Available online at http://scik.org
Advances in Fixed Point Theory, 2 (2012), No. 3, 298-312
ISSN: 1927-6303

SEQUENCES OF NONLINEAR CONTRACTIONS AND STABILITY OF FIXED POINTS

S. N. MISHRA[1,*], RAJENDRA PANT[2] AND R. PANICKER[1]

[1]Department of Mathematics, Walter Sisulu University, Mthatha 5117, South Africa

[2]Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, Maharashtra, India

Abstract. Stability of fixed points for sequences of nonlinear contractions over a variable domain is studied in a 2-metric space. The results so obtained generalize some recent results of Mishra et al. [Chaos, Soliton & Fractals 45(2012), 1012-1016] and Barbet and Nachi [Monografias del Seminario Matemático García de Galdeano **33**(2006), 51–58].

Keywords: Nonlinear contractions, fixed points, stability, 2-metric spaces.

2000 AMS Subject Classification: 54H25, 47H10.

1. Introduction

The fixed point stability has been an interesting and continuing area of research in fixed point theory since its inception in 1962, when a result about the relationship between the convergence of a sequence of contraction mappings $\{T_n\}$ of a metric space X and their fixed points was obtained by Bonsall [6] (see also Sonnenshein [31]). Subsequent results by Nadler, Jr. [22] and others (see [1, 3-5, 14-21, 24-30]) in various settings address mainly the problem of replacing the completeness of the space X (metric or

*Corresponding author

Received July 22, 2012

otherwise) by the existence of fixed points and various relaxations on the contraction constant. In most of these results, pointwise and uniform convergence play invariably a vital role in arriving at the desired conclusion. However, if the domain of definition of T_n is different for each $n \in \mathbb{N}$ (naturals), then these notions do not work. An alternative to this problem has recently been presented by Barbet and Nachi [4, 5] where some new notions of convergence have been introduced and utilized to obtain stability results in a metric space which generalize the earlier results of Bonsall [6] and Nadler [22]. These results have been further generalized by Mishra et al. [17-21]. On the other hand, the so called nonlinear contractions (or $\varphi-$contraction mappings) studied by Boyd and Wong [7] form a natural generalization of the contraction mappings. In this paper, motivated by Barbet and Nachi [4] and Boyd and Wong [7], we obtain a number of stability results in 2-metric spaces due to Gähler [8]. The results obtained here in thus compliment the results of Barbet and Nachi [4] and Mishra el al. [17-21]. We note that the results so obtained are significant in the sense that 2-metric spaces differ topologically with metric spaces(see Remark 1.4 below).

2. Preliminaries

We first recall some basics of 2-metric spaces. For details we refer to Gähler [8] and Iséki [9-11].

Definition 2.1. Let X be a nonempty set, consisting of at least three points. A 2-metric on X is a real-valued function ρ on $X \times X \times X$ which satisfies the following conditions:

(a) To each pair of distinct points $x, y \in X$ there exists a point $a \in X$ such that $\rho(x, y, a) \neq 0$.
(b) If at least two of x, y, a are equal then $\rho(x, y, a) = 0$.
(c) $\rho(x, y, a) = \rho(y, a, x) = \rho(x, a, y)$ for all $x, y, a \in X$.
(d) $\rho(x, y, a) \leq \rho(x, y, z) + \rho(x, z, a) + \rho(z, y, a)$ for all $x, y, z, a \in X$.

It is easily seen that ρ is non-negative. The pair (X, ρ) is called a 2-metric space.

Definition 2.2. A sequence $\{x_n\}$ in a 2-metric space (X, ρ) is said to be convergent with limit $z \in X$ if $\lim_{n\to\infty} \rho(x_n, z, a) = 0$ for all $a \in X$. Notice that if the sequence $\{x_n\}$ converges to z, then $\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b)$ for all $a, b \in X$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0$ for all $a \in X$. A 2-metric space (X, ρ) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.3. A 2-metric space (X, ρ) is said to be bounded if there is a constant K such that $\rho(a, b, c) \leq K$ for all $a, b, c \in X$.

Remark 2.4. The following remarks briefly capture some distinct features of topological properties of 2-metric spaces which differ from those of metric spaces. (i) Given any metric space which consist of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not always true as one can find a 2-metric space which does not have a countable basis associated with one of its arguments (see Gäher [8, Theorem 20 and Example on page 145]). (ii) It is known that a 2-metric ρ is continuous in any one of its arguments. Generally, we cannot however assert the continuity of ρ in all the three arguments. But if it is continuous in any two arguments, then it is continuous in all the three arguments (see Gäher [8, page 123]). (iii) In a complete 2-metric space a convergent sequence need not be Cauchy (see Naidu and Prasad [23, Example 0.1]). (iv) In a 2-mertic space (X, ρ) every convergent sequence is Cauchy whenever ρ is continuous. However, the converse need not be true (see Naidu and Prasad [23, Example 0.2]).

Definition 2.5. Let (X, ρ) be a 2-metric space. A mapping $T : X \to X$ is called Lipschitz (or $k-$Lipschitz) if there exists a constant $k > 0$ such that

$$\rho(Tx, Ty, a) \leq k\rho\,(x, y, a) \tag{1.1}$$

for all $x, y, a \in X$. If $0 < k < 1$, then T is called contraction (or k-contraction).

It is well known that a contraction mapping on a 2-metric space X has a unique fixed point. Initially, an additional requirement of boundedness was placed on X by Iséki et al. [12] and which was dispensed with subsequently by Rhoades [24] and Lal and Singh

[13] independently. For some recent developments on fixed points in 2−metric spaces, we refer to Aliouche and Simpson [2].

Definition 2.6. Let (X, ρ) be a 2-metric space and $T : X \to X$ a self-mapping. The mapping T is said to be nonlinear contraction or φ-contraction on X if

$$\rho(Tx, Ty, a) \leq \varphi(\rho(x, y, a)) \tag{1.2}$$

for all $x, y, a \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and $\varphi(t) < t$ for $t > 0$. We note that $\varphi(0) = 0$. For details we refer to Boyd and Wong [7].

We note that the condition (1.1) is a special case of the condition (1.2) when $\varphi(t) = kt$ with $k \in (0, 1)$.

Now onwards, X will denote a 2-metric space (X, ρ) with ρ continuous, $\mathbb{N}$, the set of naturals and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

3. Stability under (G)-convergence

Definition 3.1. [19] Let X be a 2-metrc space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called a (G)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G) if the following condition holds:

(G): $Gr(T_\infty) \subset \liminf Gr(T_n)$: for every $z \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(x_n, z, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty z, a), \text{ for all } a \in X,$$

where $Gr(T)$ denotes the graph of T.

Remark 3.2. In view of Barbet and Nachi [4], we note that:

(i): A (G)-limit need not be unique.

(ii): The property (G) is more general than pointwise convergence. However, the two notions are equivalent provided the sequence $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous when the domains of definitions are identical.

The following proposition extends a result of Barbet and Nachi [4, Proposition 1], Mishra and Pant [18, Proposition 3.1] and Mishra et al. [19, Proposition 2.2] to φ-contractions and ensures the uniqueness of a (G) limit in a $2-$metric space.

Proposition 3.3. Let X be a 2-metric space, $\{X_n\}_{\ n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{\ T_n : X_n \to X\}_{\ n\in\mathbb{N}}$ a sequence of φ-contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.

Proof. Assume that $T_\infty : X_\infty \to X$ and $T^*_\infty : X_\infty \to X$ are (G)-limit mappings of the sequence $\{T_n\}$. Hence for any point $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\underset{n\in\mathbb{N}}{\Pi} X_n$ converging to x such that $\{T_n x_n\}$ and $\{T_n y_n\}$ converge to T_∞ and T^*_∞. respectively. Therefore

$$\lim_{n\to\infty} \rho\left(T_n x_n, T_\infty x, a\right) = 0,\ \lim_{n\to\infty} \rho\left(\ T_n y_n, T^*_\infty x, a\right) = 0 \ \text{ for all } a \in X.$$

By the triangular area inequality and condition (1.2), for all $n \in \mathbb{N}$ and for any $a \in X$, we have

$$\begin{aligned}
\rho(T_\infty x, T^*_\infty x, a) &\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, a) \\
&\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n) \\
&\quad + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T^*_\infty x, a) \\
&\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n) \\
&\quad + \varphi(\rho(x_n, y_n, a)) + \rho(T_n y_n, T^*_\infty x, a) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence we deduce that $\lim\limits_{n\to\infty} \rho(T_\infty x, T^*_\infty x, a) = 0$ and the unicity of the limit is established. □

When $\varphi(t) = kt$ and $k \in (0, 1)$ in the above proposition, we get the following result.

Corollary 3.4. *[19, Proposition 2.2] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$ then T_∞ is the unique.*

The following theorem is our first stability result.

Theorem 3.5. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, $T_n : X_n \to X$ is a φ-contraction, where φ is nondecreasing. If, for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be a fixed point of T_n for each $n \in \overline{\mathbb{N}}$. Since property (G) holds and $x_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \ \text{ and } \ \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \ \text{ for all } \ a \in X.$$

If $\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0$, then there is nothing to prove. Assume that $\lim_{n\to\infty} \rho(x_n, x_\infty, a) = r$ for some $r > 0$. By the triangular area inequality, condition (1.2) and the fact that φ is nondecreasing, we get

$$\begin{aligned}
\rho(x_n, x_\infty, a) &= \rho(T_n x_n, T_\infty x_\infty, a) \\
&\le \rho(T_n x_n, T_n y_n, a) + \rho(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho(T_n y_n, T_\infty x_\infty, a) \\
&\le \varphi(\rho(x_n, y_n, a)) + \rho(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho(T_n y_n, T_\infty x_\infty, a) \\
&\le \varphi(\rho(x_n, x_\infty, a) + \rho(x_n, y_n, x_\infty) + \rho(x_\infty, y_n, a)) + \rho(T_n x_n, T_\infty x_\infty, T_n y_n) \\
&\quad + \rho(T_n y_n, T_\infty x_\infty, a).
\end{aligned}$$

Making $n \to \infty$ in the above inequality, we get

$$r \le \varphi(r) < r,$$

a contradiction. Therefore $\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0$ and the conclusion follows. ☐

Corollary 3.6. *[19, Theorem 2.3] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, $T_n : X_n \to X$ is a k-contraction. If, for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. It comes from Theorem 3.5 when $\varphi(t) = kt$ and $k \in (0, 1)$. □

The following result gives a comparison with Rhoades [24, Theorem 2] and presents a 2-metric space version of Bonsall [6, Theorem 1.2, page 6].

Corollary 3.7. *Let X be a complete 2-metric space and $\{ T_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a family of contraction mappings with the same Lipschitz constant $k < 1$ and such that the sequence $\{T_n\}_{n\in\mathbb{N}}$ converges pointwise to T_∞. Then, for all $n \in \overline{\mathbb{N}}$, T_n has a unique fixed point x_n and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Corollary 3.6 when $X_n = X$ for all $n \in \overline{\mathbb{N}}$ and the fact that X is complete. □

The existence of a fixed point for a (G)-limit mapping is characterized by the following result when it is a contraction mapping. This result also presents an analogue of [18, Theorem 3.6] to 2-metric spaces.

Theorem 3.8. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, $T_n : X_n \to X$ is a φ-contraction, where φ is nondecreasing. Assume that, for any $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then:*

$$\begin{aligned} T_\infty \text{ admits a fixed point} \quad &\Leftrightarrow \quad \{x_n\} \text{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \quad \{x_n\} \text{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Proof. The necessary part is already proved in Theorem 3.5. To prove the sufficiency, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x_\infty \in X_\infty$. By the property (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Hence for any $a \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho\left(x_\infty, T_\infty x_\infty, a\right) &\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right) \\
&\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) + \\
&\quad \rho\left(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a\right) + \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) \\
&\quad + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right) \\
&\leq \rho\left(x_\infty, x_{n_j}, a\right) + \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) + \varphi\left(\rho\left(x_{n_j}, y_{n_j}, a\right)\right) + \\
&\quad \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}\right).
\end{aligned}$$

The right hand side of the above expression tends to zero as $j \to \infty$ and hence $T_\infty x_\infty = x_\infty$, proving that x_∞ is a fixed point of T_∞. □

Remark 3.9. Under the assumptions of Theorem 3.8, and if

(i): $\liminf X_n \subset X_\infty$ (i.e., the limit of any convergent sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then:

$$T_\infty \text{ admits a fixed point} \Leftrightarrow \{x_n\} \text{ converges.}$$

(ii): $\limsup X_n \subset X_\infty$ (i.e., the cluster point of any sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞) then:

$$T_\infty \text{ admits a fixed point} \Leftrightarrow \{x_n\} \text{ admits a convergent subsequence.}$$

The following proposition extends a result of [18, Proposition 3.8] to 2-metric spaces and provides a sufficient condition under which a (G)-limit of a sequence of φ-contraction mappings is again a φ-contraction.

Proposition 3.10. Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, T_n is a φ-contraction from X_n to X. Then T_∞ is a φ-contraction.

Proof. Given two points x and y in X_∞, by the property (G) there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_n X_n$ converging respectively to x and y and such that the sequences

$\{T_n x_n\}$ and $\{T_n y_n\}$ converge respectively to $T_\infty x$ and $T_\infty y$. For any $n \in \mathbb{N}$ and $a \in X$, we deduce from condition (1.2) that

$$\begin{aligned}\rho(T_\infty x, T_\infty y, a) &\leq \rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, a)\\ &\leq \rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, T_n y_n)\\ &\quad +\rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty y, a)\\ &\leq \rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, T_n y_n)\\ &\quad +\varphi(\rho(x_n, y_n, a)) + \rho(T_n y_n, T_\infty y, a).\end{aligned}$$

Since $\limsup \varphi(\rho(x_n, y_n, a)) \leq \varphi(\rho(x, y, a))$, we conclude that $\rho(T_\infty x, T_\infty y, a) \leq \varphi(\rho(x, y, a))$. □

Corollary 3.11. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, T_n is a k-contraction from X_n to X . Then T_∞ is a k-contraction.*

Proof. This comes from Proposition 3.10 when $\varphi(t) = kt$ and $k \in (0,1)$. □

Under a compactness assumption, the existence of a fixed point of the (G)-limit mapping can be obtained from the existence of fixed points of the φ-contraction mappings T_n. The following theorem is an extension of [18, Theorem 3.10] to 2-metric spaces.

Theorem 3.12. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, T_n is a φ-contraction, where ϕ is nondecreasing. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If, for any $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the (G)-limit mapping T_∞ admits a fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be the fixed point of T_n for each $n \in \mathbb{N}$. From the compactness condition, there exists a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Now by Remark 3.9, T_∞ admits a fixed point x_∞ and by Theorem 3.5 the sequence $\{x_n\}$ converges to x_∞. □

Corollary 3.13. *[19, Theorem 2.10] Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of a 2-metric space* X *and* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a family of mappings satisfying the property (G) and such that, for any* $n \in \mathbb{N}$, T_n *is a k-contraction. Assume that* $\limsup X_n \subset X_\infty$ *and* $\bigcup_{n\in\mathbb{N}} X_n$ *is relatively compact. If, for any* $n \in \mathbb{N}$, T_n *admits a fixed point* x_n*, then the (G)-limit mapping* T_∞ *admits a fixed point* x_∞ *and the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞.

Proof. This comes from Theorem 3.12, when $\varphi(t) = kt$ and $k \in (0, 1)$. □

The following notion of convergence is weaker than (G)-convergence and has been s-tudied in [19].

Definition 3.14. [19] Let X be a 2-metrc space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called a (G$^-$) limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G$^-$), if the following condition holds:

(G$^-$): $Gr(T_\infty) \subset \limsup Gr(T_n)$: for all $z \in X_\infty$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X_n$, and which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_{j\to\infty} \rho(x_{n_j}, z, a) = 0 \text{ and } \lim_{j\to\infty} \rho(T_{n_j} x_{n_j}, T_\infty z, a) = 0 \text{ for all } a \in X.$$

The following result which is an extension of [18, Theorem 3.12] to 2-metric spaces, establishes that a fixed point of a (G$^-$)-limit mapping is a cluster point of the sequence of fixed points associated with $\{T_n\}$.

Theorem 3.15. *Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of a 2-metric space* X *and* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a family of* φ*-contraction mappings satisfying the property (G$^-$), where* φ *is nondecreasing. If, for any* $n \in \overline{\mathbb{N}}$, x_n *is a fixed point of* T_n*, then* x_∞ *is a cluster point of the sequence* $\{x_n\}_{n\in\mathbb{N}}$.

Proof. By the property (G$^-$), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ which has a subsequence $\{y_{n_j}\}$ such that $y_{n_j} \to x_\infty$ and $T_{n_j} y_{n_j} \to T_\infty x_\infty$. Therefore

$$\lim_{j\to\infty} \rho(y_{n_j}, x_\infty, a) = 0 \text{ and } \lim_{j\to\infty} \rho\left(T_{n_j} x_{n_j}, T_\infty x_\infty, a\right) = 0 \text{ for all } a \in X.$$

Since each T_{n_j} is a φ-contraction and φ is nondecreasing, for any $a \in X$ we have

$$\begin{aligned}
\rho(x_{n_j}, x_\infty, a) &= \rho(T_{n_j}x_{n_j}, T_\infty x_\infty, a) \\
&\leq \rho(T_{n_j}x_{n_j}, T_{n_j}y_{n_j}, a) + \rho(T_{n_j}y_{n_j}, T_\infty x_\infty, a) + \\
&\quad +\rho\left(T_{n_j}x_{n_j}, T_\infty x_\infty, T_{n_j}y_{n_j}\right) \\
&\leq k\rho(T_{n_j}x_{n_j}, y_{n_j}, a) + \rho(T_{n_j}y_{n_j}, T_\infty x_\infty, a) + \\
&\quad \rho\left(x_{n_j}, T_\infty x_\infty, T_{n_j}y_{n_j}\right) \\
&\leq \varphi(\rho(x_{n_j}, y_{n_j}, x_\infty) + \rho(x_{n_j}, x_\infty, a) + \rho(x_\infty, y_{n_j}, a)) \\
&\quad +\rho(T_{n_j}y_{n_j}, T_\infty x_\infty, a) + \rho(T_{n_j}x_{n_j}, T_\infty x_\infty, T_{n_j}y_{n_j}).
\end{aligned}$$

The right hand side of the above expression tends to 0 as $j \to \infty$. Thus $\{x_{n_j}\}$ converges to x_∞, the fixed point of T_∞. □

Corollary 3.16. *[19, Theorem 2.12] Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G^-). If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. This comes from Theorem 3.15, when $\varphi(t) = kt$ and $k \in (0, 1)$. □

4. Stability under (H)-convergence

Definition 4.1. [19] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then T_∞ is called an (H)$-$limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

(H): For all sequences $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that:

$$\lim_{n\to\infty} \rho(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0 \text{ for all } a \in X.$$

Remark 4.2. We remark that:

- **(a):** A (G)-limit is not necessarily an (H)-limit.
- **(b):** If $\{T_n : X \to X\}_{n\in\mathbb{N}}$ converges uniformly to T_∞ on X, then T_∞ is an (H)-limit of $\{T_n\}$.
- **(c):** The converse of (b) holds only when T_∞ is uniformly continuous on X.

The following result reveals the relationship between (G)-convergence and (H)-convergence in a 2-metric space and is an extension of [4, Proposition 9].

Proposition 4.3. [19, Proposition 2.2] Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X such that $X_\infty \subset \liminf X_n$. Let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings such that T_∞ is continuous on X_∞. If T_∞ is an (H)-limit of $\{T_n\}_{n\in\mathbb{N}}$, then T_∞ is a (G) -limit of $\{T_n\}_{n\in\mathbb{N}}$.

When $X_n = M$, a nonempty subset of X for all $n \in \overline{\mathbb{N}}$, we obtain the following comparison with uniform convergence.

Proposition 4.4. [19, Proposition 2.3] Let $\{T_n : M \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings where M is a nonempty subset of a 2 -metric space X.

- **(a):** If $\{T_n\}_{n\in\mathbb{N}}$ converges uniformly to T_∞ on M, then T_∞ is an (H)-limit of $\{T_n\}_{n\in\mathbb{N}}$.
- **(b):** The converse holds when T_∞ is uniformly continuous on M.

The following theorem which is an extension of [18, Theorem 4.1] to 2-metric spaces, is our second stability result.

Theorem 4.5. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that T_∞ is a φ-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that $\lim_{n\to\infty}\rho(x_n, y_n, a) = 0$ and $\lim_{n\to\infty}\rho(T_n x_n, T_\infty y_n, a) = 0$ for any $a \in X$. Hence for any $a \in X$,

$$\begin{aligned}\rho(x_n, x_\infty, a) &= \rho(T_n x_n, T_\infty x_\infty, a)\\ &\leq \rho(T_n x_n, T_\infty y_n, a) + \rho(T_\infty y_n, T_\infty x_\infty, a) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n)\\ &\leq \rho(T_n x_n, T_\infty y_n, a) + \varphi(\rho(y_n, x_\infty, a)) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n) \to 0 \text{ as } n \to \infty,\end{aligned}$$

and the conclusion follows. □

Corollary 4.6. *[19, Theorem 3.4] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (H) and such that T_∞ is a k-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 4.5, when $\varphi(t) = kt$ and $k \in (0, 1)$. □

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$, in Corollary 4.6, we get a special case of Rhoades [24, Theorem 3] which in turn presents a 2−metric space version of Nadler [22, Theorem 1].

Corollary 4.7. *Let X be a 2-metric space, $\{T_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings which converges uniformly to a contraction mapping $T_\infty : X \to X$. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

References

[1] S. P. Acharya, Convergence of a sequence of fixed points in a uniform space, Mat. Vesnik. **13**(1976) (28), 131–141.

[2] Abdelkrim Aliouche and Carlos Simpson, Fixed points and lines in 2-metric spaces, Advances in Mathematics 229 (2012) 668–690.

[3] V. G. Angelov, A continuous dependence of fixed points of ϕ-contractive mappings in uniform spaces, Archivum Mathematicum (Brno) **28**(1992), (3–4), 155–162.

[4] L. Barbet and K. Nachi, Sequences of contractions and convergence of fixed points, Monografias del Seminario Matemático García de Galdeano **33**(2006), 51–58.

[5] L. Barbet and K. Nachi, Convergence des points fixes de k-contractions (Convergence of fixed points of k-contractions), (2006) Preprint, University of Pau.

[6] F. F. Bonsall, Lectures on Some Fixed Point Theorems of Functional Analysis, Tata Institute of Fundamental Research, Bombay 1962.

[7] D. W. Boyd and J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. **20**(1969), 458-464.

[8] S. Gähler, 2-metrische Räume und ihre topologische Struktur, Math. Nachr. **26**(1963/64), 115–148.

[9] K. Iséki, Mathematics in 2-normed spaces, Math. Sem. Notes Kobe Univ. **4**(1974), 161–174.

[10] K. Iséki, Fixed point theorem in 2 -metric spaces, Math. Sem. Notes Kobe Univ. **3**(1975), 133–136.

[11] K. Iséki, Mathematics in 2-metric space, Bull. Korean Math. Soc. **13**2(1976), 127–135.

[12] K. Iséki, P. L. Sharma and B. K. Sharma, Contraction type mapping on 2-metric space, Math. Japonicae **21**(1976), 67–70.

[13] S. N. Lal and A.K. Singh, An analogue of Banach's contraction principle for 2-metric spaces, Bull. Austral. Math. Soc. **18**(1978), 137–143.

[14] S. N. Mishra, On sequences of mappings and fixed points, Rend. Sem. Mat. Univers. Politecn. Torino **34**(1976), 405–410.

[15] S. N. Mishra, On sequences of mappings and fixed points II, Indian J. Pure Appl. Math. **10**(1979), 699–703.

[16] S. N. Mishra, On common fixed points of multimappings in uniform spaces, Indian J. Pure Appl. Math. **13**(5)(1991), 606–608.

[17] S. N. Mishra and A. K. Kalinde, On certain stability results of Barbet and Nachi, Fixed Point Theory **12**(1)(2011), 137–144.

[18] S. N. Mishra and Rajendra Prasad, Sequences of φ-contractions and stability of fixed Points, Indian J. Math. (to appear)

[19] S. N. Mishra, S. L. Singh and Rajendra Pant, Some new results on stability of fixed points, Chaos, Solitons & Fractals 45 (2012) 1012–1016

[20] S. N. Mishra, S. L. Singh, Rajendra Pant and S. Stofile, Some new notions of convergence and stability of common fixed points in 2-metric spaces, Advances in Fixed Point Theory,**2**(2012), No. 1, 64-78.

[21] S. N. Mishra, S.L. Singh and S. Stofile, Stability of common fixed points in uniform spaces, Fixed Point Theory and Applications, 2011:37, 1-8.

[22] S. B. Nadler, Jr., Sequences of contractions and fixed points, Pacific J. Math. **27**(1968), 579–585.

[23] S. V. R. Naidu and J. Rajendra Prasad, Fixed points theorems in 2-metric spaces, Indian J. Pure Appl. Math. **17**(8)(1986), 974–993.

[24] B. E. Rhoades, Contraction type mappings on a 2 -metric space, Math. Nachr. **91**(1979), 151–156.

[25] B. E. Rhoades, Fixed point theorems in a uniform space, Publ. L'Institute Mathématique Nouvelle śerie **25**(39)(1979), 153–156.

[26] S. L. Singh, A note on the convergence of a pair of sequences of mappings, Arch. Math. **15**(1)(1979), 47–52.

[27] S. L. Singh and B. Ram, A note on the convergence of sequences of mappings and their common fixed points in a 2-metric space, Math. Sem. Not. **9**(1981), 181–185.

[28] S. L. Singh and B. Ram, A note on the convergence of sequences of mappings and their common fixed points in a 2-metric space II, J. Univ. Kuwait Sci. **10**(1993), 31–35.

[29] S. P. Singh, On a theorem of Sonnenshein, Bull. de l'Acadé mie Royale de Belgique **3**(1969), 413–414.

[30] S. P. Singh and W. Russel, A note on a sequence of contraction mappings, Can. Math. Bull. **12**(1969), 513–516.

[31] J. Sonnenshein, Opérateurs de même coefficient de contraction, Bulletin de l'Académie Royale de Belgique **52**(1966), 1078–1082.