and for the set of natural numbers if $N = \infty$. For a frame $\Phi = \{\varphi_n : n \in \mathbb{N}_N\}$ of a separable Hilbert space $H$, we denote by $T$ the analysis operator,

$$T : H \to \ell^2(\mathbb{N}_N), Tf = \sum_{n=1}^{\infty} \langle f, \varphi_n\rangle e_n,$$

where $\{e_n : n \in \mathbb{N}_N\}$ is the canonical basis of $\ell^2(\mathbb{N}_N)$. Throughout the paper The numbers $f_j := \langle f, \varphi_j\rangle$ are called frame coefficients for $f \in H$. We let $W$ denote the range of the analysis map $T(H)$. It is a linear subspace of $\ell^2(\mathbb{N}_N)$. It is quite easy to see that the set $D_\Phi := \{f \in H : \langle f, \varphi_n\rangle \neq 0, \forall n \in \mathbb{N}_N\}$ is dense in $H$.

It is known that, in the real setting, at least $2n-1$ measurements are needed to recover a signal $x \in \mathbb{R}^n$ [2]. For the complex case, the minimum number of measurements are proved to be at least $4n-4$ when $n$ is in the form of $n = 2k+1, k \in \mathbb{Z}_+$ [5]. However, for a general dimension $n$, the same question is still open. About the minimum number of observations, more details can be found in [4,6].

**Lemma 1.3.** Let $H$ be a separable Hilbert space over the complex numbers and let $\Phi = \{\varphi_n : n \in \mathbb{N}_N\}$ be a frame for $H$. Then the set $\widetilde{\Phi} := \Phi \bigcup \{\varphi_j - \varphi_{j+1}, \varphi_j + i\varphi_{j+1} : 1 \leq j < N\}$ does phase retrieval for $D_\Phi$.

**Proof**: We choose two arbitrary $f, g \in D_\Phi$ such that $|\langle f, \varphi \rangle| = |\langle g, \varphi \rangle|$ for every $\varphi \in \Phi$. Thus, we have

$$|f_j| = |g_j| \neq 0, (1 < j \leq N), \tag{3}$$

$$|f_j - f_{j+1}| = |g_j - g_{j+1}|, 1 \leq j < N, \tag{4}$$

$$|f_j - if_{j+1}| = |g_j - ig_{j+1}|, 1 \leq j < N. \tag{5}$$

Multiplying $f$ by a complex number of modulus one, we may assume that $f_1 = g_1 (\neq 0)$ holds. Suppose that $f_k = g_k (\neq 0)$ was proven for a number $1 \leq k \leq N$. Then (3), (4) and (5) for $j = k$ gives us $f_{k+1} = g_{k+1} (\neq 0)$. This verifies that $f = \alpha g$ for some $|\alpha| = 1$.

**Theorem 1.4.** Let $H$ be a separable Hilbert space over the complex numbers and let $\Phi = \{\varphi_n : n \in I\}$ be a frame for $H$. Then the set $\widetilde{\Phi} := \Phi \cup \{\varphi_m - \varphi_n, \varphi_m + i\varphi_n : m, n \in I\}$ does phase retrieval for $H$.

**Proof:** We choose two nonzero $f, g \in H$ such that $|\langle f, \varphi \rangle| = |\langle g, \varphi \rangle|$ for every $\varphi \in \Phi$. Then we infer

$$|\langle f, \varphi_n \rangle| = |\langle g, \varphi_n \rangle|, \qquad \forall n \in I.$$

Let $I_f := \{\varphi_n \in \Phi : \langle f, \varphi_n \rangle \neq 0\}$. Thus $I_f$ is countable due to the definition of frame. Let $K$ be the closed subspace in $H$ that is generated by the set $\Phi_f := \{\varphi_n : n \in I_f\}$. It is quite obvious that $K$ is a separable Hilbert space and $f, g \in D_{\Phi_f}$ . Since

$$\Phi_f \cup \{\varphi_j - \varphi_{j+1}, \varphi_j + i\varphi_{j+1} : j \in I_f\} \subset \widetilde{\Phi},$$

we have $f = \alpha g$ for some $|\alpha| = 1$ by Lemma 1.3.

We fix an orthonormal base: $\{e_j\}_{j=1}^N$. Throughout the paper $f_j := \langle f, e_j \rangle$ will denote the $jth$ coordinate of a given vector $f \in H$. It is quite easy to see that the set $D_H := \{f \in H : f_j = \langle f, e_j \rangle \neq 0, \forall j \in \mathbb{N}_N\}$ is dense in $H$.

We note that $\Phi$ does not phase retrieval $H$. An easy counterexample is obtained if we put $f = \frac{1}{\sqrt{2}}(e_1 + e_3)$ and $f = \frac{1}{\sqrt{2}}(e_1 + ie_3)$.

**Theorem 1.5.** Let $H$ be a Hilbert space with an orthonormal base $\{e_\lambda : \lambda \in \Lambda\}$ and let $\Phi = \{e_\lambda, \frac{1}{\sqrt{2}}(e_\lambda - e_{\lambda'}), \frac{1}{\sqrt{2}}(e_\lambda + ie_{\lambda'}) : \lambda, \lambda' \in \Lambda\}$. Then $\Phi$ does phase retrieval for $H$.

**Proof:** We choose two nonzero $f, g \in H$ such that $|\langle f, \varphi\rangle| = |\langle g, \varphi\rangle|$ for every $\varphi \in \Phi$. Then we infer

$$|\langle f, e_\lambda\rangle| = |\langle g, e_\lambda\rangle|, \qquad \forall \lambda \in \Lambda$$

Let $\Lambda_f := \{e_\lambda : \langle f, e_\lambda\rangle \neq 0, \lambda \in \Gamma\}$. Thus $\Lambda_f$ is countable by Bessel's Inequality and write $\Lambda_f = \{e_{\lambda_j} : j \in \mathbb{N}\}$. Let $K$ be the closed subspace in $H$ that is generated by $\Lambda_f$. It is quite obvious that $K$ is a separable Hilbert space and $f, g \in D_K$. Since

$$\{e_{\lambda_j}, \frac{1}{\sqrt{2}}(e_{\lambda_j} - e_{\lambda_{j+1}}), \frac{1}{\sqrt{2}}(e_{\lambda_j} + ie_{\lambda_{j+1}}) : 1 \leq j \leq N\} \subset \Phi,$$

we have $f = \alpha g$ for some $|\alpha| = 1$ by Lemma 1.3.

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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