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GENERALIZED ψ −WEAK CONTRACTION CONDITION FOR VARIANTS OF COMPATIBLE MAPPINGS IN G-METRIC SPACE

NEERU YADAV[1], PAWAN KUMAR[2], DHARMENDRA KUMAR[3,*], BALBIR SINGH[1]

[1]School of Physical Sciences, Department of Mathematics, Starex University, Gurugram, 122413, Haryana, India

[2]Department of Mathematics, Maitreyi College, University of Delhi, Chankya Puri, New Delhi-110021, India

[3]Department of Mathematics, Satyawati College (E), University of Delhi, Delhi, India

Abstract. In this paper first we present generalized $\psi -$weak contraction condition that contains cubic and quadratic terms of distance function $G(x, y, z)$ and then prove common fixed point theorems for compatible mappings in G-metric space. Secondly, we deal with variants of compatible mappings type (K), type (R) and type (E) in G-metric space. At the end, we provide applications of our results.

Keywords: ψ −weak contraction; compatible mappings.

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1. INTRODUCTION

The Banach fixed point theorem is the fundamental method for studying fixed point theory. In mathematical sciences and engineering this theorem provides a technique for solving a number of applied problems. Most of the concerns in applied mathematics are reduced to inequality, which in turn contributes to the fixed points of such mappings in their solutions.

Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point. Let $(\mathfrak{V}, d)$ be a complete metric space. If $\mathcal{T}: \mathfrak{V} \rightarrow$

*Corresponding authors
E-mail addresses: dharmendra_kumar215@yahoo.com
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$\mathfrak{V}$ satisfies $d(\mathcal{T}(x),\mathcal{T}(y)) \leq k\,(d(x,y))$ for all $x,y \in \mathfrak{V}$, $0 \leq k < 1$, then it has a unique fixed point. In 1969, Boyd and Wong [3] replaced the constant k in Banach contraction principle by an implicit function ψ and proved some fixed point theorems.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction in metric space and we use the same in G-metric space: A map $\mathcal{F}: \mathfrak{V} \to \mathfrak{V}$ is said to be weak contraction if for each $x,y \in \mathfrak{V}$, there exists a function $\emptyset : [0,\infty) \to [0,\infty)$, $\emptyset\,(t) > 0$ for all $t >$ 0 and $\emptyset\,(0) = 0$ such that $d(\mathcal{T}(x),\mathcal{T}(y)) \leq d(x,y) - \emptyset\,(d(x,y))$.

In connection with control function ψ: $R^+ \to R^+$ different authors have considered some of the following properties:

(i) ψ is non decreasing

(ii) $\psi(t) < 0\ for\ all\ t > 0$

(iii) $\psi(0) = 0$

(iv) ψ is continuous

(v) $\lim n \to \infty\ \psi^n(t) = 0$ for all $t \geq 0$

(vi) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all t>0, $\psi^n$is the nth iterate

(vii) $\psi(t) = 0$ iff $t = 0$

(viii) $\psi(t) > 0$ for all $t \in R^+\backslash\{0\}$

(ix) $\lim r \to t^+ \psi\cdot(t) < 0\ for\ all\ t > 0$

(x) $\lim t \to \infty\ \psi\cdot(t)=\infty$

(xi) ψ is lower semi continuous

Here we note that

(i) and (ii) implies (iii) ;

(ii) and (iv) implies (iii)

(i) and (v) implies (ii)

A function ψ satisfying (i) and (v) that is ψ is non decreasing and $\lim n \to \infty\ \psi^n(t) =$ 0 for all $t \geq 0$ is called as a comparison function.

Several fixed point theorems and common fixed point theorems have been unified on sidering a general condition by an implicit function.

2. PRELIMINARIES

In 2006, Zead Mustafa and Brailey Sims [14] introduced the notion of G-metric space as generalization of the concept of ordinary metric space.

Definition 2.1[14] A G-metric space is a pair $(\mathfrak{V}, G)$, where $\mathfrak{V}$ is a non-empty set and G is a non-negative real-valued function defined on $\mathfrak{V} \times \mathfrak{V} \times \mathfrak{V}$ such that for all $x, y, z, a \in \mathfrak{V}$, we have

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < G(x, x, y)$, for all $x, y \in \mathfrak{V}$, with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$,for all $x, y, z \in \mathfrak{V}$, with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$,for all $x, y, z, a \in \mathfrak{V}$ (rectangle inequality),

The function G is called G-metric on $\mathfrak{V}$.

Definition 2.2[15] A sequence x_n in a G-metric space $\mathfrak{V}$ is said to be convergent if there exist $x \in \mathfrak{V}$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and one says that the sequence (x_n) is G-convergent to x. We call x the limit of the sequence (x_n) and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.3[15] In a G-metric space $\mathfrak{V}$, a sequence (x_n) is said to be G-cauchy if given $\in > 0$, there is $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \in$,for all $n, m, l \geq n_0$ i.e., $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.1[15] Let $\mathfrak{V}$ be G-metric space. Then the following statements are equivalent:

(i) (x_n) is G-convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $n \to \infty$.

Proposition 2.2[15] Let $\mathfrak{V}$ be G-metric space. Then the following statements are equivalent:

(i) The sequence (x_n) is G-cauchy;

(ii) For every $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \in$, $\forall\, n, m \geq n_0$.

In 1986, Jungck [7] introduced the compatible mappings in metric space and in 2012, Choudhury et.al [5] introduced the notion of compatible mappings in G-metric space as follows:

Definition 2.4[5] Two self-mappings f and g of a G-metric space $(\mathfrak{V}, G)$ are said to be compatible if

$$lim_n G(fgx_n, gfx_n, gfx_n) = 0 \text{ or } lim_n G(gfx_n, fgx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

Now we state some properties for compatible mappings that are fruitful for further study.

Proposition 2.3[7] Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of a metric space $(\mathfrak{V}, d)$ into itself. If $\mathcal{S}t = \mathcal{T}t$, for some t in $\mathfrak{V}$, then $\mathcal{S}\mathcal{T}t = \mathcal{S}\mathcal{S}t = \mathcal{T}\mathcal{T}t = \mathcal{T}\mathcal{S}t$.

Proposition 2.4 [7] Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of a metric space $(\mathfrak{V}, d)$ into itself. Suppose that $lim_n \mathcal{S}x_n = lim_n \mathcal{T}x_n = t$ for some t in $\mathfrak{V}$. Then the following holds:

(i) $lim_n \mathcal{T}\,\mathcal{S}x_n = \mathcal{S}t$ if S is continuous at t;

(ii) $lim_n \mathcal{S}\mathcal{T}x_n = \mathcal{T}t$ if $\mathcal{T}$ is continuous at t;

(iii) $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$ and $\mathcal{S}t = \mathcal{T}t$ if $\mathcal{S}$ and $\mathcal{T}$ are continuous at t.

Now we introduce the generalized ψ - weak contraction for a pairs of mappings in the G-metric space as follows:

Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings on a G-metric space $(\mathfrak{V}, G)$ satisfying the following conditions:

(C1) $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V}), \mathcal{T}(\mathfrak{V}) \subset \mathcal{A}(\mathfrak{V})$;

(C2) $$G^3(\mathcal{S}u, \mathcal{T}v, \mathcal{T}v) \leq \psi \left\{ \begin{array}{c} G^2(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v), \\ G(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G^2(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v), \\ G(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G(\mathcal{A}u, \mathcal{T}v, \mathcal{T}v)G(\mathcal{B}v, \mathcal{S}u, \mathcal{S}u), \\ G(\mathcal{A}u, \mathcal{T}v, \mathcal{T}v)G(\mathcal{B}v, \mathcal{S}u, \mathcal{S}u)G(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \end{array} \right\}$$

for all $u, v \in \mathfrak{V}$, where ψ: $[0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$.

In this section, we prove a result for compatible mappings that satisfy generalized ψ −weak contraction in G-metric space involving cubic and quadratic terms of distance function.

Theorem 2.1 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying (C1) and (C2) and the following conditions:

(2.1) one of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ is continuous.

Assume that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible. Then $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point in $\mathfrak{V}$.

Proof. Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C1) we can find x_1 such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$, for this x_1 one can find $x_2 \in \mathfrak{V}$ such that $\mathcal{T}(x_1) = \mathcal{A}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that

$$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1}),$$
$$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}), \text{ for each } n \geq 0. \tag{2.2}$$

For brevity, we write $\sigma_{2n} = G(y_{2n}, y_{2n+1}, y_{2n+1})$

First, we prove that $\{\sigma_{2n}\}$ is non-increasing sequence and converges to zero.

Case I If n is even, taking $u = x_{2n}$ and $v = x_{2n+1}$ in (C2), we get

$$G^3(\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})$$
$$\leq \psi \left\{ \begin{matrix} G^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}) G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \\ , G(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}) G^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}) G(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) G(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}), \\ G(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) G(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}) G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{matrix} \right\}$$

Using (2.2), we have

$$G^3(y_{2n}, y_{2n+1}, y_{2n+1})$$
$$\leq \psi \left\{ \begin{matrix} G^2(y_{2n-1}, y_{2n}, y_{2n}) G(y_{2n}, y_{2n+1}, y_{2n+1}) \\ , G(y_{2n-1}, y_{2n}, y_{2n}) G^2(y_{2n}, y_{2n+1}, y_{2n+1}), \\ G(y_{2n-1}, y_{2n}, y_{2n}) G(y_{2n-1}, y_{2n+1}, y_{2n+1}) G(y_{2n}, y_{2n}, y_{2n}), \\ G(y_{2n-1}, y_{2n+1}, y_{2n+1}) G(y_{2n}, y_{2n}, y_{2n}) G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{matrix} \right\}$$

On using $\sigma_{2n} = G(y_{2n}, y_{2n+1}, y_{2n+1})$, in the above inequality, we have

$$\sigma_{2n}^3 \leq \psi\{\sigma_{2n-1}^2 \sigma_{2n}, \sigma_{2n-1} \sigma_{2n}^2, 0,0\} \tag{2.3}$$

By using rectangular inequality and property of ψ, we get

$$G(y_{2n-1}, y_{2n+1}, y_{2n+1}) \leq G(y_{2n-1}, y_{2n}, y_{2n}) + G(y_{2n}, y_{2n+1}, y_{2n+1})$$
$$= \sigma_{2n-1} + \sigma_{2n}$$

or $\quad \sigma_{2n-1} \leq \sigma_{2n-1} + \sigma_{2n}$

If $\sigma_{2n-1} < \sigma_{2n}$ and using property of ψ, then (2.3) reduces to

$\sigma_{2n}^3 < \sigma_{2n}^3$, a contradiction, therefore, $\sigma_{2n} \leq \sigma_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\sigma_{2n+1} < \sigma_{2n}$.

It follows that the sequence $\{\sigma_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \sigma_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$; then from inequality (C2) and (2.2) and (2.3), we have

$r^3 \leq \psi\,(r^3) < r^3$, a contradiction, thus we have $r = 0$. Then

$$\lim_{n\to\infty} \sigma_{2n} = \lim_{n\to\infty} G(y_{2n}, y_{2n+1}, y_{2n+1}) = r = 0. \tag{2.4}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$.

$$G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \geq \epsilon,\ \ G\big(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}\big) < \epsilon \tag{2.5}$$

Now $\quad \epsilon \leq G\big(y_{m(k)}, y_{n(k)}, y_{n(k)}\big) \leq G\big(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}\big) + G(y_{n(k)-1}, y_{n(k)}, y_{n(k)})$

Letting $k \to \infty$, and using (2.4) and (2.5), we get $\lim_{k\to\infty} G\big(y_{m(k)}, y_{n(k)}, y_{n(k)}\big) = \epsilon$

Now from the rectangular inequality, we have,

$$\big|G\big(y_{n(k)}, y_{m(k)+1}, y_{m(k)+1}\big) - G\big(y_{m(k)}, y_{n(k)}, y_{n(k)}\big)\big| \leq G\big(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1}\big).$$

Again taking limits as $k \to \infty$ and using (2.4) and (2.5), we have

$$\lim_{k\to\infty} G\big(y_{n(k)}, y_{m(k)+1}, y_{m(k)+1}\big) = \epsilon.$$

On using rectangular inequality, we have

$$\big|G\big(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}\big) - G\big(y_{m(k)}, y_{n(k)}, y_{n(k)}\big)\big| \leq G\big(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}\big).$$

Proceeding limits as $k \to \infty$ and using (2.4) and (2.5), we get

$$\lim_{k\to\infty} G\big(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}\big) = \epsilon.$$

Similarly, we have

$$\big|G\big(y_{m(k)+1}, y_{n(k)+1}, y_{n(k)+1}\big) - G\big(y_{m(k)}, y_{n(k)}, y_{n(k)}\big)\big| \leq G\big(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1}\big) + G\big(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}\big).$$

Taking limit as $k \to \infty$ in the above inequality and using (2.4) and (2.5), we have

$$\lim_{k\to\infty} G\big(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}\big) = \epsilon.$$

On putting $u = x_{m(k)}$ and $v = x_{n(k)}$ in (C2), we have

$$G^3\left(\mathcal{S}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right) \leq$$

$$\leq \psi \left\{ \begin{array}{c} G^2\left(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}\right)G\left(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right) \\ , G\left(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}\right)G^2\left(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right), \\ G\left(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}\right)G\left(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right)G\left(\mathcal{B}x_{2n+1}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}\right), \\ G\left(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right)G\left(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}\right)G\left(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}\right) \end{array} \right\}$$

Using (2.2), we obtain

$$G^3\left(y_{m(k)}, y_{n(k)}, y_{n(k)}\right)$$

$$\leq \psi \left\{ \begin{array}{c} G^2\left(y_{m(k)-1}, y_{m(k)}, y_{m(k)}\right)G\left(y_{n(k)-1}, y_{n(k)}, y_{n(k)}\right), \\ G\left(y_{m(k)-1}, y_{m(k)}, y_{m(k)}\right)G^2\left(y_{n(k)-1}, y_{n(k)}, y_{n(k)}\right) \\ G\left(y_{m(k)-1}, y_{m(k)}, y_{m(k)}\right)G\left(y_{m(k)-1}, y_{n(k)}, y_{n(k)}\right)G\left(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\right), \\ G\left(y_{m(k)-1}, y_{n(k)}, y_{n(k)}\right)G\left(y_{n(k)-1}, y_{m(k)}, y_{m(k)}\right)G\left(y_{n(k)-1}, y_{n(k)}, y_{n(k)}\right) \end{array} \right\}$$

Letting $k \to \infty$, and using property of ψ , we have

$\epsilon^3 \leq 0$, which is a contradiction.

Hence the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, \mathrm{G})$ is a complete G-metric space, therefore, $\{y_n\}$ converge to a point z in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converge to the same point z.

Now suppose that $\mathcal{A}$ is continuous. Then $\{\mathcal{A}\mathcal{A}x_{2n}\}$ and $\{\mathcal{A}\mathcal{S}x_{2n}\}$ converges to $\mathcal{A}z$ as $n \to \infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible in $\mathfrak{V}$, then by Proposition 2.4 that $\{\mathcal{S}\mathcal{A}x_{2n}\}$ converge to $\mathcal{A}z$ as $n \to \infty$.

Now we claim that $z = \mathcal{A}z$. For this put $u = \mathcal{A}x_{2n}$ and $v = x_{2n+1}$ in (C2), we get

$$G^3(\mathcal{S}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \leq$$

$$\psi \left\{ \begin{array}{c} G^2(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})G^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})G(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}), \\ G(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

or

$$G^3(\mathcal{A}z, z, z) \leq \psi \left\{ \begin{array}{c} G^2(\mathcal{A}z, \mathcal{A}z, \mathcal{A}z)G(z, z, z), \\ G(\mathcal{A}z, \mathcal{A}z, \mathcal{A}z)G^2(z, z, z), \\ G(\mathcal{A}z, \mathcal{A}z, \mathcal{A}z)G(\mathcal{A}z, z, z)G(z, \mathcal{A}z, \mathcal{A}z), \\ G(\mathcal{A}z, z, z)G(z, \mathcal{A}z, \mathcal{A}z)G(z, z, z) \end{array} \right\}$$

Therefore, we have

$G^3(\mathcal{A}z, z, z) \leq \psi\{0,0,0,0\}$, using property of ψ, we have $\mathcal{A}z = z$.

Now we claim that $z = \mathcal{S}z$. For this put $u = z$ and $v = x_{2n+1}$ in (C2), we get

$$G^3(\mathcal{S}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})$$

$$\leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{Bmatrix}$$

$$\text{or} \quad G^3(\mathcal{S}z, z, z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(z, z, z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(z, z, z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, z, z)G(z, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, z, z)G(z, \mathcal{S}z, \mathcal{S}z)G(z, z, z) \end{Bmatrix}$$

Therefore, we have

$G^3(\mathcal{S}z, z, z) \leq \psi\{0,0,0,0\}$, using property of ψ, we have $G^3(\mathcal{S}z, z, z) = 0$.

This implies that $\mathcal{S}z = z$. Since $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V})$ and hence there exists a point $p \in \mathfrak{V}$ such that $z = \mathcal{S}z = \mathcal{B}p$.

We claim that $z = \mathcal{T}u$. To prove this we put $u = z$ and $v = p$ in (C2), we get

$$G^3(\mathcal{S}z, \mathcal{T}p, \mathcal{T}p) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}p, \mathcal{T}p, \mathcal{T}p), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(\mathcal{B}p, \mathcal{T}p, \mathcal{T}p), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, \mathcal{T}p, \mathcal{T}p)G(\mathcal{B}p, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}p, \mathcal{T}p)G(\mathcal{B}p, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}p, \mathcal{T}p, \mathcal{T}p) \end{Bmatrix}$$

On simplification, and using property of ψ, we have

$$G^3(z, \mathcal{T}p, \mathcal{T}p) \leq \psi \begin{Bmatrix} G^2(z, z, z)G(z, \mathcal{T}p, \mathcal{T}p), \\ G(z, z, z)G^2(z, \mathcal{T}p, \mathcal{T}p), \\ G(z, z, z)G(z, \mathcal{T}p, \mathcal{T}p)G(z, z, z), \\ G(z, \mathcal{T}p, \mathcal{T}p)G(z, z, z)G(z, \mathcal{T}p, \mathcal{T}p) \end{Bmatrix}$$

This implies that $z = \mathcal{T}p$. Since $(\mathcal{B}, \mathcal{T})$ is compatible in $\mathfrak{V}$ and $\mathcal{B}p = \mathcal{T}p = z$, by Proposition 2.3, we have $\mathcal{B}\mathcal{T}p = \mathcal{T}\mathcal{B}p$ and hence $\mathcal{B}z = \mathcal{B}\mathcal{T}p = \mathcal{T}\mathcal{B}p = \mathcal{T}z$. Also, we have

$$G^3(\mathcal{S}z, \mathcal{T}z, \mathcal{T}z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, \mathcal{T}z, \mathcal{T}z)G(\mathcal{B}z, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}z, \mathcal{T}z)G(\mathcal{B}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{Bmatrix}$$

Therefore, we obtain

$G^3(z,\mathcal{B}z,\mathcal{B}z) \leq \psi\{0,0,0,0\}$, using property of ψ, we have

i.e., $G^3(z,\mathcal{B}z,\mathcal{B}z) \leq 0$.

This implies that $z=\mathcal{B}z$. Hence $z=\mathcal{B}z=\mathcal{T}z=\mathcal{A}z=\mathcal{S}z$. Therefore, z is a common fixed point of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Similarly, one can also complete the proof when $\mathcal{B}$ is continuous.

Next, suppose that $\mathcal{S}$ is continuous.

Then $\{\mathcal{S}\mathcal{S}x_{2n}\}$ and $\{\mathcal{S}\mathcal{A}x_{2n}\}$ converges to $\mathcal{S}z$ as $n\to\infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are compatible on $\mathfrak{V}$, it follows from the proposition 2.4 that $\{\mathcal{A}\mathcal{S}x_{2n}\}$ converges to $\mathcal{S}z$ as $n\to\infty$.

Now we claim that $z=\mathcal{S}z$. For this put $u=\mathcal{S}x_{2n}$ and $v=x_{2n+1}$ in (C2), we get

$$G^3(\mathcal{S}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})$$

$$\leq \psi\left\{\begin{array}{c} G^2(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G^2(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G\left(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1,},\mathcal{T}x_{2n+1}\right)G(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n}),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \end{array}\right\}$$

Now proceeding limit as $n\to\infty$ and using the property of ψ, we have

$$G^3(\mathcal{S}z,z,z) \leq \psi\left\{\begin{array}{c} G^2(\mathcal{S}z,\mathcal{S}z,\mathcal{S}z)G(z,z,z),\\ G(\mathcal{S}z,\mathcal{S}z,\mathcal{S}z)G^2(z,z,z),\\ G(\mathcal{S}z,\mathcal{S}z,\mathcal{S}z)G(\mathcal{S}z,z,z)G(z,\mathcal{S}z,\mathcal{S}z),\\ G(\mathcal{S}z,z,z)G(z,\mathcal{S}z,\mathcal{S}z)G(z,z,z) \end{array}\right\}$$

Therefore, we have $G^3(\mathcal{S}z,z,z) \leq \psi\{0,0,0,0\}$,using property of ψ, we have $G^3(\mathcal{S}z,z,z)=0$. This implies that $\mathcal{S}z=z$. Since $\mathcal{S}(\mathfrak{V})\subset\mathcal{B}(\mathfrak{V})$ and hence there exists a point $q\in\mathfrak{V}$ such that $z=\mathcal{S}z=\mathcal{B}q$.

We claim that $z=\mathcal{T}q$. To prove this, we put $u=\mathcal{S}x_{2n}$ and $v=q$ in (C2) we get

$G^3(\mathcal{S}\mathcal{S}x_{2n},\mathcal{T}q,\mathcal{T}q)$

$$\leq \psi\left\{\begin{array}{c} G^2(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G^2(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}q,\mathcal{T}q)G(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n}),\\ G(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}q,\mathcal{T}q)G(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})G(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q) \end{array}\right\}$$

Therefore, we get

$$G^3(z,\mathcal{T}q,\mathcal{T}q) \le \psi\left\{\begin{matrix} G^2(z,z,z)G(z,\mathcal{T}q,\mathcal{T}q), \\ G(z,z,z)G^2(z,\mathcal{T}q,\mathcal{T}q), \\ G(z,z,z)G(z,\mathcal{T}q,\mathcal{T}q)G(z,z,z), \\ G(z,\mathcal{T}q,\mathcal{T}q)G(z,z,z)G(z,\mathcal{T}q,\mathcal{T}q) \end{matrix}\right\}.$$

Using the property of ψ, we have $z = \mathcal{T}q$. Since $(\mathcal{B},\mathcal{T})$ is a compatible pair of mappings, so $\mathcal{B}q = \mathcal{T}q = z$ and by using Proposition 2.3 we have $\mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q$ and hence $\mathcal{B}z = \mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$G^3(\mathcal{S}x_{2n},\mathcal{T}z,\mathcal{T}z) \le \psi\left\{\begin{matrix} G^2(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n}), \\ G(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{matrix}\right\}$$

Proceeding limit as $n \to \infty$,weget

$$G^3(z,\mathcal{T}z,\mathcal{T}z) \le \psi\{0,0,0,0\}$$

Using the property of ψ, we have $z = \mathcal{T}z$. Since $\mathcal{T}(\mathfrak{B}) \subset \mathcal{A}(\mathfrak{B})$, therefore there exists a point $w \in \mathfrak{B}$ such that $z = \mathcal{T}z = \mathcal{A}w$.

We claim that $z = \mathcal{S}w$. On putting $u = w$ and $v = z$ in (C2) we get

$$G^3(\mathcal{S}w,\mathcal{T}z,\mathcal{T}z) \le \psi\left\{\begin{matrix} G^2(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)G^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)G(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w), \\ G(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w)G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{matrix}\right\}$$

Therefore, $$G^3(\mathcal{S}w,z,z) \le \psi\left\{\begin{matrix} G^2(z,\mathcal{S}w,\mathcal{S}w)G(z,z,z), \\ G(z,\mathcal{S}w,\mathcal{S}w)G^2(z,z,z), \\ G(z,\mathcal{S}w,\mathcal{S}w)G(z,z,z)G(z,\mathcal{S}w,\mathcal{S}w), \\ G(z,z,z)G(z,\mathcal{S}w,\mathcal{S}w)G(z,z,z) \end{matrix}\right\}$$

This implies that $\mathcal{S}w = z$. Since pair $(\mathcal{S},\mathcal{A})$ is compatible on $\mathfrak{B}$, so, $\mathcal{S}w = \mathcal{A}w = z$ and by Proposition 2.3, we have $\mathcal{A}\mathcal{S}w = \mathcal{S}\mathcal{A}w$. Thus $\mathcal{A}z = \mathcal{A}\mathcal{S}w = \mathcal{S}\mathcal{A}w = \mathcal{S}z$.

i.e., $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$. Therefore, z is a common fixed point of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Similarly, we can complete the proof when $\mathcal{T}$ is continuous.

Uniqueness: Suppose $z \ne w$ be two common fixed points of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Put $u = z$ and $v = w$ in (C2), we get

$$G^3(\mathcal{S}z,\mathcal{T}w,\mathcal{T}w) \le \psi\{0,0,0,0\}$$

$$G^3(\mathcal{S}z, \mathcal{T}w, \mathcal{T}w) \leq \psi\{0,0,00\}$$

On simplification, using the property of ψ, we have we have $G^2(z, w, w) = 0$

$i.e.,\ z = w.$ This completes the proof.

3. VARIANTS OF COMPATIBLE MAPPINGS AND FIXED POINTS

In 1986, Jungck [7] introduced more generalized commutativity, so called compatibility. In 1998, Pant [24] introduced a new notion of continuity and called it reciprocally continuous mappings. In 2001, Sahu et al. [29] introduced the notion of intimate mappings in metric spaces. Intimate mappings are more improved version of weakly commuting, semi-compatibility and R-commutativity etc. Sahu et al. [29] have also shown that intimate mappings are more general than compatible mappings. The most crucial feature of intimate mappings is that these mappings do not necessarily commute at a coincidence point. It is the generalization of compatible mappings of type (A). In 2004, Rohan et al. [27] introduced the concept of compatible mappings of type (R) by using the notion of compatible mappings and compatible mappings of type (P) together. In 2007, Singh and Singh [30] introduced the concept of compatible mappings of type (E) by rearranging terms of compatible mappings of type (P) and compatible mappings. In 2014, Jha et al. [12] introduced the concept of compatible mappings of type (K) by modification in compatible mappings of type (P) in a metric space.

In 1993, Jungck et al. [11] introduced the notion of compatible mappings of type(A) as follows:

Definition 3.1 [11] Two self mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type(A) if $lim_n d(ffx_n, gfx_n) = 0$ and $lim_n d(ggx_n, fgx_n) = 0$,
whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 1995, Pathak et al. [20] introduced the notion of compatible mappings of type(P) as follows:

Definition 3.2[20] Two self mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type(P) if $lim_n d(ffx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 1998, Pant [24] defined the notion of reciprocally continuous mappings. In fact, it is the generalization of continuous mappings.

Dentition 3.3[24] Two self mappings f and g of a metric space $(\mathfrak{V}, d)$ are called reciprocally continuous if $lim_n fgx_n = ft$ and $lim_n gfx_n = gt$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

If f and g are both continuous, then maps are reciprocally continuous, but the converse need not be true.

In similar mode we define the notions of compatible mappings of type(A), notion of compatible mappings of type(P) and reciprocally continuous mappings in setting of G-metric spaces as follows:

Definition 3.4 Two self mappings f and g of a G-metric space $(\mathfrak{V}, G)$ are said to be :

(i) Compatible of type(A) if $lim_n G(ffx_n, gfx_n, gfx_n) = 0$ and $lim_n G(ggx_n, fgx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

(ii) Compatible of type(P) if $lim_n G(ffx_n, ggx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

(iii) Reciprocally continuous if $lim_n G(fgx_n, ft, ft) = 0$ and $lim_n G(gfx_n, gt, gt) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 2001, Sahu et al. [29] introduced the notion of intimate mappings in metric space. In fact, it is the generalization of compatible mappings of type (A).

Definition 3.5[29] Let f and g are two mappings of a metric space $(\mathfrak{V}, d)$ into itself. Then f and g are said to be:

(1) g-intimate mappings if $\alpha d(gfx_n, gx_n) \leq \alpha d(ffx_n, fx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and $\alpha = lim\ sup\ or\ lim\ inf$.

(2) f-intimate mappings if $\alpha d(fgx_n, fx_n) \leq \alpha d(ggx_n, gx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and $\alpha = lim\ sup\ or\ lim\ inf$.

In similar mode we define the Intimate mappings in G-metric space as follows:

Definition 3.6 Let f and g are two mappings of a G-metric space $(\mathfrak{V}, G)$ into itself. Then f and g are said to be:

(1) g-intimate mappings if $\alpha G(gfx_n, gx_n, gx_n) \leq \alpha G(ffx_n, fx_n, fx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and $\alpha =$ limit inferior or limit superior.

(2) f-intimate mappings if $\alpha G(fgx_n, fx_n, fx_n) \leq \alpha G(ggx_n, gx_n, gx_n)$, where $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$ and $\alpha =$ limit inferior or limit superior.

In 2004, Rohan et al. [27] introduced the concept of compatible mappings of type (R) as follows:

Definition 3.7[27] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (R) if $lim_n d(fgx_n, gfx_n) = 0$ and $lim_n d(ffx_n, ggx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 2007, Singh and Singh [30] introduced the concept of compatible mappings of type (E):

Definition 3.8 [30] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (E) if $lim_n ffx_n = lim_n fgx_n = gt$ and $lim_n ggx_n = lim_n gfx_n = ft$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In 2014, Jha et al. [12] introduced the concept of compatible mappings of type (K):

Definition 3.9[12] Two self-mappings f and g of a metric space $(\mathfrak{V}, d)$ are called compatible of type (K) if $lim_n d(ffx_n, gt) = 0$ and $lim_n d(ggx_n, ft) = 0$,
whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

In similar mode we define the notions of compatible mappings of type(R), notion of compatible mappings of type (E) and compatible mappings of type (K) in setting of G-metric space as follows:

Definition 3.10 Two self mappings f and g of a G-metric space $(\mathfrak{V}, G)$ are said to be :

(i) Compatible of type(R) if $lim_n G(fgx_n, gfx_n, gfx_n) = 0$

and $lim_n G(ffx_n, ggx_n, ggx_n) = 0$,whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

(ii) Compatible of type(E) if $lim_n G(ffx_n, fgx_n, gt) = 0$

and $lim_n G(ggx_n, gfx_n, ft) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

(iii) compatible mappings of type (K) if $lim_n G(ffx_n, gt, gt) = 0$ and $lim_n G(ggx_n, ft, ft) = 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{V}$ such that $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.

We describe the relationship among compatible maps and its variants in metric space which are useful for proving our main results.

Remark 3.1 One can note that compatible mapping of type (R) is compatible mapping as well as compatible mappings of type (P).

Proposition 3.1 [30] Suppose f and g be compatible mappings of type (E) of a metric space $(\mathfrak{V}, d)$ into itself and one of f and g be continuous. Suppose $lim_n fx_n = lim_n gx_n = t$, for some t in $\mathfrak{V}$.. Then we have the following:

(a) $ft = gt$ and $lim_n ffx_n = lim_n ggx_n = lim_n fgx_n = lim_n gfx_n$.

(b)If there exists $u \in \mathfrak{V}$ such that $fu = gu = t$, then $fgu = gfu$.

Proposition 3.2 Let f and g be two mappings of a metric space $(\mathfrak{V}, d)$ into itself. If f and g are compatible mappings of type (A), then f and g are f-intimate and g-intimate.

Remark 3.2 If a pair (f, g) is f-intimate or g-intimate then it need not be necessarily compatible of type (A).

Proposition 3.3 Let f and g be two mappings of a metric space $(\mathfrak{V}, d)$ into itself Assume that f and g are g-intimate and $ft = gt = q \in \mathfrak{V}$. Then $G(gq, q, q) \leq G(fq, q, q)$.

We now prove some results in G-metric space related to compatible mappings of type (K), type (R), type (E) and intimate mappings that satisfy generalized ψ −weak contraction condition that involves cubic and quadratic terms of distance function.

Theorem 3.1 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying (C1) and (C2) and the following conditions:

(3.1) the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are reciprocally continuous,

(3.2) the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type (K).

Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. From the Theorem 2.1, we conclude the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, \mathrm{G})$ is a complete G-metric space, therefore, $\{y_n\}$ converges to a point z in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converges to the same point z. Now Since the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type(K), we have $\mathcal{A}\mathcal{A}x_{2n} \to \mathcal{S}z$, $\mathcal{S}\mathcal{S}x_{2n} \to \mathcal{A}z$ and $\mathcal{B}\mathcal{B}x_{2n} \to \mathcal{T}z$, $\mathcal{T}\mathcal{T}x_{2n} \to \mathcal{B}z$ as $n \to \infty$.

Now we claim that $\mathcal{B}z = \mathcal{A}z$. For this put $u = \mathcal{S}x_{2n}$ and $v = \mathcal{T}x_{2n+1}$ in (C2) we get

$$G^3(\mathcal{S}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1})$$
$$\leq \psi \left\{ \begin{array}{c} G^2(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ G(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) G^2(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ G(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) G(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}), \\ G(\mathcal{A}\mathcal{S}x_{2n}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) \end{array} \right\}$$

Letting $n \to \infty$ and using reciprocal continuity of the pairs $(\mathcal{A}, \mathcal{S})$and $(\mathcal{B}, \mathcal{T})$, we have $G^3(\mathcal{B}z, \mathcal{A}z, \mathcal{A}z) \leq \psi\{0,0,0,0\}$,using property of ψ , we have $G^3(\mathcal{B}z, \mathcal{A}z, \mathcal{A}z) = 0$. This implies that $\mathcal{B}z = \mathcal{A}z$.

Next, we claim that $\mathcal{S}z = \mathcal{B}z$. On putting $u = z$ and $v = \mathcal{T}x_{2n+1}$ in (C2) we get

$$G^3(\mathcal{S}z, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1})$$
$$\leq \psi \left\{ \begin{array}{c} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z) G^2(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z) G(\mathcal{A}z, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{S}z, \mathcal{S}z) G(\mathcal{B}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}, \mathcal{T}\mathcal{T}x_{2n+1}) \end{array} \right\}$$

Letting $n \to \infty$ and using reciprocal continuity of the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$, we have

$$G^3(\mathcal{S}z, \mathcal{B}z, \mathcal{B}z) \leq \psi\{0,0,0,0\}.$$

Using property of ψ, we have $G^3(\mathcal{S}z, \mathcal{B}z, \mathcal{B}z) = 0$.This implies that $\mathcal{S}z = \mathcal{B}z$.

Now we claim that $\mathcal{S}z = \mathcal{T}z$. On putting $u = z$ and $v = z$ in (C2) we get

$$G^3(\mathcal{S}z,\mathcal{T}z,\mathcal{T}z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)G^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)G(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z), \\ G(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z)G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{Bmatrix}$$

Proceeding limit as $n \to \infty$,weget

$$G^3(\mathcal{S}z,\mathcal{T}z,\mathcal{T}z) \leq \psi\{0,0,0,0\}.$$

Thus $G^3(\mathcal{S}z,\mathcal{T}z,\mathcal{T}z) = 0$, implies that $\mathcal{S}z = \mathcal{T}z$.

Now we claim that $z = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$G^3(\mathcal{S}x_{2n},\mathcal{T}z,\mathcal{T}z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ G(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n}), \\ G(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)G(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n})G(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{Bmatrix}$$

Proceeding limit as $n \to \infty$,we get

$G^3(z,\mathcal{T}z,\mathcal{T}z) \leq \psi\{0,0,0,0\}$. Uniqueness follows easily

Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

First, we prove the following theorem for compatible mappings of type (R).

Theorem 3.2 Let $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying (C1) and (C2) and the following conditions:

(3.3) One of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$ is continuous.

Assume that the pairs $(\mathcal{A},\mathcal{S})$and $(\mathcal{B},\mathcal{T})$ are compatible of type (R). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. The proof follows from Remark 2.1 and from the compatible mappings.

Finally, we prove the following theorem for pairs of compatible mappings of type (E).

Theorem 3.3 Let$\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying (C1) and (C2). Suppose that one of $\mathcal{A}$ and $\mathcal{S}$ is continuous, and one of $\mathcal{B}$ and $\mathcal{T}$ is continuous. Assume that the pairs $(\mathcal{A},\mathcal{S})$and $(\mathcal{B},\mathcal{T})$ are compatible of type(E). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. From the proof of Theorem 2.1,sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$, but $(\mathfrak{V}, \mathrm{G})$ is a complete G-metric space, therefore, $\{y_n\}$converges to a point $z$in $\mathfrak{V}$as $n \to \infty$. Consequently, the subsequences $\{\mathcal{S}x_{2n}\}, \{\mathcal{A}x_{2n}\}, \{\mathcal{T}x_{2n+1}\}$ and $\{\mathcal{B}x_{2n+1}\}$ also converges to the same point z. Now Since the pairs $(\mathcal{A}, \mathcal{S})$ are compatible of type(E) and one of $\mathcal{A}$ and $\mathcal{S}$ is continuous, then by Proposition 3.1, $\mathcal{A}z = \mathcal{S}z$.Since $\mathcal{S}(\mathfrak{V}) \subset \mathcal{B}(\mathfrak{V})$,therefore, there exists a point $q \in \mathfrak{V}$ such that $\mathcal{S}z = \mathcal{B}q$. On putting $u = z$ and $v = q$ in (C2) we get

$$G^3(\mathcal{S}z, \mathcal{T}q, \mathcal{T}q) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}q, \mathcal{T}q, \mathcal{T}q), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(\mathcal{B}q, \mathcal{T}q, \mathcal{T}q), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, \mathcal{T}q, \mathcal{T}q)G(\mathcal{B}q, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}q, \mathcal{T}q)G(\mathcal{B}q, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}q, \mathcal{T}q, \mathcal{T}q) \end{Bmatrix}$$

Therefore, we get

$G^3(\mathcal{S}z, \mathcal{T}q, \mathcal{T}q) \leq \psi\{0,0,0,0\}$, using property of ψ, we have

This implies that $\mathcal{S}z = \mathcal{T}q$. Thus we have $\mathcal{A}z = \mathcal{S}z = \mathcal{T}q = \mathcal{B}q$.

On putting $u = z$ and $v = x_{2n+1}$ in (C2) we get

$$G^3(\mathcal{S}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})$$

$$\leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})G(\mathcal{B}x_{2n+1}, \mathcal{S}z, \mathcal{S}z)G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{Bmatrix}$$

or $$G^3(\mathcal{S}z, z, z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(z, z, z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G^2(z, z, z), \\ G(\mathcal{A}z, \mathcal{S}z, \mathcal{S}z)G(\mathcal{A}z, z, z)G(z, \mathcal{S}z, \mathcal{S}z), \\ G(\mathcal{A}z, z, z)G(z, \mathcal{S}z, \mathcal{S}z)G(z, z, z) \end{Bmatrix}.$$

Therefore, we have

$G^3(\mathcal{S}z, z, z) \leq \psi\{0,0,0,0\}$,using property of ψ, we have $G^3(\mathcal{S}z, z, z) = 0$.

This implies that $\mathcal{A}z = \mathcal{S}z = z$.

Now assume that the pair$(\mathcal{B}, \mathcal{T})$are compatible of type (E) and one of $\mathcal{B}$ and $\mathcal{T}$ is continuous. Then we get $\mathcal{B}q = \mathcal{T}q = z$. By Proposition 2.1, we have $\mathcal{B}\mathcal{B}q = \mathcal{B}\mathcal{T}q = \mathcal{T}\mathcal{B}q = \mathcal{T}\mathcal{T}q$, that is $\mathcal{B}z = \mathcal{T}z$. Now we claim that $z = \mathcal{T}z$. On putting $u = x_{2n}$ and $v = z$ in (C2), we have

$$G^3(\mathcal{S}x_{2n}, \mathcal{T}z, \mathcal{T}z) \leq \psi \begin{Bmatrix} G^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})G(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ G(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})G^2(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ G(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})G(\mathcal{A}x_{2n}, \mathcal{T}z, \mathcal{T}z)G(\mathcal{B}z, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}), \\ G(\mathcal{A}x_{2n}, \mathcal{T}z, \mathcal{T}z)G(\mathcal{B}z, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})G(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{Bmatrix}$$

Proceeding limit as $n \to \infty$,weget

$$G^3(z, \mathcal{T}z, \mathcal{T}z) \leq \psi\{0,0,0,0\}$$

This implies that $z = \mathcal{T}z$.

Uniqueness follows easily. Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

At the last, we prove a common fixed point theorem for pairs of intimate mappings. In fact intimate mappings are generalizations of compatible mappings of type (A).

Theorem 3.4 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ are four self mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying (C1) and (C2) and the following conditions:

(3.4) the pair $(\mathcal{A}, \mathcal{S})$ is$\mathcal{A}$ -intimate and pair $(\mathcal{B}, \mathcal{T})$ is $\mathcal{B}$ -intimate;

(3.5) $\mathcal{A}(\mathfrak{V})$ is a complete subspace of $\mathfrak{V}$.

Then $q = \mathcal{A}q = \mathcal{S}q = \mathcal{B}q = \mathcal{T}q$, and q is unique in $\mathfrak{V}$.

Proof Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C1) we can find x_1 such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$ for this x_1 one can find $x_2 \in \mathfrak{V}$ such that $\mathcal{T}(x_1) = \mathcal{A}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that

$$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1}),$$

$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2})$, for each $n \geq 0$.

From the proof of Theorem 2.1,the sequence $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$. Since $\mathcal{A}(\mathfrak{V})$ is complete, $\exists$ a point $q \in \mathcal{A}(\mathfrak{V})$ such that $y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}) \to q$as $n \to \infty$.

Consequently, we find $p \in \mathfrak{V}$ such that $\mathcal{A}p = q$. Since $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges, which implies the convergence of $\{y_{2n}\}$, being a subsequence of the convergent sequence $\{y_n\}$. Hence $\{\mathcal{S}(x_{2n})\}, \{\mathcal{B}(x_{2n+1})\}, \{\mathcal{T}(x_{2n+1})\}, \{\mathcal{A}(x_{2n+2})\}$ converges to q.

Now we claim that $\mathcal{S}p = q$. On putting $u = p$ and $v = x_{2n+1}$ in (C2) we get

$$G^3(\mathcal{S}p, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})$$

$$\leq \psi \left\{ \begin{array}{c} G^2(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ G(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G(\mathcal{A}p, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) G(\mathcal{B}x_{2n+1}, \mathcal{S}p, \mathcal{S}p), \\ G(\mathcal{A}p, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) G(\mathcal{B}x_{2n+1}, \mathcal{S}p, \mathcal{S}p) G(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

Proceeding limit as $n \to \infty$,weget

$G^3(\mathcal{S}p, q, q) \leq \psi\{0,0,0,0\}$.

Using property of ψ, we have $G^3(\mathcal{S}p, q, q) = 0$, which implies $\mathcal{S}p = q$.

Therefore, $\mathcal{A}p = \mathcal{S}p = q$.

Since $q = \mathcal{S}p \in \mathcal{S}(\mathfrak{B}) \subset \mathcal{B}(\mathfrak{B}), \exists$ a point w in $\mathfrak{B}$ such that $\mathcal{B}w = q$.

Next, we claim that$q = \mathcal{T}w$. On putting $u = p$ and $v = w$ in (C2) we get

$$G^3(\mathcal{S}p, \mathcal{T}w, \mathcal{T}w) \leq \psi \left\{ \begin{matrix} G^2(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w), \\ G(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G^2(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w), \\ G(\mathcal{A}p, \mathcal{S}p, \mathcal{S}p) G(\mathcal{A}p, \mathcal{T}w, \mathcal{T}w) G(\mathcal{B}w, \mathcal{S}p, \mathcal{S}p), \\ G(\mathcal{A}p, \mathcal{T}w, \mathcal{T}w) G(\mathcal{B}w, \mathcal{S}p, \mathcal{S}p) G(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w) \end{matrix} \right\}$$

On simplification, we have

$$G^3(q, \mathcal{T}w, \mathcal{T}w) \leq \psi\{0,0,0,0\}$$

Thus we get $G^3(q, \mathcal{T}w, \mathcal{T}w) = 0$,which implies that $q = \mathcal{T}w$.

Hence $\mathcal{B}w = \mathcal{T}w = q$.

Since $\mathcal{A}p = \mathcal{S}p = q$ and the pair $(\mathcal{A}, \mathcal{S})$ is$\mathcal{A}$-intimate, by Proposition 3.3, we have

$$G(\mathcal{A}q, q, q) \leq G(\mathcal{S}q, q, q).$$

Next, we claim that $q = \mathcal{S}q$. On putting $u = q$ and $v = w$ in (C2) we get

$$G^3(\mathcal{S}q, \mathcal{T}w, \mathcal{T}w) \leq \psi \left\{ \begin{matrix} G^2(\mathcal{A}q, \mathcal{S}q, \mathcal{S}q) G(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w), \\ G(\mathcal{A}q, \mathcal{S}q, \mathcal{S}q) G^2(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w), \\ G(\mathcal{A}q, \mathcal{S}q, \mathcal{S}q) G(\mathcal{A}q, \mathcal{T}w, \mathcal{T}w) G(\mathcal{B}w, \mathcal{S}q, \mathcal{S}q), \\ G(\mathcal{A}q, \mathcal{T}w, \mathcal{T}w) G(\mathcal{B}w, \mathcal{S}q, \mathcal{S}q) G(\mathcal{B}w, \mathcal{T}w, \mathcal{T}w) \end{matrix} \right\}$$

Therefore,

$$G^3(\mathcal{S}q, q, q) \leq \psi\{0,0,0,0\}$$

Thus we get $G^3(\mathcal{S}q, q, q) = 0$,which further implies that $\mathcal{S}q = q$.

Hence $\mathcal{S}q = \mathcal{A}q = q$.

Similarly, we get $\mathcal{B}q = \mathcal{T}q = q$.

The uniqueness follows easily. Hence $q = \mathcal{A}q = \mathcal{S}q = \mathcal{B}q = \mathcal{T}q$, and q is unique in $\mathfrak{B}$.

4. APPLICATION

In 2002 Branciari [4] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality.

Theorem 4.1 Let $(\mathfrak{V}, d)$ be a complete metric space and $f : \mathfrak{V} \to \mathfrak{V}$ is a mapping such that, for each $x, y \in \mathfrak{V}$,

$$\int_0^{d(x,y,)} \varphi(t)\, dt \le c \int_0^{d(x,y,)} \varphi(t)\, dt$$

$c \in [0,1)$,where $\varphi : R^+ \to R^+$ is a "Lebesgue-integrable function" which is summable, nonnegative, and such that, for each $\in > 0$, $\int_0^{\in} \varphi(t) dt > 0$. Then f has a unique fixed point $z \in \mathfrak{V}$ such that, for each $x \in \mathfrak{V}$, $\lim_{n\to\infty} f^n = z$.

Now we prove the following theorem as an application of Theorem 4.1 in G-metric space.

Theorem 4.2 Let $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ be four self-mappings of a complete G-metric space $(\mathfrak{V}, G)$ satisfying the conditions (C1), and (3.1) and the following conditions:

(C3)

$$\int_0^{G^3(\mathcal{S}x,\mathcal{T}y\mathcal{T}y)} \varphi(t)\, dt \le \int_0^{M(x,y,y)} \varphi(t)\, dt$$

$$M(u,v) = \psi \begin{Bmatrix} G^2(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v), \\ G(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G^2(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v), \\ G(\mathcal{A}u, \mathcal{S}u, \mathcal{S}u)G(\mathcal{A}u, \mathcal{T}v, \mathcal{T}v)G(\mathcal{B}v, \mathcal{S}u, \mathcal{S}u), \\ G(\mathcal{A}u, \mathcal{T}v, \mathcal{T}v)G(\mathcal{B}v, \mathcal{S}u, \mathcal{S}u)G(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \end{Bmatrix}$$

for all $u, v \in \mathfrak{V}$, where ψ: $[0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$. Further, where $\varphi : R^+ \to R^+$ is a "Lebesgue-integrable over R^+function" which is summable on each compact subset of R^+, non-negative, and such that for each $\in > 0$, $\int_0^{\in} \varphi(t)dt > 0$. Moreover, assume that the pairs the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible of type (K). Then $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is unique in $\mathfrak{V}$.

Proof. The proof of the theorem follows with the same lines from the proof of the Theorem 4.1 on setting φ (t) = 1

Remark 4.1 Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting $\varphi(t) = 1$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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