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EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR THIRD ORDER BOUNDARY VALUE PROBLEM

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Abstract. This work concerned with the following third-order three point boundary value problem (BVP):

$$(P1)\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0,\ 0 < t < 1 \\ u(0) = \alpha u(1), u'(1) = \beta u'(\eta), u'(0) = 0, \end{cases}$$

where $\eta \in (0,1)$, α, $\beta \in \mathbb{R}$, f is a given function. Our main objective is to investigate the existence, uniqueness and existence of positive solutions for the boundary value problem (P1), by using Banach contraction principle, Leray Schauder nonlinear alternative, properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cone, in the case where the nonlinearity f is either superlinear or sublinear.

Keywords: Three-point boundary value problem, positive solution, Leray Schauder nonlinear alternative, Banach contraction principle, Green function, Guo-Krasnosel'skii fixed point theorem.

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1. Introduction

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Boundary value problems for differential ordinary equation of third order possess wide application in different areas of sciences such as in mechanics, physics, biology,... They arise in modeling many phenomenes like draining or coating fluid flow problems, nonlinear diffusion, thermal ignition of gases,...We refer the reader to [3,5,14].

Recently several papers appeared on third order boundary value problems, we can cite the paper of Anderson and Davis [2], Graef and Yang [6,7], Guezane-Lakoud and Khaldi [8], Sun [13], Guo and Sun [10] and Yang [15] and the excellent survey of R. Ma [11] and Agarwal et al [1] and the references therein for related results.

However, fewer results on three-point boundary value problems of third order ordinary differential equations can be found in the literature involving the polynomial growth condition on f of type:

$$|f(t,x,\overline{x})| \leq k(t)\,|x|^{p} + g(t)\,|\overline{x}|^{q} + h(t)\,, \quad (t,x,\overline{x}) \in [0,1] \times \mathbb{R}^{2},$$

$p,q > 0,\ k,g,h \in L^{1}([0,1],\mathbb{R}_{+})$, (see section 3, Theorem 3.1). This condition and the presence of the derivative u' in the expression of f leads to extra difficulties. No contributions exist, as far as we know, concerning the existence of positive solutions for the boundary value problem (P1).

In [5], Graef et al gave sufficient conditions for the existence and nonexistence of positive solutions for the following problem

$$\begin{aligned} u'''(t) + g(t)f(u(t)) &= 0,\ 0 < t < 1 \\ u(0) &= u(p) - u(1) = u''(0) = 0, \end{aligned}$$

In [8] Guezane-Lakoud et al investigated, by using Leray Schauder nonlinear alternative, the existence of a nontrivial solution for the boundary value problem

$$u'''(t) + f(t,u(t)) = 0,\ 0 < t < 1$$
$$u(0) = \alpha u'(0)\,,\ u(1) = \beta u'(\eta)\,,\ u'(1) = 0,$$

In [12],R. Ma et al considered the fourth order right focal two point boundary value problem

$$u''''(t) + f(t, u(t), u'(t)) = 0, \ 0 < t < 1$$
$$u(0) = u'(0) = 0 = u"(1) = u'''(1) = 0,$$

and show the existence and multiplicity of positive solutions by using a fixed point theorem in cones. This paper is organized as follows, in the next section we cite some definitions and Lemmas needed in our proofs. Section 3 treats the existence and uniqueness of solution by using Banach contraction principle, Leray Schauder nonlinear alternative. Section 4 is devoted to prove the existence of positive solutions with the help of Guo-Krasnoselskii theorem, then we give some examples illustrating the previous results.

2. Preliminaries

In this section we present some definitions lemmas and theorems we need in the proof of the main results. Let $E = C^1([0,1], \mathbb{R})$, with the norm $||y||_1 = ||y|| + ||y'||$, where $||.||$ denotes the norm in $C([0,1], \mathbb{R})$ defined by $||y|| = \max_{t\in[0,1]} |y(t)|$.
$E^+ = \{y \in C^1([0,1], \mathbb{R}), y(t) \geq 0, \forall t \in [0,1]\}$. We assume that
$\zeta = (1-\alpha)(1-\beta\eta) \neq 0$. The norm in $L_1[0,1]$ is denoted by $\|y\|_{L_1[0,1]} = \int_0^1 |y(t)|\, dt$ for all $y \in L_1[0,1]$. Now we start by solving an auxiliary problem.

Lemma 2.1. *Let $y \in E$. The problem*

$$(P_2) \begin{cases} u'''(t) + y(t) = 0, 0 < t < 1 \\ u(0) = \alpha u(1), u'(1) = \beta u'(\eta), u'(0) = 0. \end{cases}$$

has a unique solution

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds - \frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta-s)\, y(s)\, ds \\ &+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)+\alpha\beta\eta(1-s)+\alpha s\right) y(s)ds \end{aligned} \tag{2.1}$$

Proof. Rewriting the differential equation as $u'''(t) = -y(t)$ and integrating three times, we obtain $u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C$, the constants A, B and C are given by the three point boundary conditions.

Definition 2.2. *A function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is called carathéodory if*

(i)The map $t \to f(t,x,y)$ is measurable for all $x, y \in \mathbb{R}$.

(ii) The map $(x,y) \to f(t,x,y)$ is continuous on $\mathbb{R}^2$ for almost all $t \in [0,1]$.

To prove the existence of nontrivial solution we apply the Leray Schauder nonlinear alternative:

Lemma 2.3. *[4]. Let F be a Banach space and Ωa bounded open subset of F $0 \in \Omega$. $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$ $\lambda > 1$ such that $T(x) = \lambda x$ or there exists a fixed point $x^* \in \overline{\Omega}$.*

We recall the definition of positive solution:

Definition 2.4. *A function $u(t)$ is called positive solution of (P1) if $u(t) \geq 0, \forall t \in [0,1]$.*

We expose the well known Guo-Krasnosel'skii fixed point Theorem in cone:

Theorem 2.5. *[9] Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1 \subset \Omega_2}$ and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$$

be a completely continuous operator such that

(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, and $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$; or

(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, and $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

3. Existence and uniqueness results

First we investigate the existence of nontrivial solution by employing Lemma 2.3.

Theorem 3.1. *Assume that f is Carathéodory function, $f(t,0,0) \neq 0$ and there exist nonnegative functions $k, g, h \in L^1([0,1], \mathbb{R}_+)$ such that*

$$|f(t,x,\overline{x})| \leq k(t)|x|^p + g(t)|\overline{x}|^q + h(t), \quad (t,x,\overline{x}) \in [0,1] \times \mathbb{R}^2, \tag{3.1}$$

$$0 < \left(2 + \frac{(|\beta|+1)(2+3|\alpha|) + |\alpha\beta|}{|\zeta|}\right)\left(\|k\|_{L_1[0,1]} + \|g\|_{L_1[0,1]}\right) < \frac{1}{2} \tag{3.2}$$

$$\left(2 + \frac{(|\beta|+1)(2+3|\alpha|) + |\alpha\beta|}{|\zeta|}\right)\|h\|_{L_1[0,1]} < \frac{1}{2} \tag{3.3}$$

Then the BVP (P1)has at least one nontrivial solution $u^* \in E$.

Proof. Define the integral operator $T : E \to E$ by

$$\begin{aligned}
Tu(t) \;=\; & -\frac{1}{2}\int_0^t (t-s)^2 f(s,u(s),u'(s))\,ds \\
& -\frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta-s) f(s,u(s),u'(s))\,ds \\
& +\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)+\alpha\beta\eta(1-s)+\alpha s\right) f(s,u(s),u'(s))\,ds,
\end{aligned}$$

Set

$$M = \left(2 + \frac{(|\beta|+1)(2+3|\alpha|) + |\alpha\beta|}{|\zeta|}\right)\left(\|k\|_{L_1[0,1]} + \|g\|_{L_1[0,1]}\right)$$

and

$$N = \left(2 + \frac{(|\beta|+1)(2+3|\alpha|) + |\alpha\beta|}{|\zeta|}\right)\|h\|_{L_1[0,1]}.$$

By hypothesis (3.2) we know that $0 < M < \frac{1}{2}$. Since $f(t,0,0) \neq 0$, then there exists an interval $[\sigma, \tau] \subset [0,1]$ such that $\min\limits_{\sigma \leq t \leq r} |f(t,0,0)| > 0$ hence $N > 0$. Set $\|u\|_1^\sigma = \max(\|u\|_1^p, \|u\|_1^q)$, ($\sigma = p$ or $\sigma = q$), n the entire part of σ and $m = \left(\frac{N}{M}\right)^{\frac{1}{n}}$. Define the bounded open set Ω by $\Omega = \{u \in C[0,1] : \|u\|_1 < m\}$.

First, we prove that T is completely continuous operator in Ω.

(i) T is continuous.

Indeed, let (u_n) be a sequence that converges to u in E. Then

$$|Tu_n(t) - Tu(t)| \tag{3.4}$$

$$\begin{aligned}
&\leq \frac{1}{2}\int_0^1 (1-s)^2 \left|f\left(s,u_n\left(s\right),u_n'\left(s\right)\right)-f\left(s,u\left(s\right),u'\left(s\right)\right)\right| ds \\
&+\frac{1}{2}\left|\frac{\beta}{\zeta}\right|\left(1+2\left|\alpha\right|\right)\int_0^1 (1-s)\left|f\left(s,u_n\left(s\right),u_n'\left(s\right)\right)-f\left(s,u\left(s\right),u'\left(s\right)\right)\right| ds \\
&+\frac{1}{2\left|\zeta\right|}\int_0^1 \left(1+2\left|\alpha\right|+\left|\alpha\beta\right|(1-s)\right) \\
&\left|f\left(s,u_n\left(s\right),u_n'\left(s\right)\right)-f\left(s,u\left(s\right),u'\left(s\right)\right)\right| ds \\
&\leq \left(1+\frac{\left(\left|\beta\right|+1\right)\left(1+2\left|\alpha\right|\right)+\left|\alpha\beta\right|}{\left|\zeta\right|}\right)\left\|f\left(.,u_n\left(.\right),u_n'\left(.\right)\right)-f\left(.,u\left(.\right),u'\left(.\right)\right)\right\|
\end{aligned}$$

Moreover, we have

$$|T'u_n(t)-T'u(t)| \leq \tag{3.5}$$

$$\left(1+\frac{\left(\left|\beta\right|+1\right)\left(1+\left|\alpha\right|\right)}{\left|\zeta\right|}\right)\times\left\|f\left(.,u_n\left(.\right),u_n'\left(.\right)\right)-f\left(.,u\left(.\right),u'\left(.\right)\right)\right\|$$

Consequently,

$$\begin{aligned}
\left\|Tu_n-Tu\right\|_1 &\leq \left(2+\frac{\left(\left|\beta\right|+1\right)\left(2+3\left|\alpha\right|\right)+\left|\alpha\beta\right|}{\left|\zeta\right|}\right) \\
&\times\left\|f\left(.,u_n\left(.\right),u_n'\left(.\right)\right)-f\left(.,u\left(.\right),u'\left(.\right)\right)\right\|.
\end{aligned}$$

Condition (ii) on f implies $\left\|Tu_n-Tu\right\|_1\to 0$, as $n\to\infty$.

(ii)Let $B_r=\{u\in E;\left\|u\right\|_1\leq r\}$ be a bounded subset. We shall prove that $T\left(\Omega\cap B_r\right)$ is relatively compact:

a)For some $u\in\Omega\cap B_r$ and using (3.1) we have

$$\begin{aligned}
\left\|Tu\right\|_1 &\leq \left(2+\frac{\left(\left|\beta\right|+1\right)\left(2+3\left|\alpha\right|\right)+\left|\alpha\beta\right|}{\left|\zeta\right|}\right) \\
&\left(\left\|u\right\|_1^p\left\|k\right\|_{L_1[0,1]}+\left\|u\right\|_1^q\left\|g\right\|_{L_1[0,1]}\right)+N \\
&\leq M\max(\left\|u\right\|_1^p,\left\|u\right\|_1^q)+N,
\end{aligned}$$

then

$$\left\|Tu\right\|_1\leq Mr^\sigma+N$$

yielding that $T\left(\Omega\cap B_r\right)$ is uniformly bounded.

b) $T(\Omega \cap B_r)$ is equicontinuous. Indeed for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in \Omega$, we have by applying (3.1)

$$\begin{aligned}
&|Tu(t_1) - Tu(t_2)| \\
\leq\ & \frac{1}{2}\int_0^{t_1} \left((t_2 - s)^2 - (t_1 - s)^2\right) |f(s, u(s), u'(s))|\, ds \\
&+ \frac{1}{2}\int_{t_1}^{t_2} (t_2 - s)^2 |f(s, u(s), u'(s))|\, ds \\
&+ \frac{|\beta(1-\alpha)|}{2|\zeta|}\left(t_2^2 - t_1^2\right) \int_0^{\eta} (\eta - s) |f(s, u(s), u'(s))|\, ds \\
&+ \frac{(1-\alpha)\left(t_1^2 - t_2^2\right)}{2|\zeta|} \int_0^1 (1 - s) |f(s, u(s), u'(s))|\, ds,
\end{aligned}$$

Let us consider the function $\Phi(x) = x^2 - 2x$, we see that Φ is decreasing on $[0,1]$, consequently

$(t_2 - s)^2 - (t_1 - s)^2 \leq 2(t_2 - t_1)$, for $s \in (0, t_1)$ from which we deduce

$$\begin{aligned}
&|Tu(t_1) - Tu(t_2)| \\
\leq\ & 2(t_2 - t_1) \int_0^1 |f(s, u(s), u'(s))|\, ds \\
&+ \int_{t_1}^{t_2} |f(s, u(s), u'(s))|\, ds \\
&+ \frac{|\beta + 1|\,|(1-\alpha)|}{|\zeta|}\left(t_2^2 - t_1^2\right) \int_0^1 |f(s, u(s), u'(s))|\, ds
\end{aligned}$$

when $t_1 \to t_2$, then $|Tu(t_1) - Tu(t_2)|$ tends to 0, consequently $T(\Omega \cap B_r)$ is equicontinuous. From Arzela-Ascoli Theorem we deduce that T is completely continuous operator.

Second, we apply Leray Schauder nonlinear alternative for $T : \overline{\Omega} \to E$. Assume that $u \in \partial\Omega$, $\lambda > 1$ such $Tu = \lambda u$. First we have

$$\begin{aligned}
|Tu(t)| \leq &\left(1 + \frac{(|\beta| + 1)(1 + 2|\alpha|) + |\alpha\beta|}{|\zeta|}\right) \times \\
&\left[\max |u(t)|^p \|k\|_{L_1[0,1]} + \max |u'(t)|^q \|g\|_{L_1[0,1]} + \|h\|_{L_1[0,1]}\right]
\end{aligned}$$

$$\leq \left(1 + \frac{(|\beta| + 1)(1 + 2|\alpha|) + |\alpha\beta|}{|\zeta|}\right) \times \tag{3.6}$$

$$\left[\|u\|_1^{\sigma}\left(\|k\|_{L_1[0,1]}+\|g\|_{L_1[0,1]}\right)+\|h\|_{L_1[0,1]}\right]$$

and

$$|T'u(t)| \le \left(1+\frac{(|\beta|+1)(1+|\alpha|)}{|\zeta|}\right) \tag{3.7}$$

$$\left[||u||_1^{\sigma}\left(\|k\|_{L_1[0,1]}+\|g\|_{L_1[0,1]}\right)+\|h\|_{L_1[0,1]}\right]$$

From (3.6) and (3.7) we get

$$\begin{aligned}
\lambda m &= \lambda\,||u||_1 = ||Tu||_1 = \max_{0\le t\le 1}|(Tu)(t)| + \max_{0\le t\le 1}|(T'u)(t)| \le \\
&\left(2+\frac{(|\beta|+1)(2+3|\alpha|)+|\alpha\beta|}{|\zeta|}\right) \\
&\left[||u||_1^{\sigma}\left(\|k\|_{L_1[0,1]}+\|g\|_{L_1[0,1]}\right)+\|h\|_{L_1[0,1]}\right] \\
&= M\,\|u\|_1^{\sigma}+N.
\end{aligned}$$

Then

$$\begin{aligned}
\lambda &\le M^{\frac{((n+1)-\sigma)}{n}}N^{\frac{(\sigma-1)}{n}}+M^{\frac{1}{n}}N^{1-\frac{1}{n}} \\
&< \left(\frac{1}{2}\right)^{\frac{((n+1)-\sigma)}{n}}\left(\frac{1}{2}\right)^{\frac{(\sigma-1)}{n}}+\left(\frac{1}{2}\right)^{\frac{1}{n}}\left(\frac{1}{2}\right)^{1-\frac{1}{n}}=1,
\end{aligned}$$

consequently $\lambda<1$, this contradicts the fact that $\lambda>1$. By Lemma 2.3 we conclude that the operator T has a fixed point $u^*\in\overline{\Omega}_2$ and then the BVP (P1) has a nontrivial solution $u^*\in E$. The proof is complete.

The following Theorem deals with the uniqueness of solution

Theorem 3.2. *Assume that f is carathéodory function and there exists nonnegative functions $k_1,k_2\in L^1([0,1],\mathbb{R}_+)$ such that*

$$|f(t,x,\overline{x})-f(t,y,\overline{y})|\le k_1(t)\,|x-y|+k_2(t)\,|\overline{x}-\overline{y}|\,,\forall x,y\in\mathbb{R},t\in[0,1]\,. \tag{3.8}$$

and

$$\left(2+\frac{(|\beta|+1)(2+3|\alpha|)+|\alpha\beta|}{|\zeta|}\right)\left(\|k_1\|_{L_1[0,1]}+\|k_2\|_{L_1[0,1]}\right)<1 \tag{3.9}$$

then the BVP (P1) has a unique solution u in E.

Proof. We shall prove that T is a contraction. Let $u, v \in E$, then

$$\begin{aligned}
& |Tu(t) - Tv(t)| \\
\leq\ & \frac{1}{2}\left(1 + \frac{(|\beta| + 1)(1 + 2|\alpha|) + |\alpha\beta|}{|\zeta|}\right) \\
& \int_0^1 |f(s, u(s), u'(s)) - f(s, v(s), v'(s))|\, ds
\end{aligned}$$

Using (3.8) we obtain

$$|Tu(t) - Tv(t)| \tag{3.10}$$

$$\begin{aligned}
\leq\ & \left(1 + \frac{(|\beta| + 1)(1 + 2|\alpha|) + |\alpha\beta|}{|\zeta|}\right) \\
& \max|u(t) - v(t)| \int_0^1 k_1(s)ds + \max|u'(t) - v'(t)| \int_0^1 k_2(s)ds\Big] \\
\leq\ & \left(1 + \frac{(|\beta| + 1)(1 + 2|\alpha|) + |\alpha\beta|}{|\zeta|}\right) \|u - v\|_1 \int_0^1 (k_1(s) + k_2(s))\, ds
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
T'u(t) &= -\int_0^t (t - s) f(s, u(s), u'(s))\, ds \\
&\quad - \frac{\beta t(1 - \alpha)}{\zeta} \int_0^\eta (\eta - s) f(s, u(s), u'(s))\, ds \\
&\quad + \frac{t(1 - \alpha)}{\zeta} \int_0^1 (1 - s) f(s, u(s), u'(s))\, ds, \forall t \in [0, 1]
\end{aligned}$$

then

$$|T'u(t) - T'v(t)| \tag{3.11}$$

$$\begin{aligned}
\leq\ & \left(1 + \frac{(|\beta| + 1)(1 + |\alpha|)}{|\zeta|}\right) \\
& \left[\int_0^1 k_1(s)|u(s) - v(s)|\, ds\right. \\
& \left. + \int_0^1 k_2(s)|u'(s) - v'(s)|\, ds\right] \\
\leq\ & \left(1 + \frac{(|\beta| + 1)(1 + |\alpha|)}{|\zeta|}\right) \|u - v\|_1 \int_0^1 (k_1(s) + k_2(s))\, ds
\end{aligned}$$

adding (3.10) and (3.11), then taking the supremum after applying (3.9) it yields $\|Tu-Tv\|_1 < \|u-v\|_1$. Consequently T is a contraction, so, it has a unique fixed point which is the unique solution of the BVP (P1). The proof is complete.

4. Existence of positive solutions

In this section we investigate the positivity of solution for the boundary value problem (P1), for this rewrite the operator T as

$$Tu(t) = \int_0^1 G_1(t,s) f(s,u(s),u'(s))ds \tag{4.1}$$

$$+\frac{\beta t^2}{2(1-\beta\eta)} \int_0^1 G_2(\eta,s) f(s,u(s),u'(s))ds$$
$$+\frac{\alpha}{2\zeta} \int_0^1 G_3(\eta,s) f(s,u(s),u'(s))ds$$

where $G_i(t,s)$, $i=1,2,3$, are defined respectively by

$$G_1(t,s) = \begin{cases} \frac{1}{2}\left[t^2(1-s)-(t-s)^2\right], \ s \le t \\ \frac{1}{2}t^2(1-s), \ t \le s \end{cases} \tag{4.2}$$

$$\mathrm{G}_2(t,s) = \frac{\partial G_1(t,s)}{\partial t} \begin{cases} s(1-t), \ s \le t \\ t(1-s), \ t \le s \end{cases}$$
$$G_3(t,s) = \begin{cases} \beta s(1-t) + s(1-s)(1-\beta t), \ s \le t \\ \beta t(1-s)^2 + s(1-s), \ t \le s. \end{cases}$$

Now we give the properties of the functions $G_i(t,s)$:

Lemma 4.1. *If $\alpha > 1$ and $\beta < \frac{1}{\eta}$, then the functions $G_i(t,s)$, have the following properties*

i) $G_i(t,s) \in C([0,1]\times[0,1])$, $i=1,2,3$, $G_i(t,s) \ge 0$, $i=1,2$, and $G_3(t,s) \ge 0$ for all $t,s \in]0,1[$.

ii) If $t,s \in [\tau_1,\tau_2]$, $0<\tau_1<\tau_2<1$, then

$$\tau_1^2 G_1(1,s) \le G_1(t,s) \le G_1(1,s) \tag{4.3}$$

$$\tau G_2(s,s) \le G_2(t,s) \le \frac{1}{\tau_1} G_2(s,s) \tag{4.4}$$

where $\tau = \max(\tau_1, (1-\tau_2))$.

Proof. It is obvious that $G_i(t,s) \in C\left([0,1] \times [0,1]\right)$, moreover we have for all $t,s \in \left]0,1\right[$, $G_1(t,s)$ and $G_2(t,s)$ are nonnegative and if $\alpha < 1$ and $\beta < \frac{1}{\eta}$ then $G_3(t,s)$ is nonnegative.

ii) Let $t,s \in [\tau_1, \tau_2], 0 < \tau_1 < \tau_2 < 1$, it is easy to see that $G_1(1,s) \neq 0$ and $G_2(s,s) \neq 0$.

If $0 < \tau_1 \leq s \leq t \leq \tau_2 < 1$, then

$$\begin{aligned}
G_1(t,s) &= \frac{s}{2}\left[t^2\left(1-s\right) - \left(t-s\right)^2\right] \\
&= \frac{s}{2}\left[\left(1-s\right) - \left(t-1\right)^2\right)\right] \leq \frac{s\left(1-s\right)}{2} = G_1(1,s) \\
G_1(t,s) &= \frac{s}{2}\left[t\left(1-s\right) + \left(t-s\right)\left(1-t\right)\right] \\
&\geq t\frac{s\left(1-s\right)}{2} \geq \tau_1^2 G_1(1,s)
\end{aligned}$$

and if $0 < \tau_1 \leq t \leq s \leq \tau_2 < 1$, then

$$\begin{aligned}
G_1(t,s) &= \frac{1}{2}t^2\left(1-s\right) \leq \frac{1}{2}s\left(1-s\right) = G_1(1,s), \\
G_1(t,s) &= \frac{1}{2}t^2\left(1-s\right) = \frac{1}{2}\left[t^2 s\left(1-s\right) + t^2\left(1-s\right)\right] \\
&\geq \frac{t^2 s\left(1-s\right)}{2} \geq \tau_1^2 G_1(1,s),
\end{aligned}$$

Now we look for bounds for $G_2(t,s)$

$$\begin{aligned}
\frac{G_2(t,s)}{G_2(s,s)} &= \frac{(1-t)}{(1-s)} \leq 1 \leq \frac{1}{\tau_1} \\
\frac{G_2(t,s)}{G_2(s,s)} &\geq \left(1-\tau_2\right), 0 < \tau_1 \leq s \leq t \leq \tau_2 < 1,
\end{aligned}$$

and

$$\begin{aligned}
\frac{G_2(t,s)}{G_2(s,s)} &= \frac{t}{s} \leq \frac{1}{\tau_1}, \\
\frac{G_2(t,s)}{G_2(s,s)} &\geq \tau_1, 0 < \tau_1 \leq t \leq s \leq \tau_2 < 1,
\end{aligned}$$

since $G_2(s,s)$ are nonnegative then

$$\tau G_2(s,s) \leq G_2(t,s) \leq \frac{1}{\tau_1} G_2(s,s).$$

The proof is complete.

We make the following hypotheses:

H1) $f(t,u,v) = a(t)f_1(u,v)$ where $a \in C((0,1),(0,\infty))$ and $f_1 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.

H2) $0 < \int_0^1 G_1(s,s)a(s)ds < \infty$.

Lemma 4.2. *If $u \in E^+$, $\alpha > 1$, $0 < \beta < \frac{1}{\eta}$, then the solution of the BVP (P1) is positive and satisfies*

$$\min_{t\in(\tau_1,\tau_2)} (u(t) + u'(t)) \geq \delta \|u\|_1$$

where $\delta = \max\left(\tau_1^2, \left(\tau + \frac{\beta\tau_1}{(1-\beta\eta)}\right)\left(\frac{1}{\tau_1} + \frac{\beta}{(1-\beta\eta)}\right)^{-1}\right)$.

Proof. From hypothesis H1, we can write

$$u(t) = \int_0^1 G_1(t,s)a(s)f_1(u(s),u'(s))ds \tag{4.5}$$

$$+\frac{\beta t^2}{2(1-\beta\eta)}\int_0^1 G_2(\eta,s)a(s)f_1(u(s),u'(s))ds$$
$$+\frac{\alpha}{2\zeta}\int_0^1 G_3(\eta,s)a(s)f_1(u(s),u'(s))ds$$

Applying the right hand side of inequality (4.3) we get

$$u(t) \leq \int_0^1 G_1(1,s)a(s)f_1(u(s),u'(s))ds \tag{4.6}$$

$$+\frac{\beta}{(1-\beta\eta)}\int_0^1 G_2(\eta,s)a(s)f_1(u(s),u'(s))ds$$
$$+\frac{\alpha}{\zeta}\int_0^1 G_3(\eta,s)a(s)f_1(u(s),u'(s))ds.$$

Thus

$$\int_0^1 G_1(1,s)a(s)f_1(u(s),u'(s))ds \tag{4.7}$$

$$+\frac{\beta}{(1-\beta\eta)}\int_0^1 G_2(\eta,s)a(s)f_1(u(s),u'(s))ds$$
$$+\frac{\alpha}{\zeta}\int_0^1 G_3(\eta,s)a(s)f_1(u(s),u'(s))ds \geq \|u\|$$

On the other hand, (4.4) gives

$$u'(t) = \int_0^1 G_2(t,s)a(s)f_1(u(s),u'(s))ds \tag{4.8}$$

$$
\begin{aligned}
&+\frac{\beta t}{(1-\beta\eta)}\int_0^1 G_2(\eta,s)a(s)f_1(u(s),u'(s))ds\\
\leq\ &\left(\frac{1}{\tau_1}+\frac{\beta}{(1-\beta\eta)}\right)\int_0^1 G_2(s,s)a(s)f_1(u(s),u'(s))ds.
\end{aligned}
$$

Hence

$$
\int_0^1 G_2(s,s)a(s)f_1(u(s),u'(s))ds \geq \delta_1 \|u'\| \tag{4.9}
$$

where $\delta_1=\left(\frac{1}{\tau_1}+\frac{\beta}{(1-\beta\eta)}\right)^{-1}$. In view of the left hand side of (4.4) and (4.7), we obtain for all $t\in(\tau_1,\tau_2)$

$$
\begin{aligned}
u(t)\ \geq\ &\tau_1^2\Big[\int_0^1 G_1(1,s)a(s)f_1(u(s),u'(s))ds\\
&+\frac{\beta}{(1-\beta\eta)}\int_0^1 G_2(\eta,s)a(s)f_1(u(s),u'(s))ds\\
&+\frac{\alpha}{\zeta}\int_0^1 G_3(\eta,s)a(s)f_1(u(s),u'(s))ds\Big]
\end{aligned}
$$

$$
\geq \tau_1^2\|u\| \tag{4.10}
$$

Taking into account (4.9), it yields

$$
u'(t)\ \geq\ \left(\tau+\frac{\beta\tau_1}{(1-\beta\eta)}\right)\int_0^1 G_2(s,s)a(s)f_1(u(s),u'(s))ds
$$

$$
\geq \left(\tau+\frac{\beta\tau_1}{(1-\beta\eta)}\right)\delta_1\|u'\| \tag{4.11}
$$

Combining (4.10) and (4.11) we get

$$
\begin{aligned}
\min_{t\in(\tau_1,\tau_2)}(u(t)+u'(t))\ &\geq\ \tau_1^2\|u\|+\left(\tau+\frac{\beta\tau_1}{(1-\beta\eta)}\right)\delta_1\|u'\|\\
&\geq\ \delta\|u\|_1
\end{aligned}
$$

where $\delta=\max\left(\tau_1^2,\left(\tau+\frac{\beta\tau_1}{(1-\beta\eta)}\right)\delta_1\right)$. The proof is complete.

Define the quantities A_0 and A_∞ by

$$
A_0=\lim_{(|u|+|v|)\to 0}\frac{f_1(u,v)}{|u|+|v|},\qquad A_\infty=\lim_{(|u|+|v|)\to\infty}\frac{f_1(u,v)}{|u|+|v|}.
$$

The case $A_0=0$ and $A_\infty=\infty$ is called superlinear case and the case $A_0=\infty$ and $A_\infty=0$ is called sublinear case.

The main result of this section is the following

Theorem 4.3. *Under the hypotheses H1-H2 and if* $\alpha > 1$, $0 < \beta < \frac{1}{\eta}$ *then (P1) has at least one positive solution in the both cases superlinear as well as sublinear.*

To prove Theorem 4.3 we apply the well known Guo-Krasnosel'skii fixed point Theorem in cone (see th.2.5).

Proof. Define the cone K by

$$K = \left\{ u \in E^{+}, \min_{t \in (\tau_1, \tau_2)} (u(t) + u'(t)) \geq \delta \left\| u \right\|_1 \right\}$$

It is easy to check that K is a nonempty closed and convex subset of E, so it is a cone. Using Lemma 4.1 we see that $TK \subset K$. From the prove of Theorem 3.1, we know that T is completely continuous in E.

We prove the superlinear case.Since $A_0 = 0$, for any $\varepsilon > 0$, there exists $R_1 > 0$, such that

$$f_1(u, v) \leq \varepsilon (|u| + |v|)$$

for $0 < |u| + |v| \leq R_1$. Letting $\Omega_1 = \{u \in E, \|u\|_1 < R_1\}$, for any $u \in K \cap \partial\Omega_1$, it yields

$$Tu(t) \leq \varepsilon \left\| u \right\|_1 \left[\int_0^1 G_1(1, s) a(s) ds \right. \tag{4.12}$$

$$\left. + \frac{\beta}{\tau_1 (1 - \beta\eta)} \int_0^1 G_2(s, s) a(s) ds + \frac{\alpha}{\zeta} \int_0^1 G_3(\eta, s) a(s) ds, \right]$$

Moreover, we have

$$Tu'(t) \leq \varepsilon \left\| u \right\|_1 \left(\frac{1}{\tau_1} + \frac{\beta}{(1 - \beta\eta)} \right) \int_0^1 G_2(s, s) a(s) ds \tag{4.13}$$

From (4.12) and (4.13) we conclude

$$\left\| Tu \right\|_1 \leq \varepsilon \left\| u \right\|_1 \left[\int_0^1 G_1(1, s) a(s) ds + \right. \tag{4.14}$$

$$\left. \left(\frac{1 + 2\beta}{\tau_1 (1 - \beta\eta)} \right) \int_0^1 G_2(s, s) a(s) ds + \frac{\alpha}{\zeta} \int_0^1 G_3(\eta, s) a(s) ds \right]$$

In view of Lemma 4.1, one can choose ε such

$$\varepsilon \leq \left(\int_0^1 \left[G_1(1, s) + \left(\frac{1 + 2\beta}{\tau_1 (1 - \beta\eta)} \right) G_2(s, s) + \frac{\alpha}{\zeta} G_3(\eta, s) \right] a(s) ds \right)^{-1} \tag{4.15}$$

The inequalities (4.14) and (4.15) imply that $||Tu||_1 \leq ||u||_1, \ \ \forall u \in K \cap \partial\Omega_1$. Second, in view of $A_\infty = \infty$, then for any $M > 0$, there exists $R_2 > 0$, such that $f_1(u,v) \geq M(|u|+|v|)$ for $|u|+|v| \geq R_2$. Let $R = \max\left\{2R_1, \frac{R_2}{\delta}\right\}$ and denote by Ω_2 the open set $\{u \in E/\, ||u|| < R\}$. If $u \in K \cap \partial\Omega_2$ then

$$\min_{t\in(\tau_1,\tau_2)} (u(t)+u'(t)) \geq \delta \|u\|_1 = \delta R \geq R_2 \tag{4.16}$$

Using the left hand sides of (4.3) and (4.4) we obtain

$$\begin{aligned} Tu(t) \ \ \geq \ \ & \tau_1^2 \left[\int_0^1 G_1(1,s)a(s)f_1(u(s),u'(s))ds \right. \\ & + \frac{\beta\tau}{(1-\beta\eta)} \int_0^1 G_2(s,s)a(s)f_1(u(s),u'(s))ds \\ & \left. + \frac{\alpha}{\zeta} \int_0^1 a(s)f_1(u(s),u'(s))ds \right], \end{aligned}$$

thus

$$Tu(t) \geq \tau_1^2 M\, ||u||_1 \int_0^1 \left(G_1(s,s) + \frac{\beta\tau}{(1-\beta\eta)} G_2(s,s) + \frac{\alpha}{\zeta} G_3(\eta,s) \right) a(s)ds. \tag{4.17}$$

Moreover, we get with the help of (4.11)

$$T'u(t) \geq M\, ||u|| \left(\tau + \frac{\beta\tau_1}{(1-\beta\eta)} \right) \int_0^1 G_2(s,s)a(s)ds \tag{4.18}$$

In view of (4.17) and (4.18) we can write

$$Tu(t) + T'u(t) \tag{4.19}$$

$$\begin{aligned} \geq \ \ & M\, ||u||_1 \left[\int_0^1 \tau_1^2 \left(G_1(s,s) + \frac{\alpha}{\zeta} G_3(\eta,s) \right) a(s)ds \right. \\ & \left. + \frac{\tau\left(1+\tau_1^2\beta-\beta\eta\right)+\beta\tau_1}{(1-\beta\eta)} \int_0^1 G_2(s,s)a(s)ds \right] \\ \geq \ \ & M\, ||u||_1\, \tau_1^2 \int_0^1 G_1(s,s)a(s)ds \end{aligned}$$

Let us choose M such that

$$M \geq \left(\tau_1^2 \int_0^1 G_1(s,s)a(s)ds \right)^{-1},$$

then we get $Tu(t)+T'u(t)\geq \|u\|_1$. Hence,

$$\|Tu\|_1 \geq \|u\|_1\,,\ \forall u\in K\cap\partial\Omega_2.$$

The first part of Theorem 2.5 implies that T has a fixed point in $K\cap\left(\bar{\Omega}_2\diagdown\Omega_1\right)$ such that $R_2\leq \|u\|\leq R$. To prove the sublinear case we apply similar technics. The proof is complete.

Example 4.4. *The three point BVP*

$$\left\{\begin{array}{c} u'''=(1+t)^{-10}\left(\sin u+e^{-t}u'(t)\right)+\ln(1+t), 0<t<1 \\ u\left(0\right)=10^{-2}u\left(1\right), u'\left(1\right)=10^{-4}u'\left(\frac{1}{2}\right), u'\left(0\right)=0 \end{array}\right. \tag{4.20}$$

has a unique solution $u\in E$.

Proof. We have $\alpha=10^{-2}$, $\beta=10^{-4},\eta=\frac{1}{2}$, $\zeta=0.989\,95$ and

$$\left|f(t,x,\overline{x})-f(t,y,\overline{y})\right|\leq k(t)\left|x-y\right|+g(t)\left|\overline{x}-\overline{y}\right|$$

where $k(t)=(1+t)^{-10}$, $g(t)=(1+t)^{-10}e^{-t}$, $k,g\in L_1\left([0,1],\mathbb{R}_+\right).$ Using Theorem 3.2, it yields

$$\begin{aligned} M &= \left(2+\frac{\left(\left|\beta\right|+1\right)\left(2+3\left|\alpha\right|\right)+\left|\alpha\beta\right|}{\left|\zeta\right|}\right)\left(\|k\|_{L_1[0,1]}+\|g\|_{L_1[0,1]}\right) \\ &= 0.209\,76\times 4.0508=0.849\,70<1 \end{aligned}$$

then, we conclude that the BVP (4.20) has a unique solution u in E.

Example 4.5. *The three point BVP*

$$\left\{\begin{array}{c} u'''=10^{-2}\left(\frac{u}{4+u^2}\sin t+u'e^{-u^2}\ln(1+t)+\tan t\right),\quad 0<t<1 \\ u\left(0\right)=-2u\left(1\right), u'\left(1\right)=3u'\left(\frac{1}{2}\right), u'\left(0\right)=0 \end{array}\right. \tag{4.21}$$

has at least a nontrivial solution u in E.

Proof. We have $\alpha=-2$, $\beta=3,\eta=\frac{1}{2}$, $\zeta=\frac{-3}{2}$,

$$f(t,x,y)=10^{-2}\left(\frac{x}{4+x^2}\sin t+ye^{-x^2}\ln(1+t)+\tan t\right)$$

$f(t,0,0)=10^{-2}\tan t\neq 0$, $t\in(0,1)$ and

$$|f(t,x,y)|\leq k(t)|x|+g(t)|y|+h(t),$$

where $k(t) = 10^{-2}\sin t,\ g(t) = \frac{\ln(1+t)}{100},\ h(t) = 10^{-2}\tan t,\ k, g,\ h \in L_1([0,1], \mathbb{R}_+)$. The hypotheses of Theorem 3.1 hold, indeed:

$$\begin{aligned} M &= \left(2 + \frac{(|\beta|+1)(2+3|\alpha|)+|\alpha\beta|}{|\zeta|}\right)\left(\|k\|_{L_1[0,1]} + \|g\|_{L_1[0,1]}\right) \\ &= 27.333 \times 8.4599 \times 10^{-3} = 0.23123 < \frac{1}{2} \\ N &= \left(2 + \frac{(|\beta|+1)(2+3|\alpha|)+|\alpha\beta|}{|\zeta|}\right)\|h\|_{L_1[0,1]} \\ &= 27.333 \times 0.61563 \times 10^{-2} = 0.16827 < \frac{1}{2} \end{aligned}$$

Then BVP (4.21) has at least one nontrivial solution u in E.

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