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COMMON FIXED POINT THEOREMS IN SYMMETRIC WEAK NON-ARCHIMEDEAN $\mathscr{G}-$ FUZZY METRIC SPACES

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Abstract. In this paper, we demonstrate the weak non-Archimedean $\mathscr{G}-$ fuzzy metric instigates a Hausdorff topology. We use this new idea to get common fixed point results for a couple of $\psi-$contractive mappings.

Keywords: common fixed point; weak non-Archimedean $\mathscr{G}-$ fuzzy metric space; $\psi-$ contraction; open ball.
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1. INTRODUCTION

The idea of fuzzy metric space was presented in various manners by certain creators and further to this, the fixed point theory in this kind of spaces has been seriously examined. Here, we underline as the idea of fuzzy metric space, presented by Kramosil and Michalek [16] was changed by George and Veeramani [7] that got a Hausdorff topology for this class of fuzzy metric spaces. As of late, Mihet [17] developed the class of fuzzy contractive mappings of Gregori and Sapena [10] and demonstrated a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric space. Presently, we momentarily portray our purposes behind being

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keen on after effects of this kind. The utilizations of fixed point theorems are wonderful in various orders of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many others. Thus, numerous analysts, following the Banach contraction principle, examined the presence of more weaker contractive conditions or expanded past outcomes under moderately weak hypotheses on the metric space. The beginning stage of our paper is to follow this pattern by presenting, with the meaning of weak non-Archimedean fuzzy metric space, a more broad setting than non - Archimedean fuzzy metric space. The per user is alluded to for some conversation and applications on non-Archimedean metric spaces and its induced topology.

In this paper, we present a Hausdorff topology incited by a weak non-Archimedean $\mathscr{G}-$ fuzzy metric and a few properties. At that point, we use this new idea to get common fixed point results for a couple of generalized contractive type mappings.

2. Preliminaries

Definition: 2.1. A 3-tuple $(\chi,\mathscr{G},*)$ is called non-Archimedean $\mathscr{G}-$ fuzzy metric space if χ is an arbitrary non empty set, $*$ is a continuous $t-$ norm and $\mathscr{G}$ is a fuzzy set on $\chi\times\chi\times\chi\times[0,\infty)$ satisfying the following conditions: for each $\mu,\nu,\omega,\alpha\in\chi$ and $t,s>0$,

($\mathscr{G}$F-1) $\mathscr{G}(\mu,\mu,\nu,t)>0$ with $\mu\neq\nu$,

($\mathscr{G}$F-2) $\mathscr{G}(\mu,\mu,\nu,t)\geq\mathscr{G}(\mu,\nu,\omega,t)$ with $\nu\neq\omega$,

($\mathscr{G}$F-3) $\mathscr{G}(\mu,\nu,\omega,t)=1$ if and only if $\mu=\nu=\omega$,

($\mathscr{G}$F-4) $\mathscr{G}(\mu,\nu,\omega,t)=\mathscr{G}(p\{\mu,\nu,\omega\},t)$ where p is a permutation function,

($\mathscr{G}$F-5) $\mathscr{G}(\mu,\nu,\omega,\max\{t,s\})\geq\mathscr{G}(\mu,\alpha,\alpha,t)*\mathscr{G}(\alpha,\nu,\omega,s)$,

($\mathscr{G}$F-6) $\mathscr{G}(\mu,\nu,\omega,\cdot):[0,\infty)\to[0,1]$ is left continuous.

Definition: 2.2. In definition 2.1, if the triangular inequality ($\mathscr{G}$F-5) is replaced by the following: $\mathscr{G}(\mu,\nu,\omega,t)\geq\max\left\{\mathscr{G}(\mu,\alpha,\alpha,t)*\mathscr{G}(\alpha,\nu,\omega,\frac{t}{2}),\ \mathscr{G}(\mu,\alpha,\alpha,\frac{t}{2})*\mathscr{G}(\alpha,\nu,\omega,t)\right\}$ for all $\mu,\nu,\omega\in\chi$ and $t>0$, then $(\chi,\mathscr{G},*)$ is said to be a weak non- Archimedean $\mathscr{G}-$ fuzzy metric spaces (WNA$\mathscr{G}$FMS).

Obviously, every non-Archimedean $\mathscr{G}-$ fuzzy metric space is itself a weak non-Archimedean $\mathscr{G}-$ fuzzy metric space.

Definition: 2.3. A WNA$\mathscr{G}$FMS $(\chi,\mathscr{G},*)$ is said to be symmetric if $\mathscr{G}(\mu,\mu,\nu,\mathrm{t})=\mathscr{G}(\mu,\nu,\nu,\mathrm{t})$ for all $\mu,\nu\in\chi$ and for each $t>0$.

Example 2.4. Let (χ,d) be a metric space. Define the $t-$ norm $a*b=\min\{a,b\}$ for all $a,b\in[0,1]$. For all $\mu,\nu,\omega\in\chi$ and $t>0$, $\mathscr{G}(\mu,\nu,\omega,t)=1-e^{-\frac{d(\mu,\nu)+d(\nu,\omega)+d(\omega,\mu)}{3t}}$. Then $(\chi,\mathscr{G},*)$ is a WNA$\mathscr{G}$FMS.

Definition: 2.5. Let $(\chi,\mathscr{G},*)$ be a WNA$\mathscr{G}$FMS. We define the open ball $B(\mu,r,t)$ with centre $\mu\in\chi$ and radius $r\in(0,1)$, $t>0$ as $B(\mu,r,t)=\{\nu\in\chi:\mathscr{G}(\mu,\nu,\nu,t)>1-r\}$.

Proposition: 2.6. Every open ball is open set in $(\chi,\mathscr{G},*)$.

Proof. Consider an open ball $B(\mu,r,t)$ with centre $\mu\in\chi$ and radius $r\in(0,1)$ and $t>0$. Now, $\nu\in B(\mu,r,t)$ implies $r_s=\mathscr{G}(\mu,\nu,\nu,t)>1-r$. Let $r_u\in(0,1)$ be such that $r_s>1-r_u>1-r$. Hence there exists $r_\nu\in(0,1)$ such that $r_s*r_\nu\geq 1-r_u$.
We claim that $B(\nu,1-r_\nu,\frac{t}{2})\subset B(\mu,r,t)$.
Let $\omega\in B(\nu,1-r_\nu,\frac{t}{2})$. It implies $\mathscr{G}(\nu,\omega,\omega,\frac{t}{2})>v$. Therefore,

$$\mathscr{G}(\mu,\omega,\omega,t)\geq\mathscr{G}(\mu,\nu,\nu,t)*\mathscr{G}(\nu,\omega,\omega,\frac{t}{2})\geq r_s*r_\nu>1-r_u>1-r,$$

and so $\omega\in B(\mu,r,t)$ and hence $B(\nu,1-r_\nu,\frac{t}{2})\subset B(\mu,r,t)$. □

We deduce that the family

$\tau_{\mathscr{G}}=\{A\subset\chi:B(\mu,r,t)\subset A$ with $t>0$ and $r\in(0,1)$ for all $\mu\in A\}$ is a topology on χ.

Proposition: 2.7. Every WNA$\mathscr{G}$FMS is Hausdorff.

Proof. Let $\mu,\nu\in\chi$, with $\mu\neq\nu$. Then $\mathscr{G}(\mu,\nu,\nu,t)\in(0,1)$ for some $t>0$. For each s such that $\mathscr{G}(\mu,\nu,\nu,t)=r<r_s<1$ and find r_u such that $r_u*r_u\geq r_s$. Now, consider the open balls $B(\mu,1-r_u,t)$ and $B(\nu,1-r_u,\frac{t}{2})$. We claim that $B(\mu,1-r_u,t)\cap B(\nu,1-r_u,\frac{t}{2})=\emptyset$.
Suppose $\omega\in B(\mu,1-r_u,t)\cap B(\nu,1-r_u,\frac{t}{2})$, then

$$r=\mathscr{G}(\mu,\nu,\nu,t)\geq\mathscr{G}(\mu,\omega,\omega,t)*\mathscr{G}(\nu,\omega,\omega,\frac{t}{2})\geq r_u*r_u\geq r_s>r,$$

which is a contradiction. □

Proposition: 2.8. Let $(\chi,\mathscr{G},*)$ be a WNA$\mathscr{G}$FMS. A sequence $\{\mu_n\}$ in $(\chi,\mathscr{G},*)$ is convergent to $\mu\in\chi$ if and only if $\lim_{n\to\infty}\mathscr{G}(\mu_n,\mu,\mu,t)=1$ for all $t>0$.

Proof. Fix $t > 0$ and suppose $\{\mu_n\} \to \mu$. Then, for $r \in (0,1)$, there exists $n_0 \in N$ such that $\mu_n \in B(\mu, r, t)$ for all $n \geq n_0$. It follows that $\mathscr{G}(\mu_n, \mu, \mu, t) > 1 - r$ and hence $\mathscr{G}(\mu_n, \mu, \mu, t) \to 1$ as $n \to \infty$.

Conversely, if for each $t > 0, \mathscr{G}(\mu_n, \mu, \mu, t) \to 1$ as $n \to \infty$, then for $r \in (0,1)$, there exists $n_0 \in N$ such that $\mathscr{G}(\mu_n, \mu, \mu, t) > 1 - r$ for all $n \geq n_0$. Thus, $\mu_n \in B(\mu, r, t)$ for all $n \geq n_0$ and hence $\mu_n \to \mu$.

□

Definition: 2.9. Let $(\chi, \mathscr{G}, *)$ be a WNA$\mathscr{G}$FMS. A sequence $\{\mu_n\}$ in χ called a Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exist $n_0 \in N$ such that $\mathscr{G}(\mu_n, \mu_m, \mu_m, t) > 1 - \varepsilon$ and for all $m, n \geq n_0$.

A weak non-Archimedean $\mathscr{G}-$ fuzzy metric space $(\chi, \mathscr{G}, *)$ is called complete ($\mathscr{G}-$ complete) if every Cauchy ($\mathscr{G}-$Cauchy) sequence is convergent.

Remark: 1.

Let $(\chi, \mathscr{G}, *)$ be a WNA$\mathscr{G}$FMS and $\{\mu_n\} \in \chi$ be a sequence convergent to $\mu \in \chi$, then $\lim\limits_{n \to \infty} \mathscr{G}(\nu, \mu_n, \mu_n, t) = \mathscr{G}(\nu, \mu, \mu, t)$ for all $\nu \in \chi$ and $t > 0$. In fact, by condition (WNA$\mathscr{G}$FMS) we have,

$$\mathscr{G}(\nu, \mu_n, \mu_n, t) \geq \mathscr{G}(\nu, \mu, \mu, t) * \mathscr{G}(\mu, \mu_n, \mu_n, \frac{t}{2}) \text{ and}$$

$$\mathscr{G}(\nu, \mu, \mu, t) \geq \mathscr{G}(\nu, \mu_n, \mu_n, t) * \mathscr{G}(\mu, \mu_n, \mu_n, \frac{t}{2}).$$

Thus,

$$\mathscr{G}(\nu, \mu, \mu, t) \leq \liminf_{n \to \infty} \mathscr{G}(\nu, \mu_n, \mu_n, t) \leq \limsup_{n \to \infty} \mathscr{G}(\nu, \mu_n, \mu_n, t) \leq \mathscr{G}(\nu, \mu, \mu, t).$$

Remark: 2. Let $\psi : [0,1] \to [0,1]$ be such that

(i) ψ is non increasing and continuous,

(ii) ψ(t) > t for all t ∈[0, 1],

We denote, $\Psi = \{\psi : [0,1] \to [0,1] : \psi$ satisfies (i) - (ii) $\}$.

Lemma: 2.12. If $\psi \in \Psi$, then $\psi(1) = 1$.

Lemma: 2.13. If $\psi \in \Psi$, then $\lim\limits_{n \to \infty} \psi^n(t) = 1$ for all $t \in (0,1)$.

3. Main Results

Definition: 3.1. Let χ be a non empty set and $\mathscr{G}$ be a fuzzy set on $\chi \times \chi \times \chi \times (0,\infty)$. Let $\mathfrak{f},\mathfrak{g} : \chi \to \chi$, $(\mathfrak{f},\mathfrak{g})$ is a pair of $\psi-$ contractive mappings if there exists $\psi \in \Psi$ such that for every $\mu,\nu,\omega \in \chi, t \in (0,\infty)$ with $\mathscr{G}(\mu,\nu,\omega,t) > 0$ the following condition holds:

$$\mathscr{G}(\mathfrak{f}(\mu),\mathfrak{g}(\nu),\mathfrak{g}(\omega),t) \geq \psi\big(\mathfrak{m}(\mu,\nu,\omega,t)\big)$$

where

$$\mathfrak{m}(\mu,\nu,\omega,t) = \min\{\mathscr{G}(\mu,\nu,\omega,t),\mathscr{G}(\mu,\mathfrak{f}(\mu),\mathfrak{f}(\mu),t),\mathscr{G}(\nu,\mathfrak{g}(\nu),\mathfrak{g}(\nu),t),$$
$$\mathscr{G}(\omega,\mathfrak{g}(\omega),\mathfrak{g}(\omega),t)\}$$

Fix $\mu_0 \in \chi$ and define the sequence $\{\mu_n\}$ by $\mu_1 = \mathfrak{f}(\mu_0), \mu_2 = \mathfrak{g}(\mu_1), \cdots, \mu_{2n+1} = \mathfrak{f}(\mu_{2n}), \mu_{2n+2} = \mathfrak{g}(\mu_{2n+1}), \cdots$. We call $\{\mu_n\}$ an $(\mathfrak{f},\mathfrak{g})-$ sequence of initial point μ_0.

Lemma: 3.2. Let χ be a non empty set and $\mathscr{G}$ be a fuzzy set on $\chi \times \chi \times \chi \times (0,\infty)$. Let $\mathfrak{f},\mathfrak{g} : \chi \to \chi$, $(\mathfrak{f},\mathfrak{g})$ is a pair of $\psi-$ contractive mappings. If $\mu_0 \in \chi$ is such that $\mathscr{G}(\mu_0,\mathfrak{f}(\mu_0),\mathfrak{f}(\mu_0),t) > 0$ then $\lim_{n\to\infty} \mathscr{G}(\mu_n,\mu_{n+1},\mu_{n+1},t) = 1$, where $\{\mu_n\}$ is the $(\mathfrak{f},\mathfrak{g})-$ sequence of initial point μ_0.

Proof. If $\mathscr{G}(\mu_n,\mu_{n+1},\mu_{n+1},t) = 1$ for some $n \in N$, then $\mathscr{G}(\mu_m,\mu_{m+1},\mu_{m+1},t) = 1$ for some $m > n$. Assume that $\mathscr{G}(\mu_n,\mu_{n+1},\mu_{n+1},t) < 1$ for all $n \in N$. Clearly $\mathscr{G}(\mu_0,\mathfrak{f}(\mu_0),\mathfrak{f}(\mu_0),t) = \mathscr{G}(\mu_0,\mu_1,\mu_1,t) > 0$. Also,

$$\begin{aligned}\mathscr{G}(\mu_1,\mu_2,\mu_2,t) &= \mathscr{G}(\mathfrak{f}(\mu_0),\mathfrak{g}(\mu_1),\mathfrak{g}(\mu_1),t)\\ &\geq \psi\big(\mathfrak{m}(\mu_0,\mu_1,\mu_1,t)\big)\\ &\geq \psi\big(\mathscr{G}(\mu_0,\mu_1,\mu_1,t)\big) > 0\end{aligned}$$

$$\begin{aligned}\mathscr{G}(\mu_2,\mu_3,\mu_3,t) &= \mathscr{G}(\mathfrak{f}(\mu_1),\mathfrak{g}(\mu_2),\mathfrak{g}(\mu_2),t)\\ &\geq \psi\big(\mathfrak{m}(\mu_1,\mu_2,\mu_2,t)\big)\\ &\geq \psi\big(\mathscr{G}(\mu_1,\mu_2,\mu_2,t)\big)\\ &\geq \psi^2\big(\mathscr{G}(\mu_0,\mu_1,\mu_1,t)\big) > 0\end{aligned}$$

Generally, for each $n \in N$, we get

$$\mathscr{G}(\mu_n, \mu_{n+1}, \mu_{n+1}, t) \geq \psi^n\big(\mathscr{G}(\mu_0, \mu_1, \mu_1, t)\big),$$

By lemma (2.13) as $n \to \infty$, we deduce that $\lim_{n\to\infty} \mathscr{G}(\mu_n, \mu_{n+1}, \mu_{n+1}, t) = 1$. □

Lemma: 3.3. Let $(\chi, \mathscr{G}, *)$ be a symmetric WNA$\mathscr{G}$FMS and let $\mathfrak{f}, \mathfrak{g} : \chi \to \chi$. Assume that $(\mathfrak{f}, \mathfrak{g})$ is a pair of $\psi-$ contractive mappings. If $\mu_0 \in \chi$ is such that $\mathscr{G}(\mu_0, \mathfrak{f}(\mu_0), \mathfrak{f}(\mu_0), t) > 0$ for all $t > 0$, then $(\mathfrak{f}, \mathfrak{g})-$ sequence $\{\mu_n\}$ of initial point μ_0 is Cauchy.

Proof. If $\{\mu_n\}$ is not Cauchy, then there are $\varepsilon \in (0, \frac{1}{2})$ and $t > 0$ such that for each $\kappa \in N$ there exist $m(\kappa), n(\kappa) \in N$ with $m(\kappa) > n(\kappa) \geq \kappa$ and $\mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, t) \leq 1 - 2\varepsilon$.
By lemma (3.1), we have $\lim_{n\to\infty} \mathscr{G}(\mu_{n(\kappa)+1}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, \frac{t}{2}) = 1$. Therefore,

$$1 - 2\varepsilon \geq \mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, t) \geq \mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)+1}, \mu_{n(\kappa)+1}, t) * \mathscr{G}(\mu_{n(\kappa)+1}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, \frac{t}{2}),$$

we get

$$\begin{aligned} 1 - 2\varepsilon &\geq \limsup_{\kappa\to\infty} \mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)+1}, \mu_{n(\kappa)+1}, t) * \mathscr{G}(\mu_{n(\kappa)+1}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, \frac{t}{2}) \\ &= \limsup_{\kappa\to\infty} \mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)+1}, \mu_{n(\kappa)+1}, t) \end{aligned}$$

Analogously, we obtain

$$1 - 2\varepsilon \geq \limsup_{\kappa\to\infty} \mathscr{G}(\mu_{m(\kappa)+1}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, t),$$

$$1 - 2\varepsilon \geq \limsup_{\mu\to\infty} \mathscr{G}(\mu_{m(\kappa)+1}, \mu_{n(\kappa)+1}, \mu_{n(\kappa)+1}, t).$$

Then, we can assume that $m(\kappa)$ are odd numbers, $n(\kappa)$ are even numbers and $\mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, t) \leq 1 - \varepsilon$ for all κ.
Let $\mathfrak{q}(\kappa) = \min\{m(\kappa) : \mathscr{G}(\mu_{m(\kappa)}, \mu_{n(\kappa)}, \mu_{n(\kappa)}, t) \leq 1 - \varepsilon,\ m(\kappa)$ is odd number $\}$.

We have,

$$\begin{aligned}
1-\varepsilon &\geq \mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{n(\kappa)},\mu_{n(\kappa)},t) \\
&\geq \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{n(\kappa)},\mu_{n(\kappa)},t) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)},\frac{t}{2}) \\
&\geq \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{n(\kappa)},\mu_{n(\kappa)},t) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)-1},\frac{t}{2}) \\
&\qquad * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)},\frac{t}{4}) \\
&\geq (1-\varepsilon) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)-1},\frac{t}{2}) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)},\frac{t}{4})
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{\kappa\to\infty}(1-\varepsilon) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-2},\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)-1},\frac{t}{2}) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)-1},\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)},\frac{t}{4}) \\
&\qquad = (1-\varepsilon) * 1 * 1 = 1-\varepsilon.
\end{aligned}$$

Therefore, $\lim_{\kappa\to\infty} \mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{n(\kappa)},\mu_{n(\kappa)},t) = 1-\varepsilon$.

Now,

$$\begin{aligned}
\mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{n(\kappa)},\mu_{n(\kappa)},t) &\geq \mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{n(\kappa)+1},\mu_{n(\kappa)+1},t) * \mathscr{G}(\mu_{n(\kappa)+1},\mu_{n(\kappa)},\mu_{n(\kappa)},\frac{t}{2}) \\
&\geq \mathscr{G}(\mu_{\mathfrak{q}(\kappa)+1},\mu_{n(\kappa)+1},\mu_{n(\kappa)+1},t) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)+1},\mu_{\mathfrak{q}(\kappa)+1},\frac{t}{2}) \\
&\qquad * \mathscr{G}(\mu_{n(\kappa)+1},\mu_{n(\kappa)},\mu_{n(\kappa)},\frac{t}{2}) \\
&\geq \psi\big(\mathfrak{m}(\mu_{\mathfrak{q}(\kappa)},\mu_{n(\kappa)},\mu_{n(\kappa)},t)\big) * \mathscr{G}(\mu_{\mathfrak{q}(\kappa)},\mu_{\mathfrak{q}(\kappa)+1},\mu_{\mathfrak{q}(\kappa)+1},\frac{t}{2}) \\
&\qquad * \mathscr{G}(\mu_{n(\kappa)+1},\mu_{n(\kappa)},\mu_{n(\kappa)},\frac{t}{2})
\end{aligned}$$

As $\kappa\to\infty$, we get $1-\varepsilon \geq \psi\big(1-\varepsilon\big) * 1 * 1 > 1-\varepsilon$, which is contradiction.

Therefore $\{\mu_n\}$ is a Cauchy sequence.

$\square$

Theorem: 3.4.

Let $(\chi,\mathscr{G},*)$ be a symmetric complete WNA$\mathscr{G}$FMSand let $\mathfrak{f},\mathfrak{g}:\chi\to\chi$. Assume that $(\mathfrak{f},\mathfrak{g})$ is a pair of $\psi-$ contractive mappings and that for all $\mu,\nu,\omega\in\chi$, with $\mu\neq\nu\neq\omega$, there exists

$t > 0$ such that $0 < \mathscr{G}(\mu, \nu, \omega, t) < 1$. If there exists $\mu_0 \in \chi$ such that $\mathscr{G}(\mu_0, \mathfrak{f}(\mu_0), \mathfrak{f}(\mu_0), t) > 0$ for all $t > 0$, then $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point.

Proof. By Lemma (3.2) the $(\mathfrak{f}, \mathfrak{g})-$ sequence $\{\mu_n\}$ of initial point μ_0 is Cauchy. Since χ is complete, there exists $\mu \in \chi$ such that $\lim_{n\to\infty} \mu_n = \mu$. If $\mathfrak{f}(\mu) \neq \mu$, then there exists $t > 0$ such that $0 < \mathscr{G}(\mu, \mathfrak{f}(\mu), \mathfrak{f}(\mu), t) < 1$. From

$$\begin{aligned}
\mathscr{G}(\mu, \mathfrak{f}(\mu), \mathfrak{f}(\mu), t) &\geq \min\left\{\mathscr{G}(\mu, \mu_{2n-1}, \mu_{2n-1}, t), \mathscr{G}(\mu, \mathfrak{f}(\mu), \mathfrak{f}(\mu), t), \mathscr{G}(\mu_{2n-1}, \mu_{2n}, \mu_{2n}, t)\right\} \\
&= \mathfrak{m}(\mu, \mu_{2n-1}, \mu_{2n-1}, t) \\
&\geq \mathscr{G}(\mu, \mathfrak{f}(\mu_{2n-2}), \mathfrak{f}(\mu_{2n-2}), t) \to_{n\to\infty} \mathscr{G}(\mu, \mathfrak{f}(\mu), \mathfrak{f}(\mu), t)
\end{aligned}$$

and $\mathscr{G}(\mathfrak{f}(\mu), \mu_{2n}, \mu_{2n}, t) = \mathscr{G}(\mathfrak{f}(\mu), \mathfrak{g}(\mu_{2n-1}), \mathfrak{g}(\mu_{2n-1}), t) \geq \psi\big(\mathfrak{m}(\mu, \mu_{2n-1}, \mu_{2n-1}, t)\big)$
By Remark(1), as $n \to \infty$, we obtain

$$\mathscr{G}(\mathfrak{f}(\mu), \mu, \mu, t) \geq \psi\big(\mathscr{G}(\mathfrak{f}(\mu), \mu, \mu, t)\big) > \mathscr{G}(\mathfrak{f}(\mu), \mu, \mu, t).$$

Which is a contradiction. Therefore $\mathfrak{f}(\mu) = \mu$.
Analogously, we obtain that $\mathfrak{g}(\mu) = \mu$ and thus μ is a common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$.
Now, we prove the uniqueness of the common fixed points of $\mathfrak{f}$ and $\mathfrak{g}$.
Assume that $\mu, \nu \in \chi$ are two common fixed points of $\mathfrak{f}$ and $\mathfrak{g}$.
If $\mu \neq \nu$, then there exists $t > 0$ such that $0 < \mathscr{G}(\mu, \nu, \nu, t) < 1$ and hence

$$\mathscr{G}(\mu, \nu, \nu, t) = \mathscr{G}(\mathfrak{f}(\mu), \mathfrak{g}(\nu), \mathfrak{g}(\nu), t) \geq \psi\big(\mathfrak{m}(\mu, \nu, \nu, t)\big) = \psi\big(\mathscr{G}(\mu, \nu, \nu, t)\big) > \mathscr{G}(\mu, \nu, \nu, t).$$

Which is a contradiction. Therefore $\mu = \nu$. □

Example: 3.5.

Let $\chi = \{1, 2, \cdots\}$, $\psi(t) = \sqrt{t}$ for all $t \in [0,1]$ and define $\mathscr{G}(\mu, \nu, \omega, t)$ by $\mathscr{G}(\mu, \nu, \omega, 0) = 0$, $\mathscr{G}(\mu, \mu, \mu, t) = 1$ for all $t > 0$, $\mathscr{G}(\mu, \nu, \omega, t) = 0$ for $\mu \neq \nu \neq \omega$ and $0 < t \leq 1$, $\mathscr{G}(\mu, \nu, \omega, t) = 1$ for $\mu \neq \nu \neq \omega$ and $t > 1$.
Clearly $(\chi, \mathscr{G}, *)$ is a WNA$\mathscr{G}$FMS with $a * b = ab$ for every $a, b \in [0,1]$.
Define mapping $\mathfrak{f}, \mathfrak{g} : \chi \to \chi$ as $\mathfrak{f}(\mu) = \mu^2$ and $\mathfrak{g}(\mu) = 2\mu$.
It is trivial that $(\mathfrak{f}, \mathfrak{g})$ is a pair of $\psi-$ contractive mappings.

Now all the hypotheses of Theorem (3.1) are satisfied and then $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point.

Example: 3.6.

Let $\chi = [0,\infty)$, $\psi(t) = \sqrt{\frac{t}{4}}$ for all $t \in [0,1]$ and define $\mathscr{G}(\mu,\nu,\omega,t)$ by $\mathscr{G}(\mu,\nu,\omega,0) = 0$, $\mathscr{G}(\mu,\mu,\mu,t) = 1$ for all $t > 0$, $\mathscr{G}(\mu,\nu,\omega,t) = 0$ for $\mu \neq \nu \neq \omega$ and $0 < t \leq 1$, $\mathscr{G}(\mu,\nu,\omega,t) = \frac{t^2}{4}$ for $\mu \neq \nu \neq \omega$ and $1 < t \leq 2$, $\mathscr{G}(\mu,\nu,\omega,t) = 1$ for $\mu \neq \nu \neq \omega$ and $t > 2$.

Clearly $(\chi,\mathscr{G},*)$ is a WNA$\mathscr{G}$FMS with $a*b = ab$ for every $a,b \in [0,1]$. Define mapping $\mathfrak{f},\mathfrak{g} : \chi \to \chi$ as

$$\mathfrak{f}(\mu) = \begin{cases} \mu, & \mu \in [0,1], \\ 0, & \mu \in (1,\infty) \end{cases} \quad \text{and} \quad \mathfrak{g}(\mu) = \begin{cases} \sqrt{\mu}, & \mu \in [0,1], \\ 0, & \mu \in (1,\infty). \end{cases}$$

Apparently, it is easy to show that $(\mathfrak{f},\mathfrak{g})$ is a pair of $\psi-$ contractive mappings. Now, we note that $\psi \in \Psi$. To be precise,it is not true that $\psi(t) > t$ for all $t \in (0,1)$. Thus for all the other hypotheses of Theorem (3.1) are satisfied but $\mathfrak{f}$ and $\mathfrak{g}$ have not a unique common fixed point.

Corollary 3.7.

Let $(\chi,\mathscr{G},*)$ be a symmetric complete WNA$\mathscr{G}$FMS and let $\mathfrak{f} : \chi \to \chi$. Assume that $(\mathfrak{f},\mathfrak{f})$ is a pair of $\psi-$ contractive mappings and for all $\mu,\nu,\omega \in \chi$ with $\mu \neq \nu \neq \omega$, there exists $t > 0$ such that $0 < \mathscr{G}(\mu,\nu,\omega,t) < 1$. If there exists $\mu_0 \in \chi$ such that $\mathscr{G}(\mu_0,\mathfrak{f}(\mu_0),\mathfrak{f}(\mu_0),t) > 0$ for all $t > 0$, then $\mathfrak{f}$ has a unique common fixed point.

4. Conclusion

In this work we have managed a class of contractive mappings in fuzzy metric spaces, presented by Gregori and Sapena in [10]. We have considered these mappings in a more general setting and recognized a class of complete fuzzy metric spaces in which each fuzzy contractive mapping has a unique fixed point

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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