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APPLICATION OF SADOVSKII FIXED POINT THEOREM TO SOLUTIONS OF OPERATOR EQUATIONS IN ARBITRARY BANACH SPACES

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Abstract. We apply Sadovskii fixed point theorem for the existence of solutions of the operator equation $x - Tx = f$.

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1. Introduction and Preliminaries

We recall some definitions.

Definition 1. [1] *Let (M,ρ) denote a complete metric space and let $\mathfrak{B}$ denote the collection of nonempty and bounded subsets of M. Define the Kuratowski measure of noncompactness $\alpha : \mathfrak{B} \to \mathbb{R}^+$ by taking for $A \in \mathfrak{B}$,*

$\alpha(A)$=inf$\{\varepsilon > 0$ *A is contained in the union of a finite number of sets in $\mathfrak{B}$ each having diameter less than ε*$\}$.

If M is a Banach space the function α has the following properties for $A, B \in \mathfrak{B}$

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1. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ *is compact,*
2. $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.

Definition 2. [2] *Let K be a subset of a metric space M. A mapping $T : K \to M$ is said to be condensing if T is bounded and continuous and if*

$$\alpha(T(D)) < \alpha(D)$$

for all bounded subsets D of M for which $\alpha(D) > 0$.

We state the Sadovskii fixed point theorem.

Theorem 1. [2] *Let K be a nonempty, bounded closed and convex subset of a Banach space and let $T : K \to K$ be a condensing mapping, then T has a fixed point.*

2. Main Theorem

The main result of this paper is the following:

Theorem 2. *Let X be an arbitrary Banach space, let $f \in X$ and $T : X \to X$ be a condensing mapping, then the operator equation*

$$x - Tx = f$$

has a solution if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X, defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let the mapping $T_f : X \to X$ be defined by

$$T_f(u) = Tu + f.$$

Then u is a solution of the operator equation

$$x - Tx = f$$

if and only if u is a fixed point of T_f.
Since T is bounded and continuous, T_f is also bounded and continuous. Using the properties of the Kuratowski measure of noncompactness, for all bounded subsets D of X, we have

$$\alpha(T_f(D)) = \alpha(T(D) + \{f\}) \leq \alpha(T(D)) + \alpha(\{f\}).$$

Since $\{f\}$ is compact, $\overline{\{f\}}$ is compact, implying $\alpha(\{f\}) = 0$, giving

$$\alpha(T_f(D)) \le \alpha(T(D)) < \alpha(D).$$

Since T is condensing mapping and it follows that T_f is a condensing mapping.
Suppose T_f has a fixed point u in X. Then for all $n \in \mathbb{N}$, since T_f is a continuous mapping being condensing, we get

$$\|x_{n+1} - u\| = \|Tx_n + f - u\| = \|T_f(x_n) - T_f(u)\| \le \|x_n - u\|.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and for each $x \in X$

$$B_d[x] = \{y \in X : \ \|x - y\| \le d\}.$$

Set $C_n = \bigcap_{i \ge n} B_d[x_i]$, then C_n is a nonempty convex set for each n. Using that T is a continuous mapping and the given Picard iteration, we have

$$y \in B_d[x_n] \Rightarrow \|y - x_n\| \le d$$

$$\Rightarrow \|Ty - Tx_n\| \le d$$

$$\Rightarrow \|Ty - [x_{n+1} - f]\| \le d$$

$$\Rightarrow \|(Ty + f) - x_{n+1}\| \le d$$

$$\Rightarrow (Ty + f) \in B_d[x_{n+1}].$$

Applying this, we get the following

$$T_f(C_n) = T_f(\bigcap_{i \ge n} B_d[x_i])$$

$$\subseteq \bigcap_{i \ge n} T_f(B_d[x_i])$$

$$= \bigcap_{i \ge n} \{T_f(y) : \|y - x_i\| \le d\}$$

$$= \bigcap_{i \ge n} \{(Ty + f) : \|y - x_i\| \le d\}$$

$$\subseteq \bigcap_{i \ge n+1} B_d[x_i] = C_{n+1}.$$

Let us define

$$C = \overline{\bigcup_{n \in \mathbb{N}} C_n}.$$

Since C_n increases with n,

$$C_n \subset C_{n+1} \subset C_{n+2} \subset \ldots\ldots,$$

it follows that C is a closed, convex and bounded subset of X. Now we have

$$T_f(C) = T_f\left(\overline{\bigcup_{n \in \mathbb{N}} C_n}\right) \subseteq \overline{T_f\left(\bigcup_{n \in \mathbb{N}} C_n\right)} = \overline{\bigcup_{n \in \mathbb{N}} T_f(C_n)} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C_{n+1}} = C$$

giving $T_f : C \to C$ since T_f is continuous mapping.

Finally, applying the Sadovskii fixed point theorem to T_f and C, we obtain that T_f has a fixed point in C which proves the theorem. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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