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EXISTENCE OF COMMON FIXED POINTS FOR A PAIR OF SELF MAPS ON A CONE METRIC SPACE UNDER B.C. CONTROL CONDITION

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Abstract. In this paper, we obtain sufficient conditions for the existence of unique point of coincidence for a pair of self maps on a cone metric space satisfying certain control conditions. These results improve the fixed point theorem of Razani.et.al.[8] imposing conditions such as the cone is a lattice or lattice ordered semigroup and introducing two new control functions namely B. C. control function and S.B.C control function. An open problem is also given at the end for further investigation.

Keywords: Cone metric space, Comparison function, Lattice, Lattice ordered Semigroup, Point of coincidence, B. C. Control function and S.B.C. Control function.

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1. Introduction

The concept of a cone metric space has been introduced and properties are investigated initially by Haung and Zhang [4]. Later many authors such as ([1], [5]-[9]) obtained fixed

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point theorems on cone metric spaces. Recently, A. Razani et. al. [8] proved a fixed point theorem for a pair of self maps on a cone metric space under generalized contractions.

In this paper, we further investigate for the existence of common fixed points for a pair of self maps on a cone metric space. Consequently we obtain the result of Razani et.al. [8] as a corollary. An open problem is also given at the end for further investigations. Before we further proceed we state some definitions and results, which we need for further development.

2. Preliminaries

Definition 2.1. [4] Let E be a real Banach space. A subset P of E is called a cone if the following conditions hold.

(i) P is closed, non-empty and $P \neq \{0\}$.

(ii) $a, b \in R, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in Int\ P$(Interior of P).

Definition 2.2. [4] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3. [8] Let (X, d) be a cone metric space $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x when ever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent in X.

Definition 2.4. [8] Let $f, g : X \to X$ be two mappings. If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self maps f and g on X are said to be weakly compatible if they commute at their coincidence point. i.e., if $f(x) = g(x)$ for some $x \in X$ then $(fog)(x) = (gof)(x)$.

Definition 2.5. [8] Let (X, d) be a cone metric space and P be a cone with non empty interior. Suppose that the mappings $f, g : X \to X$ are such that the range of g contains the range of f and $f(X)$ or $g(X)$ is a complete subspace of X. In this case we shall say that the pair (f, g) is Abbas and Jungck's pair or shortly *AJ's* pair.

Definition 2.6. [8] Let P be a cone. A non decreasing function $\varphi : P \to P$ is called a comparison function if it satisfies

(i) $\varphi(0) = 0$ and $0 < \varphi(x) < x$ for all $x \in P \setminus \{0\}$.

(ii) If $x \in Int\ P$ then $x - \varphi(x) \in Int\ P$.

(iii) $\lim\limits_{n\to\infty} \varphi^n(x) = 0$ for all $x \in P \setminus \{0\}$.

Following is the result of Razani et. al. [8]

Theorem 2.7. [8] Let (X, d) be a cone metric space. Suppose (f, g) is *AJ's* pair and φ is a comparison function such that

$$d(f(x), f(y)) \leq \varphi(u) \text{ for all } x, y \in X$$

where $u \in \left\{d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{d(f(x),g(y))+d(g(x),f(y))}{2}\right\}$.

Then f and g have a unique point of coincidence in X. More over if f and g are weakly compatible, then f and g have a unique common fixed point.

3. Main results

In this section, we introduce the notion of B. C. control functions and use them to obtain sufficient conditions for the existence of unique point of coincidence for a pair of self maps on a cone metric space, satisfying certain control conditions, namely, B. C. control condition and S.B.C control condition. Before going to prove the main results we need to prove the following lemmas.

Definition 3.1. [3] A lattice is a partially ordered set S in which any two elements $a, b \in S$ have the supremum $(a \cup b)$ and the infimum $(a \cap b)$.

Definition 3.2. Let $(S, +)$ be a semi group and $(S, \cup, \cap)$ be a lattice. Then $(S, \cup, \cap, +)$ is called a lattice ordered semi group if satisfies the following conditions

(i) $a + (b \cup c) = (a + b) \cup (a + c); (a \cup b) + c = (a + c) \cup (b + c)$

(ii) $a + (b \cap c) = (a + b) \cap (a + c); (a \cap b) + c = (a + c) \cap (b + c)$ for all $a, b, c \in S$.

Definition 3.3. Let E be a real Banach space and P be a cone in E

(1) A comparison function $\varphi : P \to P$ is called a B.C. control function if

(i) $(P, \cup, \cap)$ is a lattice

(ii) φ is a lattice homomorphism on P. i.e. $\varphi(a \cup b) = \varphi(a) \cup \varphi(b)$ for all $a, b \in P$ and

(iii) $0 \leq a_n \in P$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ as $n \to \infty$ in P for every $x \in P$.

(2) A comparison function $\varphi : P \to P$ is called a S.B.C control function if

(i)$(P, \cup, \cap, +)$ is a lattice ordered semigroup

(ii) φ is a B. C. control function on P and

(iii) $\sum_{n=1}^{\infty} \varphi^n(t)$ converges in P for every $t \in P$.

Lemma 3.4. Let (X, d) be a cone metric space with cone P. Assume that P is a lattice. Let φ be a comparison function satisfying

(i) $\varphi : P \to P$ is a lattice homomorphism. i.e. $\varphi(a \cup b) = \varphi(a) \cup \varphi(b)$.

(ii) $0 \leq a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for every $x \in P$.

(That is φ is a B. C. control function on P)

Then $a, b \in P$ and $b \leq \varphi(a \cup b) \Rightarrow b \leq \varphi(a)$.

Proof. Suppose $a, b \in P$ and $b \leq \varphi(a \cup b) = \varphi(a) \cup \varphi(b)$

$$\text{Then } b \leq \varphi(a) \cup \varphi(b) \qquad \cdots (3.4.1)$$

Claim: For any positive integer k, $b \leq \varphi(a) \cup \varphi^k(b)$

The result is true for $k = 1$ by (3.4.1).

Assume it to be true for k. Then $b \leq \varphi(a) \cup \varphi^k(b)$

$$\begin{aligned} \text{Now } \varphi(b) &\leq \varphi(\varphi(a) \cup \varphi^k(b)) \\ &= \varphi^2(a) \cup \varphi^{k+1}(b) \\ &\leq \varphi(a) \cup \varphi^{k+1}(b) \end{aligned}$$

So that $b \leq \varphi(a) \cup \varphi^{k+1}(b)$

$\therefore$ By induction for every positive integer k, we have $b \leq \varphi(a) \cup \varphi^k(b)$

Thus our claim is established.

Now letting $k \to \infty$ and using (ii) we get $b \leq \varphi(a)$.

Lemma 3.5. Let P be a cone in E. Suppose $(P, \leq)$ is a lattice. Then $a, b \in P$, $\alpha \geq 0 \Rightarrow (\alpha a) \cup (\alpha b) = \alpha(a \cup b)$

Proof. We may suppose that $\alpha > 0$

$$\begin{aligned} \text{Now } 0 \leq a \leq a \cup b \text{ and } \alpha > 0 &\Rightarrow \alpha((a \cup b) - a) \geq 0 \\ &\Rightarrow \alpha(a \cup b) - \alpha a \geq 0 \\ &\Rightarrow \alpha(a \cup b) \geq \alpha a \end{aligned}$$

$$\text{Similarly } \alpha(a \cup b) \geq \alpha b$$

$$\therefore \alpha(a \cup b) \geq (\alpha a) \cup (\alpha b)$$

Further, for $x \in P$

$$\begin{aligned} \alpha a \leq x \text{ and } \alpha b \leq x \quad &\Rightarrow \quad a \leq \frac{1}{\alpha}x \text{ and } b \leq \frac{1}{\alpha}x \\ &\Rightarrow \quad a \cup b \leq \frac{1}{\alpha}x \\ &\Rightarrow \quad \alpha(a \cup b) \leq x \end{aligned}$$

$$\therefore (\alpha a) \cup (\alpha b) = \alpha(a \cup b).$$

Lemma 3.6. Let P be a cone in E. Suppose $(P, \leq, +)$ is a lattice ordered semigroup. Then $a, b \in P \Rightarrow a \cup b \cup (\frac{a+b}{2}) = a \cup b$

Proof.

$$\begin{aligned}
a\cup b\cup\big(\frac{a+b}{2}\big) &= a\cup(\frac{b}{2}+\frac{b}{2})\cup(\frac{a}{2}+\frac{b}{2})\\
&= a\cup(\frac{b}{2}+(\frac{a}{2}\cup\frac{b}{2}))\\
&\qquad \text{(since } P \text{ is a lattice ordered semigroup)}\\
&= (\frac{a}{2}+\frac{a}{2})\cup(\frac{b}{2}+(\frac{a}{2}\cup\frac{b}{2}))\\
&\le \big(\frac{a}{2}+\big(\frac{a}{2}\cup\frac{b}{2}\big)\big)\cup\big(\frac{b}{2}+\big(\frac{a}{2}\cup\frac{b}{2}\big)\big)\\
&= \big(\frac{a}{2}\cup\frac{b}{2}\big)+\big(\frac{a}{2}\cup\frac{b}{2}\big)\\
&= \frac{(a\cup b)}{2}+\frac{(a\cup b)}{2} \text{ (By lemma 3.5)}\\
&= a\cup b\\
&\le a\cup b\cup(\frac{a+b}{2})
\end{aligned}$$

$$\therefore a\cup b\cup\big(\frac{a+b}{2}\big) = a\cup b$$

Lemma 3.7. Let (X, d) be a cone metric space with cone P. Assume that P is a lattice ordered semi group. Let φ be a comparison function satisfying

(i) $\varphi : P \to P$ is a lattice homomorphism. i.e. $\varphi(a\cup b) = \varphi(a)\cup\varphi(b)$ for all $a, b \in P$.

(ii) $0 \le a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for all $x \in P$.

(That is, φ is a S.B.C. control function)

Then $a, b \in P$ and $b \le \varphi(a\cup b\cup(\frac{a+b}{2})) \Rightarrow b \le \varphi(a)$.

Proof.

$$\begin{aligned}
b &\le \varphi\Big(a\cup b\cup\big(\frac{a+b}{2}\big)\Big)\\
&= \varphi(a\cup b) \text{ (By Lemma 3.6)}
\end{aligned}$$

$$\Rightarrow b \le \varphi(a) \text{ (By Lemma 3.4)}$$

Theorem 3.8. Suppose P is a cone in a Real Banach space is such that

(i) P is lattice

(ii) $0 \leq a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for all $x \in P$.

Suppose φ is a comparison function such that $\varphi : P \to P$ is a lattice homomorphism. Suppose (X, d) is a cone metric space, $f, g : X \to X$ are such that (f, g) is *AJ's* pair and

$$d(f(x), f(y)) \leq \varphi(max\{d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y))\})$$

for all $x, y \in X$. Then f and g have a unique point of coincidence in X. More over if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, define $x_1 \in X$ such that $fx_0 = gx_1$.

Now define the sequences $\{x_n\}$ and $\{y_n\}$ in X inductively as

$y_n = f(x_n) = g(x_{n+1}), n = 0, 1, 2, \cdots$

Then,
$$\begin{aligned} d(y_n, y_{n+1}) &= d(f(x_n), f(x_{n+1})) \\ &\leq \varphi(max\{d(g(x_n), g(x_{n+1})), d(f(x_n), g(x_n)), d(f(x_{n+1}), g(x_{n+1}))\}) \\ &= \varphi(max\{d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n)\}) \\ &= \varphi(max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}) \\ &= \varphi(d(y_{n-1}, y_n) \cup d(y_n, y_{n+1})) \end{aligned}$$

Hence by Lemma 3.4. we have $d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n))$.

Consequently $d(y_n, y_{n+1}) \leq \varphi^n(d(y_0, y_1))$

For $\epsilon \gg 0$ choose a natural number n_0 and a real number δ such that

$\epsilon - \varphi(\epsilon) + \{u \in E : ||u|| < \delta\} \subset Int\ P$, $||\varphi^n(d(y_0, y_1))|| < \delta$ and consequently

$\varphi^n(d(y_0, y_1)) \ll \epsilon - \varphi(\epsilon)$ for all $n \geq n_0$.

So that $d(y_n, y_{n+1}) \ll \epsilon - \varphi(\epsilon) < \epsilon$ for all $n \geq n_0$.

Claim 1: $d(y_n, y_{n+k}) \ll \epsilon$ for all $n \geq n_0$ and $k = 1, 2, \cdots$

This is true for $k = 1$ and $n \geq n_0$

Assume it to be true for k and $n \geq n_0$

Now,
$$\begin{aligned} d(y_n, y_{n+k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k+1}) \\ &\ll \epsilon - \varphi(\epsilon) + \varphi(max\{d(y_n, y_{n+k}), d(y_{n+1}, y_n), d(y_{n+k+1}, y_{n+k})\}) \\ &\leq \epsilon - \varphi(\epsilon) + \varphi(max\{\epsilon, \epsilon - \varphi(\epsilon), \epsilon - \varphi(\epsilon)\}) \\ &\leq \epsilon - \varphi(\epsilon) + \varphi(\epsilon) = \epsilon \end{aligned}$$

Thus the claim 1 is established.

Consequently, $\{y_n\}$ is a Cauchy sequence in $f(X)$ and hence in $g(X)$.

Hence $y_n \to y$, say and $y \in g(X)$. Therefore there exists $z \in X,\ y = g(z)$

Claim 2: $y = f(z)$

$$\begin{aligned} d(f(z), y_n) &= d(f(z), f(x_n)) \\ &\leq \varphi(max\{d(y, y_{n-1}), d(f(z), y), d(y_n, y_{n-1})\}) \\ &\leq \varphi(max\{\epsilon, d(f(z), y)\})\ \forall\ n \geq n_0 \end{aligned}$$

On letting $n \to \infty$, we get $d(f(z), y) \leq \varphi(max\{\epsilon, d(f(z), y)\})$

This being true for every $\epsilon \gg 0$ we get from (ii)

$$d(f(z), y) \leq \varphi(d(f(z), y))$$

Therefore $y = f(z)$. Thus claim 2 is established.

Hence $f(z) = y = g(z)$, so that y is a point of coincidence to f and g.

Suppose w is a point of coincidence to f and g, then there exists $x \in X$ such that $f(x) = w = g(x)$.

Hence

$$\begin{aligned} d(w, y) &= d(f(x), f(z)) \\ &\leq \varphi(max\{d(g(x), g(z)), d(f(x), g(x)), d(f(z), g(z))\}) \\ &\leq \varphi(max\{d(w, y), d(w, w), d(y, y)\}) \\ &\leq \varphi(d(w, y)) \end{aligned}$$

$$\therefore w = y$$

Thus f and g have a unique point of coincidence in X. By Lemma 2.1 of [2], y is the unique common fixed point of f and g.

Theorem 3.9. Suppose P is a cone in a real Banach space E such that

(i) P is a lattice ordered semigroup $\cdots$ (3.9.1)

(ii) $0 \leq a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for all $x \in P$ $\cdots$ (3.9.2)

Suppose φ is a comparison function such that $\varphi : P \to P$ is a lattice homomorphism $\cdots(3.9.3)$

and $\sum \varphi^n(t)$ converges in P for $t \in P$. $\cdots(3.9.4)$

(That is, φ is S.B.C control function)

Suppose (X, d) is a cone metric space, and $f, g : X \to X$ are such that

(f, g) is *AJ's* pair $\cdots(3.9.5)$

and for all $x, y \in P$

$$d(f(x), f(y)) \leq \varphi\Big(max\Big\{d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), \frac{d(f(x), g(y)) + d(f(y), g(x))}{2}\Big\}\Big) \quad \cdots(3.9.6)$$

Then f and g have a unique point of coincidence in X. Also if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let the sequence $\{y_n\}$ be defined as in the proof of Theorem 3.8.

By Lemma 3.7 we can get

$$d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n))$$

Consequently ,

$$d(y_n, y_{n+1}) \leq \varphi^n(d(y_0, y_1))$$

For $\epsilon \gg 0$ choose a natural number n_0 and a real number δ such that

$$\epsilon - \varphi(\epsilon) + \{u \in E : ||u|| < \delta\} \subset IntP$$

Now, there exists n_0 such that

$$||\sum_{m=n}^{n+k} \varphi^m(d(y_0, y_1))|| < \delta \ \forall \ n \geq n_0 \text{ and } k = 1, 2, \cdots$$

$$\sum_{m=n}^{n+k} \varphi^m(d(y_0, y_1)) \ll \epsilon - \varphi(\epsilon) < \epsilon \quad \cdots(3.9.7)$$

for all $n \geq n_0$ and $k = 1, 2, \cdots$

$$\begin{aligned}
\text{Now } d(y_n, y_{n+k}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+k-1}, y_{n+k}) \\
&\leq \varphi^n(d(y_0, y_1)) + \varphi^{n+1}(d(y_0, y_1)) + \cdots + \varphi^{n+k-1}(d(y_0, y_1)) \\
&\ll \epsilon - \varphi(\epsilon) \text{ (By (3.9.7))} \\
&< \epsilon \text{ for } n \geq n_0
\end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence.

Hence $y_n \to y$ say and $y \in g(X)$. Therefore there exists $z \in X, y = g(z)$

Claim : $y = f(z)$

$$\begin{aligned}
d(f(z), y_n) &= d(f(z), f(x_n)) \\
&\leq \varphi\Big(max\Big\{d(y, y_{n-1}), d(f(z), y), d(y_n, y_{n-1}), \frac{d(f(z), y_{n-1}) + d(y, y_n)}{2}\Big\}\Big)
\end{aligned}$$

On letting $n \to \infty$ we get

$$d(f(z), y) \leq \varphi\Big(max\Big\{\epsilon, d(f(z), y), \frac{\epsilon + d(f(z), y)}{2}\Big\}\Big)$$

This being true for every $\epsilon \gg 0$ we get

$$d(f(z), y) \leq \varphi(d(f(z), y))$$

Therefore $y = f(z)$. Hence $f(z) = y = g(z)$, so that y is a point of coincidence to f and g. One can easily establish the uniqueness of point of coincidence. And also by Lemma 2.1 of [2], y is the unique common fixed point of f and g

Open Problem. Is Theorem 3.9 valid, if condition (3.9.4) is dropped?

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