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EXTENSION OF PHASE-ISOMETRIES BETWEEN THE UNIT SPHERES OF COMPLEX $\ell^p(\Gamma)$-SPACES ($p > 1$)

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Abstract. Let Γ, Δ be nonempty index sets. For $p \in (1, \infty)$, we prove that every surjective mapping $f : S_{\ell^p(\Gamma)} \to S_{\ell^p(\Delta)}$ satisfying the functional equation

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in S_{\ell^p(\Gamma)}),$$

its positive homogeneous extension is a phase-isometry which is phase equivalent a real linear isometry.

Keywords: extension of phase-isometries; unit spheres; $\ell^p(\Gamma)$ spaces.

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1. INTRODUCTION

Let X and Y be normed real or complex spaces. A mapping $f : X \to Y$ is called an *isometry* if it satisfies the equation

$$\|f(x) - f(y)\| = \|x - y\| \quad (x, y \in X).$$

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Since 1987, D. Tingley proposed the following problem in [8]: Let X and Y be real normed spaces with unit spheres $S(X)$ and $S(Y)$. Suppose that $f_0 : S(X) \to S(Y)$ be a surjective isometric mapping. Does there exist a linear isometry F from X onto Y which is extension of f_0? What is nowadays the so-called *Tingley's problem*. According to this problem which remains unsolved, more and more researchers have been results about this question in positive. There are fundamental conclusions to Tingley's problem for a wide range of Banach spaces includes sequence spaces $l^p(\Gamma)$-spaces (see[9, 11, 12]), $C_0(L)$ spaces [17], finite dimensional C^*-algebras and finite von Neumann algebras (see [18, 19]). The classical Mazur-Ulam theorem [2] state that every surjective isometry between X and Y with $f(0) = 0$ is (real) linear isometry, which is intrinsically linked to Tingley's problem.

Another significant result is the Wigner's theorem, which has several equivalen formulations, and can be observed positive answers in [4, 5]. One of the important conclusions is related to (real) linear isometries: Let H and K be real inner product spaces, Rätz's result characterizes mapping $f : H \to K$ that are phase equivalent to a linear isometry(i.e., there exists a function $\varepsilon : H \to \{-1, 1\}$ such that $\varepsilon \cdot f$ is a norm preserving real linear map) by the functional equation

$$|< f(x), f(y) >| = |< x, y >| \quad (x, y \in X).$$

In the paper [1], a real version of Wigner's theorem was revisited by using the functional equation

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\} \quad (x, y \in H). \tag{1}$$

Then it flows that there exists a plus-minus function $\varepsilon : H \to \{-1, 1\}$ such that $\varepsilon \cdot f$ is a linear isometry. In the case of complex inner product spaces, there exists a phase function $\varepsilon : H \to \mathbb{T}$ such that $\varepsilon \cdot f$ is a linear or conjugate linear isometry, respectively. Here we say that f and $\varepsilon \cdot f$ are called *phase equivalent*, f is called *phase-isometry* which satisfies the equation (1). At the end of [1] Maksa and Páles posed the following question: What are the solutions $f : H \to K$ of (1) when H and K are normed but not necessarily inner product spaces? By Wigner's theorem, Huang and Tan prove surjective phase-isometries between the real normed sequence spaces such as $\ell^p(\Gamma)$ spaces[6] and $L^p(\Gamma)$-type spaces[7]. We can easily see that every mapping is phase equivalent to a linear isometry is a phase-isometry.

Let X and Y be real or complex normed spaces with unit spheres $S(X)$ and $S(Y)$. It is given the natural positive homogeneous exstension of f by

$$F(x) = \begin{cases} \|x\| f(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{if } x \doteq 0. \end{cases}$$

Motivated by Tingley's problem, Mazur-Ulam theorem and Wigner's theorem, one of the most interesting question arised.

Problem 1.1. *Let $f : S(X) \to S(Y)$ is a surjective phase-isometry. Is it true that F is phase-isometry from X onto Y, which is the natural positive homogeneous exstension of f?*

In the proof of [14], Huang and Jin observed that a surjective phase-isometry between the unit spheres of two real L^p-spaces for $p > 0$, its positive homogeneous exstension is a phase-isometry which is phase equivalent to a linear isometry.

In this paper, we answer Problem 1.1 on complex $\ell^p(\Gamma)$-type spaces with $p > 1$. That is to say, we show that every phase-isometry f between the unit complex $\ell^p(\Gamma)$-type spaces with $p > 1$ is a plus-minus real linear isometry. In order to do this, we also give the representation theorem of surjective phase-isometry between two $\ell^p(\Gamma)$-type spaces with $p > 1$.

2. Results

Throughout this paper, X will be a Banach space over complex field, $S(X)$ will denote the unit spheres of X, respectively. We consider use the symbol Γ and Δ to represent nonempty index set. We shall note $\mathbb{T} = \{\alpha : |\alpha| = 1, \alpha \in \mathbb{C}\}$. For $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

We mainly concern the standard notation $\ell^p(\Gamma)$, where $p \in (1, \infty)$ and Γ is a nonempty index set. It will denote the Banach space of all functions $x : \Gamma \to \mathbb{C}$ such that $\sum_{\gamma \in \Gamma} |x_\gamma|^p < \infty$. That is

$$\ell^p(\Gamma) = \{x = \sum_{\gamma \in \Gamma} x_\gamma e_\gamma : \|x\| = (\sum_{\gamma \in \Gamma} |x_\gamma|^p)^{\frac{1}{p}} < \infty\}$$

where e_γ is the vector in $\ell^p(\Gamma)$ having 1 at the γ-th entry and otherwise 0. The unit sphere of $\ell^p(\Gamma)$ is $\{x \in \ell^p(\Gamma) : \|x\| = 1\}$ and denoted by $S_{\ell^p(\Gamma)}$. For every $x \in \ell^p(\Gamma)$, we denote the *support* of x by Γ_x, i.e.,

$$\Gamma_x = \{\gamma \in \Gamma : x_\gamma \neq 0\}.$$

Then x can be rewritten in the form $x = \sum_{\gamma\in\Gamma_x} x_\gamma e_\gamma$. Let $x, y \in \ell^p(\Gamma)$, we say that x is *orthogonal to* y, denoted by $x \perp y$, if $\Gamma_x \cap \Gamma_y = \emptyset$. It has been known that if $p \in (1,\infty) \setminus \{2\}$, equality

$$\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p)$$

holds for $x, y \in \ell^p(\Gamma)$ if and only if $x \perp y$.

Theorem 2.1. *Let H and K be complex Hilbert spaces, and let $f : S(H) \to S(K)$ be a phase-isometry. Then the positive homogeneous extension F of f is a phase-isometry, and there exists a plus-minus function $\varepsilon : H \to \{-1, 1\}$ such that $\varepsilon \cdot F$ is a real linear isometry.*

Proof: Elementary observations show that $f : S(H) \to S(K)$ is a phase-isometry if and only if f is a norm preserving map such that

$$|\mathrm{Re}\langle f(x), f(y)\rangle| = |\mathrm{Re}\langle x, y\rangle| \quad (x, y \in S(H)).$$

Hence

$$\begin{aligned} |\mathrm{Re}\langle F(x), F(y)\rangle| &= |\mathrm{Re}\langle \|x\| f(\frac{x}{\|x\|}), \|y\| f(\frac{y}{\|y\|})\rangle| \\ &= \|x\|\|y\||\mathrm{Re}\langle f(\frac{x}{\|x\|}), f(\frac{y}{\|y\|})\rangle| \\ &= \|x\|\|y\||\mathrm{Re}\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle| = |\mathrm{Re}\langle x, y\rangle| \quad (x, y \in H). \end{aligned}$$

It is clearly that $F : H \to K$ is surjective phase-isometry. In order to complete this result, we need the unpublished paper[**?**], Theorem 2.1, the following we include their proof for the readers' convenience.

Let $x \in H$ and $a \in \mathbb{R}$. Then

$$|a| \cdot \|x\|^2 = |\mathrm{Re}\langle ax, x\rangle| = |\mathrm{Re}\langle F(ax), F(x)\rangle| \leq \|F(ax)\| \cdot \|F(x)\| = |a| \cdot \|x\|^2.$$

By the equality condition in the Cauchy-Schwartz inequality, it follows that $F(ax) = bF(x)$ for some $b \in \mathbb{R}$. Since F is norm preserving, we have $b = \pm a$, and so $F(ax) = \pm aF(x)$ for each $x \in H$ and each $a \in \mathbb{R}$. By the axiom of choice, there exists a phase function $\varepsilon : H \to \{-1, 1\}$ such that $\varepsilon \cdot F$ is a real homogeneous mapping. Indeed, there is a set $L \subset S(H)$ such that for every nonzero vector $x \in H$, there exists uniquely determined $y \in L$ and $s \in \mathbb{R}$ such that $x = sy$. Define $f_0 : H \to K$ by

$$f_0(0) = 0,\ f_0(x) = f_0(sy) = sF(y), \quad \forall x = sy \in X \setminus \{0\}.$$

Now f_0 is well defined, real homogeneous and $F(x) = \pm f_0(x)$ for each $x \in H$. Without loss of generality we can assume that F is real homogeneous.

Let x and y be nonzero vectors such that $\mathrm{Re}\langle x, y\rangle = 0$. Clearly, we have

$$|\mathrm{Re}\langle F(x+y), F(x)\rangle| = |\mathrm{Re}\langle x+y, x\rangle| = \|x\|^2,$$

$$|\mathrm{Re}\langle F(x+y), F(y)\rangle| = |\mathrm{Re}\langle x+y, y\rangle| = \|y\|^2.$$

Set $\alpha := \|x\|^{-2}(\mathrm{Re}\langle F(x+y), F(x)\rangle)$ and $\beta := \|y\|^{-2}(\mathrm{Re}\langle F(x+y), F(y)\rangle)$. It is a routine matter to show that $\alpha, \beta \in \{-1, 1\}$ and

$$\begin{aligned}
&\|F(x+y) - \alpha F(x) - \beta F(y)\|^2 \\
&= \|x+y\|^2 + \|x\|^2 + \|y\|^2 - 2\alpha \mathrm{Re}\langle F(x+y), F(x)\rangle - 2\beta \mathrm{Re}\langle F(x+y), F(y)\rangle \\
&= 0.
\end{aligned}$$

This means precisely that

$$F(x+y) = \alpha F(x) + \beta F(y), \quad \alpha, \beta \in \{-1, 1\}.$$

Fix a unit vector $e \in H$, and set $Z := \{z \in H : \mathrm{Re}\langle z, e\rangle = 0\}$. By the above observations, we immediately obtain that

$$F(z+e) = \alpha(z)F(z) + \beta(z)F(e), \quad \alpha(z), \beta(z) \in \{-1, 1\}$$

for each $z \in Z \setminus \{0\}$. Define a mapping $g : H \to K$ as following:

$$g(0) = 0,\ g(ae) = aF(e),\ g(z) = \beta(z)\alpha(z)F(z),\ g(z+ae) = g(z) + g(ae)$$

for each $z \in Z \setminus \{0\}$ and each $a \in \mathbb{R}$. Obviously, the restricted mapping $g|_Z : Z \to K$ is a phase-isometry. Then

$$|\mathrm{Re}\langle g(z_1), g(z_2)\rangle| = |\mathrm{Re}\langle z_1, z_2\rangle|$$

and

$$|1+\mathrm{Re}\langle g(z_1),g(z_2)\rangle| = |\mathrm{Re}\langle g(z_1+e),g(z_2+e)\rangle| = |\mathrm{Re}\langle z_1+e,z_2+e\rangle| = |1+\mathrm{Re}\langle z_1,z_2\rangle|$$

for all $z_1,z_2 \in Z$. Then the restricted mapping $g|_Z : Z \to K$ satisfies the following property:

$$\mathrm{Re}\langle g(z_1),g(z_2) = \mathrm{Re}\langle z_1,z_2\rangle, \quad (z_1,z_2 \in Z).$$

Then, by the above equation and the norm-preserving property of g, we get that

$$\|g(z_1+z_2)-g(z_1)-g(z_2)\|^2 = \|(z_1+z_2)-z_1-z_2\|^2 = 0$$

which yields that g is additive. Given $z \in Z\setminus\{0\}$ and $a \in \mathbb{R}\setminus\{0\}$, we get

$$|a\|z\|^2+1| = |\mathrm{Re}\langle z+e,az+e\rangle| = |\mathrm{Re}\langle g(z+e),g(az+e)\rangle|$$
$$= |1+\mathrm{Re}\langle g(z),g(az)\rangle| = |1+a\alpha(tz)\beta(tz)\beta(z)\alpha(z)\|z\|^2|,$$

which implies that $\alpha(az)\beta(az) = \beta(z)\alpha(z)$, and thus $g|_Z$ is real homogeneous. This shows that $g|_Z : Z \to K$ is a real linear isometry, and so also is the mapping $g : H \to K$.

It suffices to prove that $g(x) = \pm F(x)$ for every $x \in H$. Given $z \in Z\setminus\{0\}$ and $a \in \mathbb{R}\setminus\{0\}$,

$$F(z+ae) = aF(a^{-1}z+e) = \alpha(a^{-1}z)F(z)+\beta(a^{-1}z)aF(e)$$

where $\alpha(a^{-1}z),\beta(a^{-1}z) \in \{-1,1\}$. Since g and F are real homogeneous, it follows that

$$\alpha(a^{-1}z)\beta(a^{-1}z) = \beta(z)\alpha(z)$$

as desired. This completes the proof. □

Lemma 2.2. *Let X and Y be complex Banach spaces. Suppose that $f : S(X) \to S(Y)$ is a surjective mapping satisfying equation (1). Then $f(-x) = -f(x)$ for all $x \in X$.*

Proof: Fix $0 \neq x \in S(X)$ and we can find $y \in S(X)$ such that $f(y) = -f(x)$. Since f satisfies equation (1),

$$\{\|x+y\|,\|x-y\|\} = \{\|f(x)+f(y)\|,\|f(x)-f(y)\|\} = \{0,2\},$$

which implies $y = \pm x$. In the case $y = x$, we obtain $f(x) = 0$, which is impossible.

□

Now we can state the result that every phase-isometry between two unit spheres of complex $\ell^p(\Gamma)$-type spaces for $p \in (1,\infty)\setminus\{2\}$ preserves orthogonal elements in both directions.

Lemma 2.3. *Let $X = \ell^p(\Gamma)$, $Y = \ell^p(\Delta)$ with $p > 1, p \neq 2$, and $f : S(X) \to S(Y)$ be a phase-isometry. Then $x \perp y \in S(X) \Leftrightarrow f(x) \perp f(y) \in S(Y)$.*

Proof: Select $x, y \in S(X)$. It is known that $x \perp y$ if and only if

$$\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p) = 4,$$

and $f(x) \perp f(y)$ if and only if

$$\|f(x)+f(y)\|^p + \|f(x)-f(y)\|^p = 2(\|f(x)\|^p + \|f(y)\|^p) = 4.$$

This completes the proof, since f is a phase-isometry. □

We continue our study with a specific version of [16, Lemma 2.4] for the behaviour of a surjective phase-isometry between two unit spheres on a complex number who's model 1 multiple of some element of the canonical basis.

Lemma 2.4. *Let $X = \ell^p(\Gamma)$, $Y = \ell^p(\Delta)$ with $p > 1, p \neq 2$, and $f : S(X) \to S(Y)$ be a surjective phase-isometry. Then for each $\gamma_0 \in \Gamma$, we have $\Delta_{f(\alpha e_{\gamma_0})} = \Delta_{f(e_{\gamma_0})}$ is a singleton for each $\alpha \in \mathbb{T}$. Moreover, one the following statements holds:*
(a) *$f(\alpha e_{\gamma_0}) = \pm \alpha f(e_{\gamma_0})$ for every $\alpha \in \mathbb{T}$;*
(b) *$f(\alpha e_{\gamma_0}) = \pm \bar{\alpha} f(e_{\gamma_0})$ for every $\alpha \in \mathbb{T}$.*

Proof: Take $\gamma_0 \in \Gamma$ and $\alpha \in \mathbb{T}$. If there are two distinct points $\delta_1, \delta_2 \in \Delta_{f(\alpha e_{\gamma_0})}$, we can find $x_1, x_2 \in S(X)$ such that $f(x_1) = e_{\delta_1}$ and $f(x_2) = e_{\delta_2}$. By Lemma 2.3 we have $f(\alpha e_{\gamma_0}) \perp f(e_\gamma)$ for all $\gamma \in \Gamma \setminus \{\gamma_0\}$. By applying Lemma 2.3 to f^{-1} we deduce that $x_1 \perp x_2, x_1 \perp e_\gamma$ and $x_2 \perp e_\gamma$ for all $\gamma \neq \gamma_0$, which is impossible. Therefore, we get $\Delta_{f(\alpha e_{\gamma_0})}$ is a singleton, and hence $\Delta_{f(\alpha e_{\gamma_0})} = \Delta_{f(e_{\gamma_0})}$.

Next, we show that $f(\alpha e_{\gamma_0}) = \beta f(e_{\gamma_0})$ for some $\beta \in \{\pm\alpha, \pm\bar{\alpha}\}$. Let us write $f(\alpha e_{\gamma_0}) = \beta f(e_{\gamma_0})$ for some $|\alpha| = |\beta|$. Now we get

$$\begin{aligned} &\{|1+\alpha|, |1-\alpha|\} = \{\|e_{\gamma_0} + \alpha e_{\gamma_0}\|, \|e_{\gamma_0} - \alpha e_{\gamma_0}\|\} \\ = \ & \{\|f(e_{\gamma_0}) + f(\alpha e_{\gamma_0})\|, \|f(e_{\gamma_0}) - f(\alpha e_{\gamma_0})\|\} \\ = \ & \{|1+\beta|, |1-\beta|\}, \end{aligned}$$

which assures that $s \in \{\pm\alpha, \pm\bar{\beta}\}$ as desired.

Suppose now that $f(\theta e_{\gamma_0}) = \pm\theta f(e_{\gamma_0})$ and $f(\lambda e_{\gamma_0}) = \pm\overline{\lambda} f(e_{\gamma_0})$ for some $\theta, \lambda \in \mathbb{T} \setminus \{\pm 1, \pm i\}$. Then we have

$$\begin{aligned} &2 + 2|\mathrm{Re}(\theta\overline{\lambda})| = \|\theta e_{\gamma_0} + \lambda e_{\gamma_0}\|^2 \vee \|\theta e_{\gamma_0} - \lambda e_{\gamma_0}\|^2 \\ = \ & \|f(\theta e_{\gamma_0}) + f(\lambda e_{\gamma_0})\|^2 \vee \|f(\theta e_{\gamma_0}) - f(\lambda e_{\gamma_0})\|^2 \\ = \ & |\theta + \overline{\lambda}|^2 \vee |\theta - \overline{\lambda}|^2 = 2 + 2|\mathrm{Re}(\theta\lambda)|. \end{aligned}$$

It can be easily deduced that

$$|\mathrm{Re}(\theta)\mathrm{Re}(\lambda) + \mathrm{Im}(\theta)\mathrm{Im}(\lambda)| = |\mathrm{Re}(\theta)\mathrm{Re}(\lambda) - \mathrm{Im}(\theta)\mathrm{Im}(\lambda)|$$

which is impossible since $\theta, \lambda \in \mathbb{T} \setminus \{\pm 1, \pm i\}$. It follows that either $f(\theta e_{\gamma_0}) = \pm\theta f(e_{\gamma_0})$ for all $\theta \in \mathbb{T}$ or $f(\theta e_{\gamma_0}) = \pm\overline{\theta} f(e_{\gamma_0})$ for all $\theta \in \mathbb{T}$. □

The next result is given the representation theorem of surjective mapping satisfying equation (1) between two unit spheres of complex $\ell^p(\Gamma)$-type spaces.

Proposition 2.5. *Let $X = \ell^p(\Gamma)$, $Y = \ell^p(\Delta)$ with $p > 1, p \neq 2$, and $f : S(X) \to S(Y)$ be a surjective phase-isometry. Then for each $x = \sum_{\gamma\in\Gamma} x_\gamma e_\gamma \in S(X)$, we have $f(x) = \sum_{\gamma\in\Gamma_x} |x_\gamma| f(\frac{y_\gamma}{|x_\gamma|} e_\gamma)$, where $y_\gamma = \pm x_\gamma$ for each $\gamma \in \Gamma_x$.*

Proof: According to Lemma 2.4, we note

$$\Gamma_1 := \{\gamma \in \Gamma : f(\alpha e_\gamma) = \pm\alpha f(e_\gamma), \forall \alpha \in \mathbb{T}\}$$

$$\Gamma_2 := \{\gamma \in \Gamma : f(\alpha e_\gamma) = \pm\overline{\alpha} f(e_\gamma), \forall \alpha \in \mathbb{T}\},$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let us take $x \in S(X)$. By Lemma 2.3, we can write

$$f(x) = \sum_{\gamma \in \Gamma_x} y_\gamma f(e_\gamma) = \sum_{\gamma \in \Gamma_x \cap \Gamma_1} y_\gamma f(e_\gamma) + \sum_{\gamma \in \Gamma_x \cap \Gamma_2} y_\gamma f(e_\gamma).$$

Fixed $\gamma \in \Gamma_x \cap \Gamma_1$. Since f is a phase-isometry, then

$$\begin{aligned} & 1 - |x_\gamma|^p + (1 + |x_\gamma|)^p = \|x + \frac{x_\gamma}{|x_\gamma|} e_\gamma\|^p \vee \|x - \frac{x_\gamma}{|x_\gamma|} e_\gamma\|^p \\ = \ & \|f(x) + f(\frac{x_\gamma}{|x_\gamma|} e_\gamma)\|^p \vee \|f(x) - f(\frac{x_\gamma}{|x_\gamma|} e_\gamma)\|^p \\ = \ & (1 - |y_\gamma|^p + |y_\gamma + \frac{x_\gamma}{|x_\gamma|}|^p) \vee (1 - |y_\gamma|^p + |y_\gamma - \frac{x_\gamma}{|x_\gamma|}|^p) \\ \leq \ & 1 - |y_\gamma|^p + (1 + |y_\gamma|)^p, \end{aligned}$$

which shows that $(1+|x_\gamma|)^p - |x_\gamma|^p \leq (1+|y_\gamma|)^p - |y_\gamma|^p$. Since the function $\varphi(t) = (1+t)^p - t^p$ is strictly increasing on $(0, +\infty)$ for $p > 1$, it follows that $|x_\gamma| \leq |y_\gamma|$ for each $\gamma \in \Gamma_x \cap \Gamma_1$. Similarly, it is also true for each $\gamma \in \Gamma_x \cap \Gamma_2$,

$$\begin{aligned} & 1 - |x_\gamma|^p + (1 + |x_\gamma|)^p = \|x + \frac{x_\gamma}{|x_\gamma|} e_\gamma\|^p \vee \|x - \frac{x_\gamma}{|x_\gamma|} e_\gamma\|^p \\ = \ & \|f(x) + f(\frac{x_\gamma}{|x_\gamma|} e_\gamma)\|^p \vee \|f(x) - f(\frac{x_\gamma}{|x_\gamma|} e_\gamma)\|^p \\ = \ & (1 - |y_\gamma|^p + |y_\gamma + \frac{\overline{x_\gamma}}{|x_\gamma|}|^p) \vee (1 - |y_\gamma|^p + |y_\gamma - \frac{\overline{x_\gamma}}{|x_\gamma|}|^p) \\ \leq \ & 1 - |y_\gamma|^p + (1 + |y_\gamma|)^p. \end{aligned}$$

The equation $\|f(x)\| = \|x\| = 1$ assures that $|x_\gamma| = |y_\gamma|$ for each $\gamma \in \Gamma_x$. This establishes

$$(|y_\gamma + \frac{x_\gamma}{|x_\gamma|}|) \vee (|y_\gamma + \frac{x_\gamma}{|x_\gamma|}|) = 1 + |y_\gamma|,$$

and hence $y_\gamma = \pm x_\gamma$ for each $\gamma \in \Gamma_x \cap \Gamma_1$. A similar argument holds for $\gamma \in \Gamma_x \cap \Gamma_2$, we get $y_\gamma = \pm \overline{x_\gamma}$ for each $\gamma \in \Gamma_x \cap \Gamma_2$. We deduce from the definition of Γ_1 and Γ_2 that $y_\gamma f(e_\gamma) = |x_\gamma| f(\pm \frac{x_\gamma}{|x_\gamma|} e_\gamma)$ for each $\gamma \in \Gamma_x$. □

Lemma 2.6. *Let $X = \ell^p(\Gamma)$, $Y = \ell^p(\Delta)$ with $p > 1, p \neq 2$, and $f : S(X) \to S(Y)$ be a surjective phase-isometry. Let x and y be nonzero orthogonal vectors in $S(X)$. Then there exist two real number $\alpha(Ax, By), \beta(Ax, By) \in \{-1, 1\}$ such that*

$$f(Ax + By) = A\alpha(Ax, By) f(x) + B\beta(Ax, By) f(y)$$

where $|A|^p+|B|^p=1, A,B\in\mathbb{R}$.

Proof: Since $f(-x)=-f(x)$ for all $x\in S(X)$, we can assume that $A,B>0$. By Proposition 2.5, we write

$$f(x)=\sum_{\gamma\in\Gamma_x}|x_\gamma|f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma),\ f(y)=\sum_{\gamma\in\Gamma_y}|y_\gamma|f(\frac{y'_\gamma}{|y_\gamma|}e_\gamma),$$
$$f(Ax+By)=A\sum_{\gamma\in\Gamma_x}|x_\gamma|f(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)+B\sum_{\gamma\in\Gamma_y}|y_\gamma|f(\frac{y''_\gamma}{|y_\gamma|}e_\gamma)$$

where $x'_\gamma, x''_\gamma\in\{x_\gamma,-x_\gamma\}$ for every $\gamma\in\Gamma_x$ and $y'_\gamma, y''_\gamma\in\{y_\gamma,-y_\gamma\}$ for every $\gamma\in\Gamma_y$. It is easy to check that

$$\begin{aligned}
&\{(1+A)^p+B^p,(1-A)^p+B^p\}\\
=\ &\{\|Ax+By+x\|^p,\|Ax+By-x\|^p\}\\
=\ &\{\|f(Ax+By)+f(x)\|^p,\|f(Ax+By)-f(x)\|^p\}\\
=\ &\{\Big\|\sum_{\gamma\in\Gamma_x}|x_\gamma|[f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)\pm Af(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)]\Big\|^p+B^p\}.
\end{aligned}$$

This shows that

$$(1+A)^p\in\{\Big\|\sum_{\gamma\in\Gamma_x}|x_\gamma|[f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)\pm Af(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)]\Big\|^p\}.$$

Suppose that

$$\begin{aligned}
(1+A)^p &= \Big\|\sum_{\gamma\in\Gamma_x}|x_\gamma|[f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)+Af(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)]\Big\|^p\\
&\le \sum_{\gamma\in\Gamma_x}|x_\gamma|^p(1+A)^p=(1+A)^p,
\end{aligned}$$

which implies that $\|f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)+Af(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)\|=\|f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)\|+\|Af(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)\|$ for all $\gamma\in\Gamma_x$. Furthermore, it is not hard to check that $f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)=f(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)$ for all $\gamma\in\Gamma_x$ since X is strictly convex. Similarly, we cliam that for all $\gamma\in\Gamma_x$, $f(\frac{x'_\gamma}{|x_\gamma|}e_\gamma)=-f(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)$. It means that $\sum_{\gamma\in\Gamma_x}|x_\gamma|f(\frac{x''_\gamma}{|x_\gamma|}e_\gamma)=\pm f(x)$. Similar conclusion yields $\sum_{\gamma\in\Gamma_y}|y_\gamma|f(\frac{y''_\gamma}{|y_\gamma|}e_\gamma)=\pm f(y)$, which concludes the proof. □

Corollary 2.7. *Especially, we take* $A=B=\frac{1}{\|x+y\|}=2^{-\frac{1}{p}}$. *It means that we can write* $f(Ax+Ay)=A\alpha(x,y)f(x)+A\beta(x,y)f(y)$.

As a consequence of the above result, we will show the main conclusion of this paper.

Theorem 2.8. *Let $X=\ell^p(\Gamma)$, $Y=\ell^p(\Delta)$ with $p>1, p\neq 2$, and $f:S(X)\to S(Y)$ be a surjective phase-isometry. Then its positive homogeneous extension F of f is phase equivalent a real linear isometry.*

Proof: The previous arguments show that when $p=2$ by Theorem 2.1, thus we only need to consider the case $p>0, p\neq 2$. Fixed $\gamma_0\in\Gamma$, as a consequence of Lemma 2.4, we can assume that $f(\alpha e_{\gamma_0})=\alpha f(e_{\gamma_0})$ for each $\alpha\in\mathbb{T}$, the other statement's proof is very similar. Set $Z:=\{x\in\ell^p(\Gamma): x\perp e_{\gamma_0}\}$ and $W:=\{y\in\ell^p(\Delta): y\perp f(e_{\gamma_0})\}$. It is not hard to prove $S(X)=\{az+te_{\gamma_0}: z\in S(Z), |a|^p+|t|^p=1, a\in\mathbb{R}, t\in\mathbb{C}\}$.

By considering the Proposition 2.5 that the restricted mapping $f|_Z: S(Z)\to S(W)$ is a surjective phase-isometry. By Corollary 2.7 we can therefore write

$$f(Az+Ae_{\gamma_0})=A\alpha(z,e_{\gamma_0})f(z)+A\beta(z,e_{\gamma_0})f(e_{\gamma_0}),\ \alpha(z,e_{\gamma_0}),\beta(z,e_{\gamma_0})\in\{-1,1\}$$

where $A=\frac{1}{\|z+e_{\gamma_0}\|}=2^{-\frac{1}{p}}$ for each $z\in S(Z)$. Define a mapping $g:S(Z)\to S(W)$ given by

$$g(z)=\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})f(z),$$

for each $z\in S(Z)$. It is easily seen that $g(z)=\pm f(z)$ for each $z\in S(Z)$. Since f is a phase-isometry, for each $z\in S(Z)$,

$$\begin{aligned}
&\frac{1}{2}\{2^p\}=\frac{1}{2}\{\|(z+e_{\gamma_0})+(-z+e_{\gamma_0})\|^p, \|(z+e_{\gamma_0})-(-z+e_{\gamma_0})\|^p\}\\
&=\{\|f(Az+Ae_{\gamma_0})+f(-Az+Ae_{\gamma_0})\|^p, \|f(Az+Ae_{\gamma_0})-f(-Az+Ae_{\gamma_0}))\|^p\}\\
&=\frac{1}{2}\{\|g(z)+f(e_{\gamma_0})+g(-z)+f(e_{\gamma_0})\|^p, \|g(z)-g(-z)\|^p\}\\
&=\frac{1}{2}\{|\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})-\alpha(-z,e_{\gamma_0})\beta(-z,e_{\gamma_0})|^p+2^p,\\
&\qquad |\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})+\alpha(-z,e_{\gamma_0})\beta(-z,e_{\gamma_0})|^p\}
\end{aligned}$$

which implies that $\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})=\alpha(-z,e_{\gamma_0})\beta(-z,e_{\gamma_0})$. This means that $g(-z)=-g(z)$, and so $g:S(Z)\to S(W)$ is a surjective phase-isometry. Next we show that $g:S(Z)\to S(W)$ is a surjective isometry. For $z_1,z_2\in S(Z)$, since g is a phase-isometry, we have

$$\{\|g(z_1)+g(z_2)\|^p, \|g(z_1)-g(z_2)\|^p\}=\{\|z_1+z_2\|^p, \|z_1-z_2\|^p\}$$

and

$$\begin{aligned}
&\frac{1}{2}\{\|z_1+z_2\|^p+2^p,\|z_1-z_2\|^p\}\\
&=\{\|f(Az_1+Ae_{\gamma_0})+f(Az_2+Ae_{\gamma_0})\|^p,\|f(Az_1+Ae_{\gamma_0})-f(Az_2+Ae_{\gamma_0})\|^p\\
&=\{\|\beta(z_1,e_{\gamma_0})f(Az_1+Ae_{\gamma_0})\pm\beta(z_2,e_{\gamma_0})f(Az_2-Ae_{\gamma_0})\|^p\}\\
&=\frac{1}{2}\{\|g(z_1)+g(z_2)\|^p+2^p,\|g(z_1)-g(z_2)\|^p\}.
\end{aligned}$$

Hence we obtain that $\|g(z_1)-g(z_2)\|=\|z_1-z_2\|$, which implies g is a surjective isometry. From Yi's result[12], the restriction of G to Z is a real linear isometry, where $G:Z\to W$ is the natural positive homogeneous extension of g. It means that for $z_1,z_2\in S(Z)$, and $a_1,a_2\in\mathbb{R}$, we have

$$\|a_1g(z_1)-a_2g(z_2)\|=\|G(a_1z_1)-G(a_2z_2)\|=\|a_1z_1-a_2z_2\|.$$

Now we shall fristly show a function $\widetilde{f}:S(X)\to S(Y)$ is a surjective isometry, which is given by the following for every $z\in S(Z)$, $|a|^p+|t|^p=1$, $a\in\mathbb{R}$ and $t\in\mathbb{C}$:

$$\widetilde{f}(az+te_{\gamma_0})=ag(z)+tf(e_{\gamma_0}),$$

Choose $x_1,x_2\in S(X)$, where $x_1=a_1z_1+t_1e_{\gamma_0}$, $x_2=a_2z_2+t_2e_{\gamma_0}$, $a_1,a_2\in\mathbb{R}$ and $t_1,t_2\in\mathbb{C}$, we can obtain

$$\begin{aligned}
\|\widetilde{f}(x_1)-\widetilde{f}(x_2)\|^p &= \|a_1g(z_1)+t_1f(e_{\gamma_0})-(a_2g(z_2)+t_2f(e_{\gamma_0}))\|^p\\
&= \|a_1z_1-a_2z_2\|^p+|t_1-t_2|^p=\|x_1-x_2\|^p,
\end{aligned}$$

which implies that $\widetilde{f}$ is a isometry. Obviously, $\widetilde{f}(-x)=-\widetilde{f}(x)$ for all $x\in S(X)$.

As we commented above, it follows to prove that $f(x)=\pm\widetilde{f}(x)$ for every $x\in S(X)$. In the case of $a=0$ or $t=0$, we have $\widetilde{f}(te_{\gamma_0})=tf(e_{\gamma_0})$ or $\widetilde{f}(az)=ag(z)$ respectively. So we only need to consider $a\in\mathbb{R}\setminus\{0\},t\in\mathbb{C}\setminus\{0\}$. Given $z\in S(Z)$. By the above result and Lemma 2.6, we can write

$$\widetilde{f}(az+te_{\gamma_0})=a\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})f(z)+tf(e_{\gamma_0}),\ \alpha(z,e_{\gamma_0}),\beta(z,e_{\gamma_0})\in\{-1,1\},$$

$$f(az+te_{\gamma_0})=a\alpha(az,te_{\gamma_0})f(z)+\beta(az,te_{\gamma_0})tf(e_{\gamma_0}),\ \ \alpha(az,te_{\gamma_0}),\beta(az,te_{\gamma_0})\in\{-1,1\}.$$

It is equivalent to check that

$$\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0}) = \alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}).$$

Since f is a phase-isometry,

$$\begin{aligned}
& \{|a+A|^p+|t+A|^p, |a-A|^p+|t-A|^p\} \\
= & \{\|az+te_{\gamma_0}+Az+Ae_{\gamma_0}\|^p, \|az+te_{\gamma_0}-(Az+Ae_{\gamma_0})\|^p\} \\
= & \{\|f(az+te_{\gamma_0})+f(Az+Ae_{\gamma_0})\|^p, \|f(az+te_{\gamma_0})-f(Az+Ae_{\gamma_0})\|^p\} \\
= & \{\|\beta(az,te_{\gamma_0})f(az+te_{\gamma_0})\pm\beta(z,e_{\gamma_0})f(Az+Ae_{\gamma_0})\|^p\} \\
= & \{|a\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0})+A\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})|^p+|t+A|^p, \\
& |a\alpha(z,te_{\gamma_0})\beta(z,te_{\gamma_0})-A\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})|^p+|t-A|^p\}.
\end{aligned}$$

If $|t+A| \neq |t-A|$ or $t \neq ib$ for some $b \in \mathbb{R}\setminus\{0\}$, then we get the desired equation

$$\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0}) = \alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}).$$

Now assume that $t = ib$ for some $b \in \mathbb{R}\setminus\{0\}$. Choose $\alpha \in \mathbb{T}\setminus\{\pm1,\pm i\}$. Following a similar argument as above, we get

$$\begin{aligned}
& \{|a+A|^p+|t+A\alpha|^p, |a-A|^p+|t-A\alpha|^p\} \\
= & \{\|az+te_{\gamma_0}+Az+A\alpha e_{\gamma_0}\|^p, \|(az+te_{\gamma_0})-(Az+A\alpha e_{\gamma_0})\|^p\} \\
= & \{\|f(az+te_{\gamma_0})+f(Az+A\alpha e_{\gamma_0})\|^p, \|f(az+te_{\gamma_0})-f(Az+A\alpha e_{\gamma_0})\|^p\} \\
= & \{\|\beta(az,te_{\gamma_0})f(az+te_{\gamma_0})\pm\beta(z,\alpha e_{\gamma_0})f(Az+A\alpha e_{\gamma_0})\|^p\} \\
= & \{|a\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0})+A\alpha(z,\alpha e_{\gamma_0})\beta(z,e_{\gamma_0})|^p+|t+A\alpha|^p, \\
& |a\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0})-A\alpha(z,\alpha e_{\gamma_0})\beta(z,\alpha e_{\gamma_0})|^p+|t-A\alpha|^p\}
\end{aligned}$$

Since $|t-A\alpha| \neq |t+A\alpha|$, we obtain

$$\alpha(az,te_{\gamma_0})\beta(az,te_{\gamma_0}) = \alpha(z,\alpha e_{\gamma_0})\beta(z,\alpha e_{\gamma_0}) = \alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}).$$

It is clearly that $F(x) = \pm\widetilde{F}(x)$ for all $x \in X$. By Yi's result [12] again, we show the natural positive homogeneous extension $\widetilde{F}$ of $\widetilde{f}$ is a real linear isometry from X onto Y.

This completes the proof. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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