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SOME COMMON FIXED POINT THEOREM FOR GENERALIZED WEAK CONTRACTION IN COMPLETE METRIC SPACE

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Abstract: In this paper, we prove some fixed point theorems in metric space by using generalized weak contractive mapping. Our main result is motivated by C- contraction also our results are generalization of many previous known results.

Key words: Metric space, fixed point, Common fixed point, generalized weak contractive mapping

2000 AMS Subject Classification: 47H10, 54H25

INTRODUCTION

Fixed point theorem is an important tool in analysis. Its play an important role for solving theproblems that are boundary valued problems, computer optimization theory, engineering sciences as well as medical sciences. The first result in this direction was given by Banach in 1922 it is well known that Banach's contraction mapping theorem which states is

A mapping $T: X \to X$ where (X, d) is a metric space, is said to be a contraction if there exists $0 < k < 1$ such that for all $x, y \in X$,

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$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T . A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

After the time ago many researchers develop this theorem in much direction such as on taking different type of contractions or spaces. One of them is more interesting result is given by Kannan [10,11] which as follows

If $T: X \rightarrow X$ where (X, d) is a complete metric space, satisfies the inequality

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

Where $0 < k < \frac{1}{2}$and $x, y \in X$, then T has a unique fixed point .

The mapping T need not be continuous has been established through examples [11]. The mappings satisfying (1.2) are called Kannan type mappings. There is a large literature dealing with Kannan type mappings and generalization some of which are noted in [8],[17] and [19].

Beside this similar contractive condition has been introduced by Chatterjee[6] .We call this contraction a C-contraction (borrowing the name from the name of its author).

Definition 1.1 C-contraction [6]

A mapping $T: X \rightarrow X$ where (X, d) is a metric space is called a C-contraction if there exists $0 < k < \frac{1}{2}$Such that for all $x, y \in X$ the following inequality holds:

$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Tx)]$ (1.3)

Theorem1.1[6] A C-contraction defined on a complete metric space has a unique fixed point.

In establishing theorem 1.1 there is no requirement of continuity of the C-contraction.

It has been established in [15] that inequalities (1.1),(1.2) and (1.3) are independent of one another. C-contraction and its generalizations have been discussed in a number of works some of which are noted in [4] ,[8],[9] and [19].

Banach's contraction mapping theorem has been generalized in a number of recent papers . As for example asmptotic contraction has been introduced by Kirk [12] and generalized Banach contraction conjecture has been proved in [1] and [14].

Particularly a weaker contraction has been introduced in Hilbert space in [2]. The following is the corresponding definition in metric space.

Definition 1.2 Weakly contractive mapping

A mapping $T: X \to X$ where $(X, d)$is a complete metric space is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \psi\big(d(x, y)\big), \quad (1.4)$$

Where $x, y \in X, \psi: [0, \infty) \to [0, \infty)$ is continuous and non decreasing,

$\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$.

If we take $\psi(x) = kx$ where $0 < k < 1$ then (1.4) reduces to (1.1).

There are number of works in which weakly contractive mapping have been considered. Some of these works are noted in [3],[7],[13] and [16].

Definition 1.3 Weakly C-contraction :

A mapping $T: X \to X$, where(X, d) is a metric space is said to be Weakly C-contractive or a weak C-contraction if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)) \qquad (1.5)$$

Where $\psi : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Motivated by the above result we give an generalized weakly contraction , defines as follows

Definition 1.4 Generalized weakly contraction:

A mapping$T: X \to X$, where$(X, d)$is a metric space is said to be Generalized Weaklycontractive or a Generalized weak contraction if for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha[\max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}]$$

$$- \psi\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\big) \quad (1.6)$$

Where $\alpha \in [0, 1), \psi : [0, \infty)^5 \to [0, \infty)$ is a continuous mapping such that $\psi(x_1, x_2, x_3, x_4, x_5) = 0$ if and only if one of x_1, x_2, x_3, x_4, x_5 is equal to 0.

Definition 1.5 weakly compatible

Let A and S be two mapping of a metric space (X, d) then it is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = t \quad \text{and} \quad \lim_{n\to\infty} Sx_n = t \qquad \text{for some } t \in X.$$

Let A and S be two mapping of a metric space (X, d) then it is said to be weakly compatible if they commute at coincidence point , that is $Ax = Sx$ implies that ,

$$ASx = SAx \quad \forall\, x \in X$$

It is easy to see that compatible mapping commute at there coincidence point .Note that a compatible maps are weakly compatible but converges need not be true.

In this paper we prove some fixed point and common fixed point theorems for generalized weakly contraction. Our result is generalization of various previously known results.

MAIN RESULTS

Theorem 2 Let$T : X \to X$, where$(X, d)$is a complete metric space be a generalized weak contraction. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ and for all $n \geq 0$, $x_{n+1} = Tx_n$

If $x_n = Tx_{n-1}$ then x_n is a fixed point of T.

So we assume $x_n \neq x_{n+1}$

Putting $x = x_{n-1}$ and $y = x_n$ in (1.6) we have for all $n = 0,1,2, \ldots ....$

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \alpha[\max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}), d(x_{n-1}, x_n)\}]$$

$$- \psi\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}), d(x_{n-1}, x_n)\big)$$

$$= \alpha[\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_n)\}]$$

$$- \psi\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_n)\big)$$

$$= \alpha\,[\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\}]$$

$$- \psi\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\big)$$

Since T is generalized weakly contraction, this gives that

$$\psi\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\big) = 0$$

And

$$d(x_n, x_{n+1}) \leq \alpha[\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\}] \quad (2.1)$$

Now here arise three cases:

Case I:- If we choose

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$$

Then (2.1) can be written as

$$d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_n)$$

Similarly we can write,

$$d(x_{n-1}, x_n) \leq \alpha\, d(x_{n-2}, x_{n-1})$$

Follows the same manner, we can write

$$d(x_1, x_2) \leq \alpha\, d(x_0, x_1)$$

That is

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1)$$

For any $m > n,\ m, n \in N$ and by triangle inequality

$$d(x_n, x_m) \leq \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \alpha^{n+2} d(x_0, x_1) + \dots\dots\dots + \alpha^{n-(m+1)} d(x_0, x_1)$$

$$d(x_n, x_m) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1)$$

Since $0 \leq \alpha < 1$ and as $n \to \infty$ then

$$\lim_{n\to\infty} d(x_n, x_m) = 0$$

That is $\{x_n\}$ is a Cauchy sequence, converges to $x \in X$.

Case – 2:- If we take

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\} = d(x_n, x_{n+1})$$

Then (2.1) can be written as

$$d(x_n, x_{n+1}) \leq \alpha\, d(x_n, x_{n+1})$$

Since $0 \leq \alpha < 1$, which give contradiction.

Case-3:- If we take

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0, d(x_{n-1}, x_n)\} = d(x_{n-1}, x_{n+1})$$

Then (2.1) can be written as

$$d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_{n+1})$$

By triangle inequality

$$d(x_n, x_{n+1}) \leq \alpha\, [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n)$$

Let $\frac{\alpha}{1-\alpha} = q$ then

$$d(x_n, x_{n+1}) \le q\, d(x_{n-1}, x_n)$$

From the case $-$ 1, gives that $\{x_n\}$ is a Cauchy sequence, converges to $x \in X$.

Finally we say that from the above $\text{Case} - 1{,}2{,}3\{x_n\}$ is a Cauchy sequence and converges to $x \in X$due to the complete metric space (X, d) .

Let $x_n \to z$ as $n \to \infty$

Then

$$d(z, Tz) \le d(z, x_{n+1}) + d(x_{n+1}, Tz)$$

$$\le d(z, x_{n+1}) + d(Tx_n, Tz)$$

$$\le d(z, x_{n+1}) + \alpha[\max\{d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n), d(x_n, z)\}]$$

$$- \psi\big(d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n), d(x_n, z)\big)$$

$$\le d(z, x_{n+1}) + \alpha[\max\{d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(z, x_{n+1}), d(x_n, z)\}]$$

$$- \psi\big(d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(z, x_{n+1}), d(x_n, z)\big)$$

$$d(z, Tz) \le \alpha d(z, Tz)$$

Which contradiction.So $z = Tz$.

That is z is a fixed point of T in X.

Uniqueness: Let w is another fixed point of T in X such that $z \ne w$, then we have

$$d(z, w) = d(Tz, Tw)$$

$$d(Tz, Tw) \le d(Tz, Tx_n) + d(Tx_n, Tw)$$

From (1.6) and as $n \to \infty$ we have

$$d(Tz, Tw) \le 0$$

Which contradiction.

So $z = w$ that is, z is unique fixed point of T in X.

Theorem 3 : Let (X, d) be a complete metric space and let $S, T: X \to X$ be a generalized weak contraction such that

$$d(Sx, Tx) \leq \alpha[\max\{d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx), d(x, y)\}]$$

$$-\psi\big(d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx), d(x, y)\big) \qquad (3.1)$$

For $x, y \in X$ and $0 \leq \alpha < 1$. Then S and T has unique fixed point in X.

Proof : For any arbitrary x_0 in X, we choose $x_1, x_2 \in X$ such that,

$Sx_0 = x_1$ and $Tx_1 = x_2$

In general we can define a sequence $\{x_n\}$ such that,

$x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$

Consider,

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \alpha[\max\{d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n}), d(x_{2n}, x_{2n+1})\}$$

$$-\psi\big(d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n}), d(x_{2n}, x_{2n+1})\big)$$

$$\leq \alpha[\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1})\}$$

$$-\psi\big(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1})\big)$$

$$\leq \alpha[\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\}$$

$$-\psi\big(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\big)$$

Since S, T are Generalized weakly contraction, this gives that,

$$\psi\big(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\big) = 0$$

And

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha \left[\max \begin{Bmatrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1}) \end{Bmatrix}\right] \quad (3.2)$$

Now here following three cases arise :

Case I :- If we choose

$$\max \begin{Bmatrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1}) \end{Bmatrix} = d(x_{2n}, x_{2n+1})$$

Then (3.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

Similarly we can write,

$$d(x_{2n}, x_{2n+1}) \leq \alpha d(x_{2n-1}, x_{2n})$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha^{2n+1} d(x_0, x_1)$$

For $n \leq m$, we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \dots \dots \dots \dots . + d(x_{2m-1}, x_{2m})$$

In general,

$$d(x_n, x_m) \leq \{\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots \dots \dots \dots . + \alpha^m\} d(x_0, x_1)$$

$$d(x_n, x_m) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1)$$

Since $0 \leq \alpha < 1$ and as $n \to \infty$ then

$$\lim_{n\to\infty} d(x_n, x_m) = 0$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Case II :-If we take

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\} = d(x_{2n+1}, x_{2n+2})$$

Then (3.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n+1}, x_{2n+2})$$

Since $0 \leq \alpha < 1$ which give contraction.

Case III :- If we take

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+2})$$

Then (3.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+2})$$

Using triangle inequality

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha\, [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha}{1-\alpha}\, d(x_{2n}, x_{2n+1})$$

Let $\frac{\alpha}{1-\alpha} = h$

$$d(x_{2n+1}, x_{2n+2}) \leq h\, d(x_{2n}, x_{2n+1})$$

Form the case I gives that $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Finally we say that the above Case – I,II,III $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$due to complete metric space (X, d)

Thus $x_n \to u$ as $n \to \infty$

$Sx_{2n} \to u$and $Tx_{2n+1} \to u$ as $n \to \infty$

u is fixed point of S and T in X.

Since $ST = TS$ this give,

$$u = Tu = TSu = Su = u$$

u is common fixed point of S and T .

Uniqueness: Let us assume that , v is another fixed point of S and T in Xdifferent from u . Then

$Tu = u$ and$Tv = v$ also $Su = u$ and $Sv = v$

$$d(u, v) = d(Su, Tv)$$

From (3.1)

$$d(Su, Tv) \leq \alpha[\max\{d(u, Su), d(v, Tv), d(u, Tv), d(v, Su), d(u, v)\}]$$

$$-\psi\big(d(u, Su), d(v, Tv), d(u, Tv), d(v, Su), d(u, v)\big)$$

$$= \alpha[\max\{d(u, u), d(v, v), d(u, v), d(v, u), d(u, v)\}]$$

$$-\psi\big(d(u, u), d(v, v), d(u, v), d(v, u), d(u, v)\big)$$

$$d(Su, Tv) \leq \alpha d(u, v)$$

$$d(u, v) \leq \alpha d(u, v)$$

Which contraction

uis unique common fixed point ofS and T in X .

Theorem 4: Let (X, d) be a complete metric space . Suppose that S, R and T be the generalized weak contractionfrom X into itself satisfies condition

$$d(SRx, TRy) \leq \alpha[\max\{d(x, SRx), d(y, TRy), d(x, TRy), d(y, SRx), d(x, y)\}$$

$$-\psi\big(d(x, SRx), d(y, TRy), d(x, TRy), d(y, SRx), d(x, y)\big) \quad (4.1)$$

For $x, y \in X$ and$0 \leq \alpha < 1$. Then S , R and T has unique fixed point in X.

Proof : For any arbitrary x_0 in X, we choose $x_1, x_2 \in X$ such that,

$SRx_0 = x_1$ and$TRx_1 = x_2$

In general we can define a sequence $\{x_n\}$ such that,

$x_{2n+1} = SRx_{2n}$ and $x_{2n+2} = TRx_{2n+1}$

Consider

$$d(x_{2n+1}, x_{2n+2}) = d(SRx_{2n}, TRx_{2n+1})$$

$$\leq \alpha \left[\max\left\{\begin{matrix} d(x_{2n}, SRx_{2n}), d(x_{2n+1}, TRx_{2n+1}), \\ d(x_{2n}, TRx_{2n+!}), d(x_{2n+1}, SRx_{2n}), d(x_{2n}, x_{2n+1}) \end{matrix}\right\}\right]$$

$$- \psi\left(\begin{matrix} d(x_{2n}, SRx_{2n}), d(x_{2n+1}, TRx_{2n+1}), \\ d(x_{2n}, TRx_{2n+1}), d(x_{2n+1}, SRx_{2n}), d(x_{2n}, x_{2n+1}) \end{matrix}\right)$$

$$\leq \alpha \left[\max\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1}) \end{matrix}\right\}\right]$$

$$- \psi\left(\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1}) \end{matrix}\right)$$

$$\leq \alpha \left[\max\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ , d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1}) \end{matrix}\right\}\right]$$

$$- \psi\left(\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1}) \end{matrix}\right)$$

Since R, S and T are Generalized weakly contraction, this gives that,

$$\psi\big(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\big) = 0$$

And

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha \left[\max\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1}) \end{matrix}\right\}\right] \quad (4.2)$$

Now here following three cases arise :

Case I :- If we choose

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$$

Then (4.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

Similarly we can write,

$$d(x_{2n}, x_{2n+1}) \leq \alpha d(x_{2n-1}, x_{2n})$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}) = \alpha^{2n+1} d(x_0, x_1)$$

For $n \leq m$, we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \dots \dots \dots \dots . + d(x_{2m-1}, x_{2m})$$

In general

$$d(x_{2n}, x_{2m}) \leq \{\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots \dots \dots \dots . + \alpha^m\} d(x_0, x_1)$$

$$d(x_{2n}, x_{2m}) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1)$$

Since $0 \leq \alpha < 1$ and as $n \to \infty$ then

$$\lim_{n\to\infty} d(x_{2n}, x_{2m}) = 0$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Case II :-If we take

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\} = d(x_{2n+1}, x_{2n+2})$$

Then (4.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n+1}, x_{2n+2})$$

Since $0 \leq \alpha < 1$ which give contraction.

Case III :- If we take

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0, d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+2})$$

Then (4.2) can be written as

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+2})$$

Using triangle inequality

$d(x_{2n+1}, x_{2n+2}) \leq \alpha [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha}{1-\alpha} d(x_{2n}, x_{2n+1})$$

Let $\frac{\alpha}{1-\alpha} = h$

$$d(x_{2n+1}, x_{2n+2}) \leq h \, d(x_{2n}, x_{2n+1})$$

Form the case -Igives that $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Finally we say that the above Case – I,II,III$\{x_n\}$ is a Cauchy sequence which converges to $u \in X$ due to complete metric space (X, d).

So we can write

$$d(x_{2n+1}, x_{2n+2}) \leq h \, d(x_{2n}, x_{2n+1})$$

As $n \to \infty$

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) \to 0$$

That is the sequence $\{x_n\}$ is Cauchy sequence, converges to u.

Now we also have

$$\lim_{n\to\infty} SRx_{2n} = SR \lim_{n\to\infty} x_{2n} = SRu \text{ and} \lim_{n\to\infty} TRx_{2n+1} = TR \lim_{n\to\infty} x_{2n+1} = TRu$$

Implies that

$$SRu = TRu = u$$

$$Su = Ru = Tu = u$$

That is common fixed point of S, T, R.

Uniqueness:-

Let us assume that v is another fixed point of S, T and R different from u. Now

$$d(u,v) = d(SRu, TRv)$$

$$\leq \; d(SRu, x_{2n}) + d(SRx_{2n+1}, TRv)$$

From (4.1) and as $n \to \infty$, we have

$$d(u,v) \leq \; \alpha\, d(u,v)$$

Which contradiction.

So that u is unique common fixed point of S, T and R.

Theorem 5: Let (X, d) be a complete metric space and suppose that A, B, S and T be the mapping form X into itself satisfies the condition,

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$

(ii) $\{A, S\}$and$\{B, T\}$ are weakly compatible.

(iii) SorT is continuous.

(iv) $d(Ax, By) \leq \alpha \left[\max\left\{\begin{matrix} d(Sx, Ax), d(Ty, By), \\ d(Sx, By), d(Ty, Ax), d(Sx, Ty) \end{matrix}\right\}\right]$

$$-\psi\left(\begin{matrix} d(Sx, Ax), d(Ty, By), \\ d(Sx, By), d(Ty, Ax), d(Sx, Ty) \end{matrix}\right) \qquad (5.1)$$

For $x, y \in X$, $0 \leq \alpha < 1$. Then A, B, S and T have unique fixed point in X .

Proof: For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$in X, such that

$Ax_{2n} = Tx_{2n+1} = y_{2n}$and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for all $n = 0,1,2, \ldots \ldots \ldots$..

Now

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

Form (iv)

$$d(Ax_{2n}, Bx_{2n+1}) \leq \alpha\left[\max\left\{\begin{matrix} d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \\ d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n}), d(Sx_{2n}, Tx_{2n+1}) \end{matrix}\right\}\right]$$

$$-\psi\begin{pmatrix} d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \\ d(Sx_{2n}, Bx_{2n+1}), d(Tx_{2n+1}, Ax_{2n}), d(Sx_{2n}, Tx_{2n+1}) \end{pmatrix}$$

$$\leq \alpha\left[\max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n}) \end{matrix}\right\}\right]$$

$$-\psi\begin{pmatrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n}) \end{pmatrix}$$

$$\leq \alpha\left[\max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}) \\ , d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n}) \end{matrix}\right\}\right]$$

$$-\psi\begin{pmatrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n}) \end{pmatrix}$$

Since A, B, S and T are Generalized weakly contraction, this gives that,

$$\psi\begin{pmatrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n}) \end{pmatrix} = 0$$

And

$$d(y_{2n}, y_{2n+1}) \leq \alpha\left[\max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n}) \end{matrix}\right\}\right] \qquad (5.2)$$

Now here following three cases arise :

Case I :- If we choose

$$\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n})$$

Then (5.2) can be written as

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n})$$

Similarly we can write,

$$d(y_{2n-1}, y_{2n}) \leq \alpha d(y_{2n-2}, y_{2n-1})$$

In general we can write,

$$d(y_{2n}, y_{2n+1}) \leq \alpha^{2n+1} d(y_0, y_1)$$

For $n \leq m$, we have

$$d(y_{2n}, y_{2m}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \cdots \ldots \ldots \ldots \ldots . + d(y_{2m-1}, y_{2m})$$

$$d(y_{2n}, y_{2m}) \leq \{\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots \ldots \ldots \ldots . + \alpha^m\} d(y_0, y_1)$$

$$d(y_{2n}, y_{2m}) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1)$$

Since $0 \leq \alpha < 1$ and as $n \to \infty$ then

$$\lim_{n\to\infty} d(y_{2n}, y_{2m}) = 0$$

Hence $\{y_n\}$ is a Cauchy sequence which converges to $\in X$.

Case II :-If we take

$$\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n}, y_{2n+1})$$

Then (5.2) can be written as

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n}, y_{2n+1})$$

Since $0 \leq \alpha < 1$which give contraction.

Case III :- If we take

$$\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n+1})$$

Then (5.2) can be written as

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n+1})$$

Using triangle inequality

$$d(y_{2n}, y_{2n+1}) \leq \alpha[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha}{1-\alpha} d(y_{2n-1}, y_{2n})$$

Let $\frac{\alpha}{1-\alpha} = h$

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n})$$

Form the case I gives that $\{y_n\}$ is a Cauchy sequence which converges to $u \in X$

Finally we say that the above Case – I,II,III$\{y_n\}$ is a Cauchy sequence which converges to $u \in X$By the continuity of S and T$\{x_n\}$ is also convergent sequence which converges to $u \in X$. Hence (X, d) is complete metric space. uis fixed point of A, B, S and T .

Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible,implies that u is fixed point of A, B, S and T

Uniqueness : Let us assume that , v is another fixed point of A, B, S and T in X different from u . Then

$Au = u$ and$Av = v$ also $Bu = u$ and $Bv = v$

$$d(u, v) = d(Au, Bv)$$

From (iv)

$$d(Au, Bv) \leq \alpha \left[\max\left\{\begin{matrix} d(Su, Au), d(Tv, Bv), \\ d(Su, Bv), d(Tv, Au), d(Su, Tv)\end{matrix}\right\}\right]$$

$$-\psi\begin{pmatrix} d(Su, Au), d(Tv, Bv), \\ d(Su, Bv), d(Tv, Au), d(Su, Tv)\end{pmatrix}$$

$$= \alpha \left[\max\left\{\begin{matrix} d(u, u), d(v, v), \\ d(u, v), d(v, u), d(u, v)\end{matrix}\right\}\right]$$

$$-\psi\begin{pmatrix} d(u, u), d(v, v), \\ d(u, v), d(v, u), d(u, v)\end{pmatrix}$$

$$d(u, v) \leq \alpha d(u, v)$$

Which contraction

u is unique fixed point of A , B, S and T in X.

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