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NORM PRESERVING FUNCTION AND b-NORM PRESERVING FUNCTION

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Abstract. In this paper, the concept of norm preserving function and b-norm preserving function are presented. The properties and relation between norm preserving function and b-norm preserving function are discussed.

Keywords: norm preserving function; B-Banach space; B-norm preserving function; metric preserving function.

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1. Introduction

Metric space is a basic and important topological space. At the beginning of the 20th century, French mathematician M.R. Frechet found that many analytical results, from a more abstract point of view, involve the distance relationship between functions, thus abstracting the concept of metric space.

Subsequently, as an extension of metric space, the concept of b-metric space was given by Bakhtin [1]. In the framework of b-metric, we can deal with many analytical problems, and have made many important achievements. For example, Czerwink extended the famous Banach contraction mapping principle in b-metric spaces, M.B. Zada et al. [2] applied fixed point

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theorems in b-metric space to fractional differential equations. Typical b-metric spaces, such as $L^p[a,b](0<p<1)$ or $l^p(0<p<1)$, are important in theory and applications.

Since $L^p[a,b](0<p<1)$ and $l^p(0<p<1)$ not only have topological structure, but also have good linear structure, we have reasons to conduct a more detailed study on them. Recently in [3-4], Monica etc. introduced the concept of b-Banach space, which is an extension of Banach space, and a special case of b-metric space. We recognize that the most typical examples of this kind of spaces are $L^p[a,b](0<p<1)$ and $l^p(0<p<1)$.

In 1935, Wilson.W.A proposed a special class of functions, that is metric preserving functions. Later Bakhtin proposed the concept of b-metric preserving functions. These two kinds of functions are of great significance, Juza observed that real numbers can be topologized to obtain a class of incomplete discrete metric spaces by metric preserving functions and b-metric preserving functions. Recent discussions on metric preserving functions and b-metric preserving functions can be seen in [5-9] and references therein.

Inspired by these results on metric preserving function and b-metric preserving function, we introduce the concept norm preserving functions and b-norm preserving functions in this paper. The properties of norm preserving functions and b-norm preserving functions are presented and the relation of these two functions are discussed.

2. PRELIMINARIES

In this section, let's revisit the concept of normed linear space and b-normed linear space, in addition we also revisit some definitions related to them, such as b-metric space and metric preserving function, see in [2-9].

Definition 2.1 Let X be a vector space over a field K (either C or R). A functional $\|\cdot\|:X\to[0,+\infty)$ is said to be a norm if the following conditions are satisfied:

(1) $\|x\|\geq 0$, and $\|x\|=0$ if and only if $x=0$;

(2) $\|\lambda x\|=|\lambda|\,\|x\|$;

(3) $\|x+y\|\leq\|x\|+\|y\|$.

for all $x,y,z\in X$ and $\lambda\in K$. A pair $(X,\|\cdot\|)$ is called a normed linear space.

Example 2.2 Let $L^p[a,b](p>1)$ be the set of all real-valued Lebesgue measurable function x on $[a,b]$ for which $\int_{[a,b]} |x(t)|^p\,dt < \infty$. For each $x \in L^p[a,b]$, define

$$\|x\| = \left[\int_a^b |x(t)|^p\,\mathrm{d}t\right]^{\frac{1}{p}}.$$

Then $(L^p[a,b],\ \|.\|)\ \ (p>1)$ is a normed linear space.

Definition 2.3 Let X be a set and we define a functional $d : X \times X \to \mathbb{R}_+$ is called a metric if for any $x,y,z \in X$, the following conditions hold:

(1) $d(x,y) = 0$ if and only if $x = y = 0$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq d(x,z) + d(y,z)$.

Then (X,d) is called a metric space.

In a normed space $(X,\|\cdot\|)$, let $\forall x,y \in X, d(x,y) = \|x-y\|$, then d a distance induced by $\|\cdot\|$ and (X,d) as metric space.

Definition 2.4 Let (X,d) be a metric space. For each $f : [0,\infty) \to [0,\infty)$ define a function $d_f : X^2 \to [0,\infty)$ as follows $d_f(x,y) = f(d(x,y))$ for each $x,y \in X$. We call a function $f : [0,\infty) \to [0,\infty)$ metric preserving iff for each metric space (X,d) the function d_f is a metric on X.

Example 2.5 Define $f : [0,\infty) \to [0,\infty)$ by

$$f(x) = \begin{cases} 0 & \text{if } x\text{=0}, \\ 1 & \text{if } x \text{ is irrational}, \\ 2 & \text{otherwise}. \end{cases}$$

Then f is metric preserving.

Definition 2.6 Let X be a vector space over a field K (either C or R) and let $s \geq 1$ be a given real number. A functional $\|\cdot\| : X \to [0,+\infty)$ is said to be a b-norm if the following conditions are satisfied:

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\lambda x\| = |\lambda|\,\|x\|$;

(3) $\|x+y\| \leq s(\|x\| + \|y\|)$.

for all $x,y,z \in X$ and $\lambda \in K$. A pair $(X,\|\cdot\|)$ is called a b-normed linear space.

Example 2.7 Let $L^p[a,b](0<p<1)$ be the set of all real-valued Lebesgue measurable function x on $[a,b]$ for which $\int_{[a,b]}|x(t)|^p\,dt<\infty$. For each $x\in L^p[a,b]$, define

$$\|x\|=\left[\int_a^b|x(t)|^p\,\mathrm{d}t\right]^{\frac{1}{p}}.$$

Then $(L^p[a,b],\ \|.\|)\ \ (0<p<1)$ is a b-normed linear space with $s=2^{\frac{1}{p}-1}$.

Definition 2.8 : Let X be a set and we define a functional $d:X\times X\to\mathbb{R}_+$ is called a b-metric if for any $x,y,z\in X$, and $s\geq 1$, the following conditions hold:

(1) $d(x,y)=0$ if and only if $x=y=0$;

(2) $d(x,y)=d(y,x)$;

(3) $d(x,y)\leq s[d(x,z)+d(y,z)]$.

Then (X,d) is called a b-metric space.

Definition 2.9 Let (X,d) be a b-metric space. For each $f:[0,\infty)\to[0,\infty)$ define a function $d_f:X^2\to[0,\infty)$ as follows $d_f(x,y)=f(d(x,y))$ for each $x,y\in X$. We call a function $f:[0,\infty)\to[0,\infty)$ b-metric preserving iff for each b-metric space (X,d) the function d_f is a b-metric on X.

Example 2.10 Define $f:[0,\infty)\to[0,\infty)$ by

$$f(x)=x^2$$

Then f is b-metric preserving.

Also, we know that f defined in Example 2.10 is not metric preserving.

3. Norm Preserving Function

Definition 3.1 $f:[0,\infty)\to[0,\infty)$ is called a norm preserving function if for each normed linear space $(X,\|\cdot\|)$, $f(\|\cdot\|)$ is a norm on X.

Theorem 3.2 A norm preserving function $f:[0,\infty)\to[0,\infty)$ has the following properties,

(1) positive homogeneous: $f(\lambda a)=\lambda f(a)$ for $\lambda\geq 0, a\geq 0$;

(2) subadditivity: $f(a+b)\leq f(a)+f(b)$ for $a,b\geq 0$;

(3) positive definiteness: $f(a)\geq 0$ equality holds if and only if $a=0$.

Corollary 3.3 All norm preserving function $f:[0,\infty)\to[0,\infty)$ is convex.

Proof. For all norm preserving function $f:[0,\infty)\to[0,\infty)$, $\lambda\geq 0$, $a,b\geq 0$ by the Theorem 3.2(2), we have

$$f[\lambda a+(1-\lambda)b]\leq f(\lambda a)+f[(1-\lambda)b].$$

According to Theorem 3.2(1) , we have

$$f(\lambda a)+f[(1-\lambda)b]\leq \lambda f(a)+(1-\lambda)f(b).$$

Therefore f is convex.

Example 3.4 Assume $a,b\in R$ satisfy $a<0<b$, $A=[a,b]$. Then the Minkowski function of A is

$$p(x)=\inf\{\lambda>0|\frac{x}{\lambda}\in A\}=\begin{cases}\frac{x}{a} & \text{if } x\leq 0,\\ \frac{x}{b} & \text{if } x\geq 0.\end{cases}$$

It easy to verify that $p(x)$ is a norm preserving function.

Especially, when $a=-1,b=1$, $p(x)=|x|$.

Definition 3.5 If a nonnegative real number triple (a,b,c) satisfies $a\leq b+c, b\leq a+c$ and $c\leq a+b$, then (a,b,c) is called a triangle triple, and Δ is the set of all triangle triples.

Definition 3.6 For function $f:[0,\infty)\to[0,\infty)$, if $\exists a>0$, such that for $\forall x>0$ we have $f(x)\in[a,2a]$, so f is said to be tightly bounded.

In what follows, we'll present some necessary and sufficient conditions for norm preserving functions.

Theorem 3.7 If $f:[0,\infty)\to[0,\infty)$ is positive homogeneous, subadditivity and positive definite. Then the following conclusions are equivalent

(1) f is a norm preserving function.

(2) For $\forall(a,b,c)\in\Delta$, we have $(f(a),f(b),f(c))\in\Delta$.

Proof.

(1)$\Rightarrow$(2) Because of f is norm preserving function, $\|\cdot\|$ and $f(\|\cdot\|)$ are norm. According to the triangle inequality of norm, $\exists x,y,z\in X$ such that

$$\|x\|+\|y\|\geq\|x+y\|, f(\|x\|)+f(\|y\|)\geq f(\|x+y\|),$$

Choose $a = \|x\|, b = \|y\|, c = \|x+y\|$, we obtain

$$f(a)+f(b) \geq f(c),$$

that is $(f(a), f(b), f(c)) \in \Delta$.

(2)⇒(1) For $\forall (a,b,c) \in \Delta$, we have $(f(a), f(b), f(c)) \in \Delta$. Choose $a = \|x\|, b = \|y\|, c = \|x+y\|$, we have $f(\|x\|)+f(\|y\|) \geq f(\|x+y\|)$, i.e., $f(\|\cdot\|)$ satisfies the norm triangle inequality. Because f is positive definite and positive homogeneous, then $f(\|\cdot\|)$ is also positive definite and positive homogeneous.

To sum up, f is a norm preserving function.

Theorem 3.8 If $f : [0,\infty) \to [0,\infty)$ is positive definite, positive homogeneous, sub-additive and increasing, then f is a norm preserving function.

Proof. Firstly, for $\forall \lambda > 0$, by the positive homogeneous of f we have

$$f(\|\lambda x\|) = f(\lambda \|x\|) = \lambda f(\|x\|),$$

so $f(\|\cdot\|)$ satisfied the positive homogeneous.

Secondly, let $a = \|x\|, b = \|y\|, c = \|x+y\|$, then from the subadditivity of f we know that $f(a)+f(b) \geq f(a+b)$ is true, notice that $c < a+b$ then according to the incremental of f, we have $f(a+b) \geq f(c)$, such that

$$f(\|x\|)+f(\|y\|) \geq f(\|x+y\|),$$

Finally, by the definition of f and $f(\|0\|) = f(0) = 0$, we know $f(\|\cdot\|)$ is positive definite.

In conclusion, f is a norm preserving function.

Theorem 3.9 If $f : [0,\infty) \to [0,\infty)$ is positive definite, positive homogeneous, tightly bounded, then f is a norm preserving function.

Proof. By the tightly boundedness of f, we know that $\exists a > 0$, such that for $\forall x \geq 0$ have $f(x) \in [a, 2a]$. So for a triplet (a,a,a) we have

$$f(a) \leq 2a = a+a = f(b)+f(c),$$

such that

$$(f(a), f(b), f(c)) \in \Delta.$$

According to theorem 3.8, f is a norm preserving function.

4. b-NORM PRESERVING FUNCTION

In this section, we'll establish the definition of b-norm preserving function, and discuss some properties of b norm preserving function.

Definition 4.1 Let $f : [0,\infty) \to [0,\infty)$. f is called a b-norm preserving function if for each b-normed linear space $(X, \|\cdot\|)$, $f(\|\cdot\|)$ is a b-norm on X.

To prove the main results in this section, the following Lemma is crucial.

Lemma 4.2 A b-norm preserving function $f : [0,\infty) \to [0,\infty)$ has the following properties,

(1) positive homogeneous, $f(\lambda a) = \lambda f(a)$ for $\lambda, a \geq 0$;

(2) quasi-subadditivity, $f(a+b) \leq s[f(a)+f(b)]$ for $a,b \geq 0$, $s \geq 1$;

(3) positive definiteness, $f(a) \geq 0$ for $a \geq 0$ and equality holds if and only if $a = 0$.

Definition 4.3 If a nonnegative real number triple (a,b,c) satisfies $\exists s \geq 1$, such that we have $a \leq s(b+c), b \leq s(a+c)$ and $c \leq s(a+b)$, then (a,b,c) is called a quasi triangle triple, and Δ_s is the set of all quasi triangle triples.

Theorem 4.4 If $f : [0,\infty) \to [0,\infty)$ is positive homogeneous, quasi-subadditivity and positive definite. Then the following conditions are equivalent

(1) f is a b-norm preserving function.

(2) For $\forall (a,b,c) \in \Delta_s$, we have $(f(a), f(b), f(c)) \in \Delta_s$.

Theorem 4.5 If $f : [0,\infty) \to [0,\infty)$ is positive definite, positive homogeneous, quasi-subadditive and increasing, then f is a b-norm preserving function.

Theorem 4.6 If $f : [0,\infty) \to [0,\infty)$ is a norm preserving function, then f is a b-norm preserving function.

Proof. Let $\|\cdot\|$ be a b-norm. Since f is a norm preserving function, $f(\|\cdot\|)$ satisfies (1) and (2) of the definition of b-norm.

Let $a = \|x+y\|, b = \|x\|, c = \|y\|$, we have $a \leq s(b+c)$. Take $n > s$, we have $a \leq n(b+c) = nb+nc$, so

$$(a, nb+nc, nb+nc) \in \Delta.$$

Therefore,

$$f(a) \leq f(nb+nc)+f(nb+nc) = 2f(nb+nc).$$

Moreover, due to the subadditivity and positive homogeneity of f, we have

$$2f(nb+nc) \leq 2[f(nb)+f(nc)] = 2n[f(b)+f(c)].$$

Let $s' = 2n$, then $f(\|x+y\|) \leq s'[f(\|x\|)+f(\|y\|)]$. Hence $f(\|\cdot\|)$ satisfied (3) of the definition of b-norm, i.e., f is a b-norm preserving function.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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