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# CONVERGENCE OF AN INERTIAL-TYPE KRASNOSEL'SKII-MANN-TYPE ITERATIVE SEQUENCE TO FIXED POINTS OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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**Abstract.** Let $H$ be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive self mapping. We introduce a new inertial-type iterative sequence which converges to a fixed point of T. Our results are new and complement several important existing results in the literature.

**Keywords:** norm preserving function; B-Banach space; B-norm preserving function; metric preserving function.

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## 1. Introduction

Let $C$ be a nonempty subset of a real Hilbert space H and $T : C \to C$, be a self map of C. We denote by $F(T) := \{x \in C : Tx = x\}$, the set of fixed points of T. Then T is called:

(i) Nonexpansive(see eg [1]) if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in H$.

(ii) Lipschitzian if there exists a real constant $L > 0$ such $||Tx - Ty|| \leq L||x - y||$ for all $x, y \in H$.

It is clear that every nonexpansive mapping is an $L-$Lipschitzian (and hence continuous) mapping with $L = 1$ .

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The construction of fixed points of nonexpansive mappings and their generalizations has generated much interest among researchers in the past seventy years. This is because fixed points of nonexpansive mappings find so many applications in practice in several areas including image recovery, computer tomography and again largely to its intimate connection with the class of accretive operators.

The accretive operators were introduced independently in 1967 by Browder [2] and Kato [3]. An early fundamental result in the theory of accretive operators, due to Browder [2], states that the initial value problem $\frac{du}{dt} + Au = 0,\ u(0) = u_0$ is solvable if A is Lipschitzian and accretive.

It is also well known (see e.g [4]) that if $A : C \to C$ is an accretive operator, then for any real constant $\lambda > 0$, the resolvent of A denoted by $J_A^\lambda$ and given by $J_A^\lambda := (I + \lambda A)^{-1}$, is a nonexpansive operator. It is easily verifiable that the fixed points of $J_A^\lambda$ are the zeros of A. Therefore, the study of fixed points of nonexpansive mappings brings together several areas of application unified by the theory of accretive operators.

In [5], W. R. Mann introduced an iteration scheme for construction of fixed points of nonexpansive mappings. The Mann's sequence is generated from an arbitrary $x_0 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{1}$$

where $\{\alpha_n\} \subset (0,1)$ is a real sequence. Under the condition $\sum \alpha_n(1 - \alpha_n) = \infty$, the author proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In the recent past, many authors (see eg [6], [7], [8]) have studied iteration schemes known as the 'inertial iteration schemes'. These schemes are characterized by the addition of a certain term known as the inertial term to well known convergent iteration schemes with the purpose of speeding them up. The inertial term usually takes the form $\alpha_n(x_n - x_{n-1})$, and satisfies certain conditions. Some of the inertial schemes can be found in [6],[7], where the authors stated and proved the following theorems:

**Theorem 1 [Alvarez and Attouch][6]:** Let $\{x_n\} \subset H$ be a sequence such that

$$x_{n+1} = J^A_{\lambda_n}(x_n + \alpha_n(x_n - x_{n-1})),\ n = 1,2,$$

where $A : H \to P(H)$ is a maximal monotone operator with $S := A^{-1}(0) \neq \emptyset$, and the parameters $\alpha_n$ and $\lambda_n$ satisfy

$(i)$ *there exists* $\lambda > 0$ *such that* $\forall n \in N,\ \lambda_n \geq \lambda$.

$(ii)$ *there exists* $\alpha \in [0,1]$ *such that* $\forall n \in N,\ 0 \leq \alpha_n \leq \alpha$.

If the following condition holds

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty,$$

then there exists $x^* \in S$ such that $\{x_n\}$ converges weakly to $x^*$ in H as $n \to \infty$.

In [7], Mainge introduced the classical inertial Mann algorithm

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = (1-\lambda_n)y_n + \lambda_n T y_n \end{cases}$$

for each $n \in N$. He proved that under the conditions:

$(i)$ $\alpha_n \in [0,\alpha]$ for each $n \geq 1$, where $\alpha \in [0,1)$;

$(ii)$ $\sum \alpha_n \|x_n - x_{n-1}\|^2 < \infty$

$(iii)$ $inf_{n\geq 1}\lambda_n > 0$ *and* $sup_{n\geq 1}\lambda_n < 1$,

$\{x_n\}$ converges weakly to a fixed point of T.

More recently in 2015, Bot and Csetnek (see [8]) replaced conditions (i) and (iii) by the conditions:

$(i')$ $\delta > \frac{\alpha^2(1+\alpha)+\alpha\sigma}{1-\alpha^2}$ $(iii')$ $0 < \lambda \leq \lambda_n \leq \theta := \frac{\delta-\alpha[\alpha(1+\alpha)+\alpha\delta+\sigma]}{\delta[1+\alpha(1+\alpha)+\alpha\delta+\sigma]}$

Although the authors did not explicitly list condition (ii) as one of the hypotheses, it was implicitly used in the convergence results. The authors explained that condition (ii) is easily implementable since at each stage, $x_n$ and $x_{n-1}$ are known, so that $\alpha_n$ can be chosen in such a manner that it is dominated by $\frac{1}{\|x_n-x_{n-1}\|^2}$ multiple of a summable sequence.

**Remark 1:** The inertial algorithms above are either defined on the whole Hilbert space H or on closed affine subsets of H. It is worth noting that many important results in Analysis hold on closed convex subsets of H. Moreover, the class of convex subsets of a Hilbert spaces is larger than the class of affine subsets. For example in H=R(reals), all the subsets $[c,d]$, for any $c,d \in R$ are convex but not affine. However, all affine subsets are convex.

Motivated by the above results and remark 1, we introduce an inertial-type Krasnosel'skii-Mann type algorithm on a convex subset of a real Hilbert space as follows:

Let C be a convex subset of a real Hilbert space, H. From arbitrary $x_0, x_1 \in C$ generate a sequence $\{x_n\}$ by rule

$$\begin{cases} y_n = x_n + t_n(x_{n-1} - x_n) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases} \tag{2}$$

Observe that our algorithm is well defined in C. We shall prove that our algorithm converges weakly to fixed points of nonexpansive mappings in C under mild conditions.

## 2. Preliminaries

Before we state and prove our main results, we give a definition and some lemmas which will be useful in the sequel:

**Definition:** Let $E$ be a Banach space. A mapping $T : D(T) \subseteq E \to E$ is said to be demiclosed at a point $u \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to $u$, then $Tx = u$ (see for example [9]).

**Lemma 1 (see[10]):** Let $H$ be a real Hilbert space and $C \subseteq H$ be a nonempty closed convex subset of H. If $T : C \to C$ is a nonexpansive mapping, then $I - T$ is demiclosed at zero.

**Lemma 2 (see[11]):** Let $H$ be a real Hilbert space. Then for all $x, y \in H$, and for any real number, $\lambda$, the following well-known identity holds:

$||(1-\lambda)x+\lambda y||^2 = (1-\lambda)||x||^2+\lambda||y||^2-\lambda(1-\lambda)||x-y||^2$

## 3. Main Results

We now state and prove our main results.

**Theorem 2:** Let C be a nonempty closed convex subset of a real Hilbert space H and let $T: C \to C$ be a nonexpansive mapping with a nonempty fixed points set, F(T). Then the inertial-type Krasnosel'skii-Mann-type sequence $\{x_n\}$ generated from arbitrary $x_0,\ x_1 \in C$ by

$$\begin{cases} y_n = x_n + t_n(x_{n-1}-x_n) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are real sequences in $(0,1)$ satisfying:
(a) $0 < r \leq \alpha_n \leq t < 1$ for some real constants $r, t \in (0,1)$,
(b) $t_n \leq d$ for some real constant $d \in (0,1)$,
converges weakly to an element of F(T).

**Proof:** Let $p \in F(T)$. Using (2) and lemma 2, we have

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\alpha_n)y_n + \alpha_n T y_n - p||^2 \\
&= ||(1-\alpha_n)(y_n-p) + \alpha_n(Ty_n-p)||^2 \\
&= (1-\alpha_n)||(y_n-p)||^2 + \alpha_n||(Ty_n-p)||^2 - \alpha_n(1-\alpha_n)||Ty_n-y_n||^2 \\
&\leq (1-\alpha_n)||(y_n-p)||^2 + \alpha_n||y_n-p||^2 - \alpha_n(1-\alpha_n)||Ty_n-y_n||^2 \\
&= ||y_n-p||^2 - \alpha_n(1-\alpha_n)||Ty_n-y_n||^2 \\
&= \|x_n+t_n(x_{n-1}-x_n)-p\|^2 - \alpha_n(1-\alpha_n)||Ty_n-y_n||^2 \\
&= \|(1-t_n)(x_n-p)+t_n(x_{n-1}-p)\|^2 - \alpha_n(1-\alpha_n)||Ty_n-y_n||^2 \\
&= (1-t_n)\|x_n-p\|^2 + t_n\|x_{n-1}-p\|^2 - t_n(1-t_n)\|x_n-x_{n-1}\|^2 \\
&\quad -\alpha_n(1-\alpha_n)||Ty_n-y_n||^2
\end{aligned}$$

$$
\begin{aligned}
&= ||x_n - p||^2 + t_n[||x_{n-1} - p||^2 - ||x_n - p||^2] - t_n(1-t_n)||x_n - x_{n-1}||^2 \\
&\quad -\alpha_n(1-\alpha_n)||Ty_n - y_n||^2 \qquad (3)
\end{aligned}
$$

We can estimate $||x_{n-1} - p||^2 - ||x_n - p||^2$ in two ways, viz: (i) $||x_{n-1} - p||^2 - ||x_n - p||^2 < 0$ or (ii) $||x_{n-1} - p||^2 - ||x_n - p||^2 \geq 0$. If (i) holds (i.e $||x_{n-1} - p||^2 < ||x_n - p||^2$), then substituting this is (3), we get $||x_{n+1} - p||^2 < ||x_n - p||^2$. This is an absurdity. Hence (ii) holds. This implies $||x_{n+1} - p||^2 \leq ||x_n - p||^2$. Therefore $\{||x_n - p||^2\}$ is a monotone decreasing sequence so that $\lim ||x_n - p||^2$ exists. From this, we have that $||x_n - p||$ and hence $||x_n||$ are bounded. Therefore, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to $z \in H$. Since H is an Opial space, a standard argument (see eg [12]) yields that $\{x_n\}$ converges weakly to z.

Since (ii) holds, $\lim ||x_n - p||^2$ exists, we have from (3) and (a) that

$$
\begin{aligned}
r(1-t)\lim ||Ty_n - y_n||^2 &\leq \lim \alpha_n(1-\alpha_n)||Ty_n - y_n||^2 \\
&\leq \lim[[||x_n - p||^2 - ||x_{n+1} - p||^2] + [||x_{n-1} - p||^2 - ||x_n - p||^2]] \to 0
\end{aligned}
$$

This implies

$$
\lim ||Ty_n - y_n|| = 0 \qquad (4)
$$

Again, from (3), since (ii) holds and $\lim ||x_n - p||^2$ exists, we have

$$
\begin{aligned}
\lim t_n(1-t_n)||x_n - x_{n-1}||^2 &\leq \lim[[||x_n - p||^2 - ||x_{n+1} - p||^2] \\
&\quad +[||x_{n-1} - p||^2 - ||x_n - p||^2]] \to 0 \qquad (5)
\end{aligned}
$$

Using (b) and (5), we have

$$
\begin{aligned}
||y_n - x_n|| &= t_n||x_n - x_{n-1}|| \\
&= [[\frac{t_n(1-t_n)}{1-t_n}||x_n - x_{n-1}||]^2]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad [[\frac{t_n(1-t_n)}{1-d}]^2\|x_n - x_{n-1}\|^2]^{\frac{1}{2}} \\
&= \quad [\frac{[t_n(1-t_n)]^2}{[1-d]^2}\|x_n - x_{n-1}\|^2]^{\frac{1}{2}} \\
&\leq \quad [\frac{t_n(1-t_n)}{[1-d]^2}\|x_n - x_{n-1}\|^2]^{\frac{1}{2}} \to 0 \qquad (6)
\end{aligned}
$$

Using the nonexpansiveness of T, (4) and (6), we have

$$
\begin{aligned}
||x_n - Tx_n|| &= ||x_n - Ty_n + Ty_n - y_n + y_n - Tx_n|| \\
&\leq ||x_n - y_n|| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\
&\leq 2||x_n - y_n|| + \|y_n - Ty_n\| \to 0
\end{aligned}
$$

Using this and the fact that $I-T$ is demiclosed at 0, we have that $z \in F(T)$. Setting $z = p$ above, our proof is complete.

**Remark 2:** The conditions imposed on our iteration parameters are simpler than those in the existing inertial algorithms in the literature. Moreover, our results hold in closed convex subsets (rather than in closed affine subsets as is obtainable in the literature).

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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