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APPROXIMATING COMMON FIXED POINTS OF A THREE-STEP ITERATION FOR FOUR ASYMPTOTICALLY NON-EXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. In the present paper, we study a three-step iterative algorithm for four asymptotically non-expansive mappings in CAT(0) spaces.The result presented in this paper improve and extend the corresponding results of [Y.Niwongsa and B.Panyank: Noor Iteration for Asymptotically Nonexpansive Mappings in $CAT(0)$ Spaces [9]].Consquently an interesting example is provided to illustrate the main results.

Keywords: Asymptotically non-expansive mapping; fixed points; Noor iteration; uniformly convex Banach space; CAT(0) space.

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1. Introduction

A metric space X is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. Fixed point theory in $CAT(0)$ spaces was first studied by Kirk; see the references [3] and [4] and the references therein. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed

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point. Then the fixed point theory for single-valued and multivalued mappings in $CAT(0)$ spaces has been rapidly developed and much papers have appeared; see the references [1] and [9].

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in Y$ is a map c for closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t^{'})) = \left| t - t^{'} \right|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted by $[x, y]$.The space (X, d) is said to be geodesic space if every two points of X are joined by geodesic and X is said to be uniquely geodesic if there is one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ is a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X(the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a $CAT(0)$ space if all geodesic triangle satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X, let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is satisfy the $CAT(0)$ inequality if, for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{E^2}(\overline{x}, \overline{y})$.

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies that

$$\text{(CN)} \qquad d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [1]. By using the (CN) inequality, it is easy to see that the $CAT(0)$ spaces are uniformly convex. In fact geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

The notion of asymptotic centers in a Banach space can be extended to a $CAT(0)$ space as well, by replacing $\|.\|$ with $d(.,.)$ .It is known that,in a $CAT(0)$ space, $A(K, \{x_n\})$ consists of exactly one point.

Definition 1.1. Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to X$ be a mapping. T is said to be asymptotically non-expansive mappings if there is a sequence $\{k_n\}$ of positive numbers with the property $lim_{n\to\infty} k_n = 1$ and such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y).$$

The following are some elementary facts about $CAT(0)$ space.

Definition 1.2. A mapping $T : X \to X$ is called

(1) Nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$.

(2) Asymptotically nonexpansive if there exists $k_n \in [0,\infty)$ for all $n \in N$ with $lim_{n\to\infty}k_n = 0$ such that $d(T^n x, T^n y) \leq (1+k_n)d(x,p)$ for all $x,y \in X$.

(3) Quasi-nonexpansive if $d(Tx,p) \leq d(x,p)$ for all $p \in F(T)$, where F(T) is the set of all fixed points of T.

(4) Asymptotically quasi-nonexpansive if there exists $k_n \in [0,\infty)$for all $n \in N$ with $lim_{n\to\infty}k_n = 0$ such that $d(T^n x,p) \leq (1+k_n)d(x,p)$ for all $p \in F(T)$.

Remark 1.3. From Definition 1.2, it is clear that the classes of quasi-nonexpansive mappings and asymptotically nonexpansive mappings include nonexpansive mappings when their fixed point sets are not empty, whereas the class of asymptotically quasi-nonexpansiveness is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings when their fixed point sets are not empty. The reverse of these implications may not be true; see the references [6,7] and the references therein.

Xu and Noor [11] introduced a three-step iterative scheme as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ are real sequences in $[0,1]$.

Lemma 1.4. [8] Let (X,d) be a CAT(0) space.

(i) (X,d) is uniquely geodesic.

(ii) Let p,x,y be points of X. Let $\alpha \in [0,1]$. Let m_1 ,m_2 denotes respectively, the points of $[p,x]$ and $[p,y]$ satisfying $d(p,m_1) = \alpha d(p,x)$ and $d(p,m_2) = \alpha d(p,y)$. Then

$$d(m_1,m_2) = \alpha d(p,x)$$

(iii) Let $x,y \in X$,$x \neq y$ and $z,w \in [x,y]$, such that $d(x,z) = d(x,w)$. Then $z = w$.

(iv) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \; and \; d(y,z) = (1-t)d(x,y)$$

and

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z). \tag{1.1}$$

(vi)

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y).$$

Diaz and Metcalf [2] introduced the concept of quasi-nonexpansive mappings. Gobel and Kirk gave the definition of asymptotically nonexpansive mapping.

Lemma 1.5. *If K is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequences in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 1.6. [10] *Let X be a complete $CAT(0)$ space and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[b,c]$ for some $b, c \in (0,1)$ and that $\{x_n\}$, $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq r,$$

$$\limsup_{n\to\infty} d(y_n, x) \leq r,$$

and

$$\lim_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x)) = r \ for \ \ some \ r \geq 0.$$

Then $lim_{n\to\infty} d(x_n, y_n) = 0$.

2. Main results

Theorem 2.1. *Let K be a nonempty closed, bounded and convex subset of a complete $CAT(0)$ space X. Let $R, S, T, U : K \to K$ be four asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ be real sequences in $[0,1]$. For a given $x_1 \in K$ consider the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ defined by*

$$\begin{cases} z_n = (1-\gamma_n)U^n x_n \oplus \gamma_n R^n x_n \\ y_n = (1-\beta_n)R^n x_n \oplus \beta_n T^n z_n, \\ x_{n+1} = (1-\alpha_n)R^n x_n \oplus \alpha_n S^n y_n, \quad n \geq 1. \end{cases}$$

Suppose that $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$.

Proof. Let $x^* \in F(T)$. It follows that

$$\begin{aligned} d(z_n, x^*) &= d\left\{(1-\gamma_n)U^n x_n \oplus \gamma_n R^n x_n, x^*\right\} \\ &\leq (1-\gamma_n)d(U^n x_n, x^*) + \gamma_n d(R^n x_n, x^*) \\ &\leq (1-\gamma_n)k_n d(x_n, x^*) + \gamma_n k_n d(x_n, x^*) \\ &\leq d(x_n, x^*). \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
d(y_n, x^*) &= d\{(1-\beta_n)R^n x_n \oplus \beta_n T^n z_n, x^*\} \\
&\le (1-\beta_n)d(R^n x_n, x^*) + \beta_n d(T^n z_n, x^*) \\
&\le (1-\beta_n)k_n d(x_n, x^*) + \beta_n k_n d(x_n, x^*) \\
&\le d(x_n, x^*).
\end{aligned} \tag{2.2}
$$

In view of (1.1), we obtain that

$$
\begin{aligned}
d(x_{n+1}, x^*) &= d\{(1-\alpha_n)R^n x_n \oplus \alpha_n S^n y_n, x^*\} \\
&\le (1-\alpha_n)d(R^n x_n, x^*) + \alpha_n d(S^n y_n, x^*) \\
&\le (1-\alpha_n)k_n d(x_n, x^*) + \alpha_n k_n d(y_n, x^*) \\
&\le d(x_n, x^*).
\end{aligned}
$$

This implies that $d(x_n, x^*)$is bounded and nonincreasing for all $x^* \in F(T)$. It follows that $\lim_{n\to\infty} d(x_n, x^*)$ exists.

Theorem 2.2. *Let K be a nonempty closed, bounded and convex subset of a complete $CAT(0)$ space X. Let $R, S, T, U : K \to K$ be four asymptotically nonexpansive mappings with the sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ be real sequences in [0,1] satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined in Theorem 2.1. If $F = F(R)\cap F(S)\cap F(T)\cap F(U) \ne \phi$,

$$
d(x, Sy) \le d(Rx, Sy), \quad \forall x, y \in K, \ d(x, Rx) \le d(Ux, Rx), \quad \forall x \in K. \tag{2.3}
$$

Then

$$
\lim_{n\to\infty} d(R^n x_n, x_n) = \lim_{n\to\infty} d(S^n y_n, x_n) = \lim_{n\to\infty} d(T^n z_n, x_n) = \lim_{n\to\infty} d(U^n x_n, x_n) = 0, \quad \forall x^* \in F.
$$

Proof. In view of Theorem 2,1, we see that $\lim_{n\to\infty} d(x_n, x^*)$ exists. Let $\lim_{n\to\infty} d(x_n, x^*) = r$. If $r = 0$, then nothing to prove. Assume that $r > 0$. Now we show that $\lim_{n\to\infty} d(R^n x_n, S^n y_n) = 0$. From (2.2), we have

$$
d(y_n, x*) \le d(x_n, x*).
$$

Taking $\lim sup_{n\to\infty}$ both the sides, we obtain that

$$
\limsup_{n\to\infty} d(y_n, x^*) \le \limsup_{n\to\infty} d(x_n, x^*) = r.
$$

Observe that

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_{n+1}, x^*) \\ &= \lim_{n\to\infty} d((1-\alpha_n)R^n x_n \oplus \alpha_n S^n y_n, x^*) \\ &= \lim_{n\to\infty} [(1-\alpha_n) d(R^n x_n, x^*) + \alpha_n d(S^n y_n, x^*)]. \end{aligned}$$

It follows from Lemma 1.6 that

$$\lim_{n\to\infty} d(R^n x_n, S^n y_n) = 0.$$

In view of (2.3), we arrive at

$$\begin{aligned} d(R^n x_n, x_n) &\leq d(R^n x_n, S^n y_n) + d(S^n y_n, x_n) \\ &\leq 2d(R^n x_n, S^n y_n) \to 0, \ as \ n \to \infty. \end{aligned}$$

Now we observe that, for each $n \geq 1$,

$$d(z_n, x^*) \leq d(x_n, x^*).$$

Taking $\limsup_{n\to\infty}$ both the sides, we obtain that

$$\limsup_{n\to\infty} d(z_n, x^*) \leq \limsup_{n\to\infty} d(x_n, x^*) = r.$$

Notice that

$$\begin{aligned} r &= \lim_{n\to\infty} d(y_n, x^*) \\ &= \lim_{n\to\infty} d\left\{(1-\beta_n)R^n x_n \oplus \beta_n T^n z_n, x^*\right\} \\ &= \lim_{n\to\infty} \left\{(1-\beta_n) d(R^n x_n, x^*) + \beta_n d(T^n z_n, x^*)\right\}. \end{aligned}$$

It follows from Lemma 1.6 that

$$\lim_{n\to\infty} d(R^n x_n, T^n z_n) = 0.$$

And hence

$$d(T^n z_n, x_n) \leq d(T^n z_n, R^n x_n) + d(R^n x_n, x_n) \to 0 \ as \ n \to \infty. \tag{2.4}$$

On the other hand, we have

$$\limsup_{n\to\infty} d(U^n x_n, x^*) = \limsup_{n\to\infty} d(x_n, x^*) = r$$

and

$$\limsup_{n\to\infty} d(R^n x_n, x^*) = \limsup_{n\to\infty} d(x_n, x^*) = r.$$

Also

$$\begin{aligned} d(x_n, x^*) &= d(x_n, T^n z_n) + d(T^n z_n, x^*) \\ &\leq d(x_n, T^n z_n) + k_n d(z_n, x^*). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain that

$$\lim_{n\to\infty} d(z_n, x^*) = r.$$

Now

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n, x^*) \\ &= \lim_{n\to\infty} d((1-\gamma_n)U^n x_n \oplus \gamma_n R^n x_n, x^*) \\ &= \lim_{n\to\infty} [(1-\gamma_n)d(U^n x_n, x^*) + \gamma_n d(R^n x_n, x^*)]. \end{aligned}$$

Using Lemma 1.6, we obtain that

$$\lim_{n\to\infty} d(R^n x_n, U^n x_n) = 0.$$

It follows from (2.3) that

$$d(U^n x_n, x_n) \le d(U^n x_n, R^n x_n) + d(R^n x_n, x_n) \to 0 \ as \ n \to \infty.$$

It follows that

$$\lim_{n\to\infty} d(R^n x_n, x_n) = \lim_{n\to\infty} d(S^n y_n, x_n) = \lim_{n\to\infty} d(T^n z_n, x_n) = \lim_{n\to\infty} d(U^n x_n, x_n) = 0,$$

This completes the proof.

Remark.2.3. If $U = R = I$, the identity operator on K, then the iterative sequences in Theorem 2.1 are reduced to the iterative scheme defined by Ishikawa for $CAT(0)$ spaces

$$\begin{cases} y_n = (1-\beta_n)x_n) \oplus \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n S^n x_n, \quad n \ge 1, \end{cases}$$

Remark.2.4. If $U = R = T = I$, the identity operator on K, then the sequences in Theorem 2.1 are reduced to the usual Mann iterative scheme for $CAT(0)$ spaces

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n S^n y_n, \quad n \ge 1.$$

Corollary 2.5. *Let K be a nonempty closed, bounded and convex subset of a complete CAT(0) space X. Let $R, S, T, U : K \to K$ be four asymptotically nonexpansive mappings with sequence $\{x_n\}$ satisfying $\{k_n\} \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined in Theorem 2.1. If $F = F(R) \cap F(S) \cap F(T) \cap F(U) \ne \phi$, then $\{x_n\}$ converges weakly to a common fixed point of the mappings R,S,T and U.*

Example 2.6. Let K be the real line with usual norm $\|.\|$ and $K = [0, 1]$. Define $R, S, T, U : K \to K$ by $R(x) = \sin x$, $S(x) = \frac{x}{3}$, $T(x) = \frac{x}{2}$ and $U(x) = x$ for all $x \in K$. Obviously R(0)=S(0)=T(0)=U(0)=0, i.e., 0 is the common fixed point of R, S, T and U. Now we check that R is asymptotically nonexpansive. In fact if $x \in [0,1]$, then

$$d(Rx, Ry) = \|\sin x - \sin y\| = 2\|\cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2}\| \leq 2\|\frac{x-y}{2}\| = \|x-y\|.$$

Thus R is asymptotically nonexpansive with constant sequence $\{k_n\}$ for each $n \geq 1$. Similarly we can show that S, T and U are asymptotically nonexpansive with constant sequence $\{k_n\}$ for each $n \geq 1$.

$$\begin{aligned}
d(z_n, x^*) &= d\{(1-\gamma_n)Ux \oplus \gamma_n Rx, x^*\} \\
&\leq (1-\gamma_n)d(Ux, x^*) + \gamma_n d(Rx, x^*) \\
&\leq \gamma_n k_n d(x, 0) + (1-\gamma_n)k_n d(sinx, 0) \\
&\leq \gamma_n k_n \|x\| + (1-\gamma_n)k_n \|\sin x\| \\
&\leq \gamma_n k_n \|x\| + (1-\gamma_n)k_n \|x\| \\
&\leq x \\
&= d(x, x^*).
\end{aligned}$$

similarly, we have

$$\begin{aligned}
d(x_{n+1}, x^*) &= d\{(1-\alpha_n)Rx \oplus \alpha_n Sy, x^*\} \\
&\leq (1-\alpha_n)d(Rx, x^*) + \alpha_n d(Sy, x^*) \\
&\leq \alpha_n k_n d(\sin x, 0) + (1-\alpha_n)k_n d(\frac{x}{3}, 0) \\
&\leq \alpha_n k_n \|\sin x\| + (1-\alpha_n)k_n \left\|\frac{x}{3}\right\| \\
&\leq \alpha_n k_n \|x\| + (1-\alpha_n)k_n \|x\| \\
&\leq x \\
&= d(x, x^*).
\end{aligned}$$

Hence Theorem 2.1 is satisfied.

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