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COMMON FIXED POINT THEOREMS IN TRIANGULAR INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In triangular intuitionistic fuzzy metric spaces, we propose a common fixed point theorem for four mappings under various compatible mapping conditions. On the basis of this, we can further extend the theorem to the common fixed point theorems of four finite families of mappings.

Keywords: triangular intuitionistic fuzzy metric space; compatible mappings;pairwise commuting mappings; common fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The concept of fuzzy sets was initially introduced by Zadeh[9] in 1965. The emergence of fuzzy set theory provides a powerful mathematical tool for describing and studying fuzzy phenomena. Thereafter the concept of a fuzzy metric space was introduced by Kramosil and Michalek[16] and was revised by George and Veeramani8[6] in 1994. Atanassov[8] generalized fuzzy sets and introduced the concept of intuitionistic fuzzy sets in 1986. Nearly two decades later in 2004 Park[5] introduced and discussed the concept of intuitionistic fuzzy metric Spaces, which is based on the concept of intuitionistic fuzzy sets and fuzzy metric spaces.

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Fixed point theory is an important part of functional analysis, and various compatibility conditions are often used to prove fixed point theorems. In the 90's, Murthy[11] introduced the concept of compatible mappings of type (A), while Pathak [13] introduced the concept of compatible mappings of type (P). After that, various compatible conditions have been proposed, such as: compatible of type (K)[4], compatible of type (R)[17] and compatible of type (E)[18].

In this paper, we introduce some common fixed point theorems in an triangular intuitionistic fuzzy metric space[3] under various compatible conditions.

2. Preliminaries

The definitions closely related to the theorems introduced later in the article are described below.

Definition 2.1. [15] *A binary operation* $* : [0,1]\times[0,1]\to[0,1]$ *is continuous t-norm if* $*$ *is satisfying the following conditions:*

(1) $*$ *is commutative and associative, that is for all* $a,b,c\in[0,1]$, $a*b=b*a$, $(a*b)*c=a*(b*c)$.

(2) $*$ *is continuous.*

(3) $a*1=1*a=a$ *for all* $a\in[0,1]$.

(4) $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$, *and* $a,b,c,d\in[0,1]$.

Example 1. *Two typical examples of continuous t-norm are* $a*b=ab$ *and* $a*b=min\{a,b\}$.

Definition 2.2. [15] *A binary operation* $\diamond : [0,1]\times[0,1]\to[0,1]$ *is continuous t-conorm if* $\diamond$ *is satisfying the following conditions:*

(1) $\diamond$ *is commutative and associative, that is for all* $a,b,c\in[0,1]$, $a\diamond b=b\diamond a$, $(a\diamond b)\diamond c=a\diamond(b\diamond c)$.

(2) $\diamond$ *is continuous.*

(3) $a\diamond 0=0\diamond a=a$ *for all* $a\in[0,1]$.

(4) $a\diamond b\le c\diamond d$ *whenever* $a\le c$ *and* $b\le d$, *and* $a,b,c,d\in[0,1]$.

Example 2. *Two typical examples of continuous t-conorm are* $a\diamond b=min\{a+b,1\}$ *and* $a\diamond b=max\{a,b\}$.

Definition 2.3. [5] *A 5-tuple $(X,M,N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M,N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x,y,z \in X, s,t > 0$,*

(1) $M(x,y,t) + N(x,y,t) \leq 1$.
(2) $M(x,y,t) > 0$.
(3) $M(x,y,t) = 1$ *if and only if* $x = y$.
(4) $M(x,y,t) = M(y,x,t)$.
(5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$.
(6) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ *is continuous.*
(7) $N(x,y,t) < 1$.
(8) $N(x,y,t) = 0$ *if and only if* $x = y$.
(9) $N(x,y,t) = N(y,x,t)$.
(10) $N(x,y,t) * N(y,z,s) \geq N(x,z,t+s)$.
(11) $N(x,y,\cdot) : (0,\infty) \to [0,1)$ *is continuous.*

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.4. [5] *Every fuzzy metric space $(X,M,*)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e. $x \diamond y = 1 - [(1-x)*(1-y)]$ for any $x,y \in X$.*

Remark 2.5. [5] *In intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is non-decreasing and $N(x,y,\cdot)$ is non-increasing for all $x,y \in X$.*

Remark 2.6. [19] *Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space, then M and N are continuous functions on $X \times X$.*

Example 3. *Let (X,d) be a metric space. Denote $a * b = min\{a,b\}$ and $a \diamond b = max\{a,b\}$ for all $a,b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:*

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by a metric d is the standard intuitionistic fuzzy metric.

Definition 2.7. [3] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. The intuitionistic metric (M, N) is triangular if it satisfies the condition:*

$$(\frac{1}{M(x,y,t)} - 1) \leq (\frac{1}{M(x,z,t)} - 1) + (\frac{1}{M(y,z,t)} - 1),$$
$$N(x,y,t) \leq N(x,z,t) + N(y,z,t)$$

for every $x, y, z \in X$ and every $t > 0$.

Definition 2.8. [5] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set $B(x,r,t) = \{y \in X : M(x,y,t) > 1-r, N(x,y,t) < r\}$ is called the open ball with center x and radius r with respect to t.*

Definition 2.9. [5] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if for $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x,r,t)$ for all $n \geq n_0$.*

Lemma 2.10. [5] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if for all $t > 0$,*

$$M(x_n, x, t) \to 1 \quad \textit{and} \quad N(x_n, x, t) \to 0 \quad \textit{as} \quad n \to \infty.$$

Definition 2.11. [5] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$.*

Definition 2.12. [5] *The intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent.*

Definition 2.13. *Let f, g be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. The mappings f and g are said to be weakly compatible if for all $t > 0$,*

$$\lim_{n\to\infty} M(fgx, gfx, t) = 1 \quad \textit{and} \quad \lim_{n\to\infty} N(fgx, gfx, t) = 0$$

whenever $fx = gx$.

Definition 2.14. *Let f, g be self-mappings of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$. The mappings f and g are said to be compatible if for all $t>0$,*

$$\lim_{n\to\infty} M(fgx_n,gfx_n,t)=1 \quad \text{and} \quad \lim_{n\to\infty} N(fgx_n,gfx_n,t)=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z\in X$.

Definition 2.15. *Let f, g be self-mappings of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$. $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z\in X$. For all $t>0$,*

(1) [11] *f and g are said to be compatible of type (A) if*

$$\lim_{n\to\infty} M(fgx_n,ggx_n,t)=1, \quad \lim_{n\to\infty} N(fgx_n,ggx_n,t)=0,$$
$$\lim_{n\to\infty} M(gfx_n,ffx_n,t)=1, \quad \lim_{n\to\infty} N(gfx_n,ffx_n,t)=0.$$

(2) [13] *f and g are said to be compatible of type (P) if*

$$\lim_{n\to\infty} M(ffx_n,ggx_n,t)=1, \quad \lim_{n\to\infty} N(ffx_n,ggx_n,t)=0.$$

(3) [4] *f and g are to be compatible of type (K) if*

$$\lim_{n\to\infty} M(ffx_n,gz,t)=1, \quad \lim_{n\to\infty} N(ffx_n,gz,t)=0,$$
$$\lim_{n\to\infty} M(ggx_n,fz,t)=1, \quad \lim_{n\to\infty} N(ggx_n,fz,t)=0.$$

(4) [17] *f and g are said to be compatible of type (R) if*

$$\lim_{n\to\infty} M(fgx_n,gfx_n,t)=1, \quad \lim_{n\to\infty} N(fgx_n,gfx_n,t)=0,$$
$$\lim_{n\to\infty} M(ffx_n,ggx_n,t)=1, \quad \lim_{n\to\infty} N(ffx_n,ggx_n,t)=0.$$

(5) [18] *f and g are said to be compatible of type (E) if*

$$\lim_{n\to\infty} M(ffx_n,fgx_n,t)=\lim_{n\to\infty} M(ffx_n,gz,t)=\lim_{n\to\infty} M(fgx_n,gz,t)=1,$$
$$\lim_{n\to\infty} N(ffx_n,fgx_n,t)=\lim_{n\to\infty} N(ffx_n,gz,t)=\lim_{n\to\infty} N(fgx_n,gz,t)=0,$$
$$\lim_{n\to\infty} M(ggx_n,gfx_n,t)=\lim_{n\to\infty} M(ggx_n,fz,t)=\lim_{n\to\infty} M(gfx_n,fz,t)=1,$$
$$\lim_{n\to\infty} N(ggx_n,gfx_n,t)=\lim_{n\to\infty} N(ggx_n,fz,t)=\lim_{n\to\infty} N(gfx_n,fz,t)=0.$$

(6) [14] *f and g are said to be f-compatible if*

$$\lim_{n\to\infty} M(fgx_n, ggx_n, t) = 1, \quad \lim_{n\to\infty} N(fgx_n, ggx_n, t) = 0.$$

(7) [14] *f and g are said to be g-compatible if*

$$\lim_{n\to\infty} M(gfx_n, ffx_n, t) = 1, \quad \lim_{n\to\infty} N(gfx_n, ffx_n, t) = 0.$$

(8) [10] *f and g are said to be semicompatible if*

$$\lim_{n\to\infty} M(fgx_n, gz, t) = 1, \quad \lim_{n\to\infty} N(fgx_n, gz, t) = 0.$$

Definition 2.16. [1] *Let f, g be self-mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. The mappings f and g are said to be reciprocally continuous if for all $t > 0$,*

$$\lim_{n\to\infty} M(fgx_n, fz, t) = 1, \quad \lim_{n\to\infty} N(fgx_n, fz, t) = 0,$$
$$\lim_{n\to\infty} M(gfx_n, gz, t) = 1, \quad \lim_{n\to\infty} N(gfx_n, gz, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.17. [12] *Two finite families of self-mappings $\{f_i\}_{i=1}^{m}$ and $\{g_k\}_{k=1}^{n}$ of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ are said to be pairwise commuting if:*

(1) $f_i f_j = f_j f_i$, $\forall i, j \in \{1, 2, ..., m\}$.

(2) $g_k g_l = g_l g_k$, $\forall k, l \in \{1, 2, ..., n\}$.

(3) $f_i g_k = g_k f_i$, $\forall i \in \{1, 2, ..., m\}$ *and* $\forall k \in \{1, 2, ..., n\}$.

3. Main Results

Theorem 3.1. *Let $(X, M, N, *, \diamond)$ be an triangular intuitionistic fuzzy metric space. ϕ, ψ are continuous functions with $\phi(r+s) \le \phi(r) + \phi(s)$, $\psi(r+s) \le \psi(r) + \psi(s)$, $\phi(r) \le r$ and $\psi(r) \le r$ whenever $r, s > 0$. Let F, G, S, T be the self-mappings of X satisfying the following conditions:*

(1) $F(X) \subseteq T(X)$, $G(X) \subseteq S(X)$.

(2) $\forall x \neq y \in X$, $t > 0$, $a,b,c,d,e : X \times X \to (0,1)$ *such that* $a+b+c+2d+2e < 1$ *and*

$$\begin{aligned}\frac{1}{M(Fx,Gy,t)} - 1 \leq\ & a(Sx,Ty)\phi(\frac{1}{M(Sx,Ty,t)} - 1) + b(Sx,Ty)\phi(\frac{1}{M(Fx,Sx,t)} - 1)\\ & + c(Sx,Ty)\phi(\frac{1}{M(Gy,Ty,t)} - 1) + d(Sx,Ty)\phi(\frac{1}{M(Fx,Ty,t)} - 1)\\ & + e(Sx,Ty)\phi(\frac{1}{M(Sx,Gy,t)} - 1),\end{aligned}$$

$$\begin{aligned}N(Fx,Gy,t) \leq\ & a(Sx,Ty)\psi(N(Sx,Ty,t)) + b(Sx,Ty)\psi(N(Fx,Sx,t))\\ & + c(Sx,Ty)\psi(N(Gy,Ty,t)) + d(Sx,Ty)\psi(N(Fx,Ty,t))\\ & + e(Sx,Ty)\psi(N(Sx,Gy,t)).\end{aligned}$$

(3) One of $F(X)$, $G(X)$, $S(X)$ *and* $T(X)$ *is a complete subspace of* X.

Then F *and* S *have a point of coincidence,* G *and* T *have a point of coincidence.*

(4) If (F,S), (G,T) *satisfy any of the following conditions:*

(i) (F,S), (G,T) *are weakly compatible;*

(ii) (F,S), (G,T) *are compatible;*

(iii) (F,S), (G,T) *are compatible of type* (A)*;*

(iv) (F,S), (G,T) *are compatible of type* (P)*;*

(v) (F,S), (G,T) *are compatible of type* (K)*;*

(vi) (F,S), (G,T) *are compatible of type* (R)*;*

(vii) (F,S), (G,T) *are compatible of type* (E)*;*

(viii) (F,S) *is* F*-compatible,* (G,T) *is* G*-compatible;*

(ix) (F,S) *is* S*-compatible,* (G,T) *is* T*-compatible;*

(x) (F,S), (G,T) *are semicompatible and reciprocally continuous;*

then F, G, S, T *have a unique common fixed point.*

Proof. $\forall x_0 \in X$, since $F(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $Fx_0 = Tx_1$. As $G(X) \subseteq S(X)$, for this point x_1, there exists $x_2 \in X$ such that $Gx_1 = Sx_2$. Inductively, we can construct a sequence $\{y_n\}$ in X such that $y_{2n} = Fx_{2n} = Tx_{2n+1}$, $y_{2n+1} = Gx_{2n+1} = Sx_{2n+2}$ for all $n \in$

$\{0,1,2,...\}$. For all $t>0$,

$$\begin{aligned}
&\frac{1}{M(y_{2n},y_{2n+1},t)}-1=\frac{1}{M(Fx_{2n},Gx_{2n+1},t)}-1\\
&\leq a(Sx_{2n},Tx_{2n+1})\phi(\frac{1}{M(Sx_{2n},Tx_{2n+1},t)}-1)+b(Sx_{2n},Tx_{2n+1})\phi(\frac{1}{M(Fx_{2n},Sx_{2n},t)}-1)\\
&+c(Sx_{2n},Tx_{2n+1})\phi(\frac{1}{M(Gx_{2n+1},Tx_{2n+1},t)}-1)+d(Sx_{2n},Tx_{2n+1})\phi(\frac{1}{M(Fx_{2n},Tx_{2n+1},t)}-1)\\
&+e(Sx_{2n},Tx_{2n+1})\phi(\frac{1}{M(Sx_{2n},Gx_{2n+1},t)}-1)\\
&\leq a(y_{2n-1},y_{2n})\phi(\frac{1}{M(y_{2n-1},y_{2n},t)}-1)+b(y_{2n-1},y_{2n})\phi(\frac{1}{M(y_{2n},y_{2n-1},t)}-1)\\
&+c(y_{2n-1},y_{2n})(\frac{1}{M(y_{2n+1},y_{2n},t)}-1)+d(y_{2n-1},y_{2n})(\frac{1}{M(y_{2n},y_{2n},t)}-1)\\
&+e(y_{2n-1},y_{2n})\phi(\frac{1}{M(y_{2n-1},y_{2n+1},t)}-1),
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&N(y_{2n},y_{2n+1},t)=N(Fx_{2n},Gx_{2n+1},t)\\
&\leq a(Sx_{2n},Tx_{2n+1})\psi(N(Sx_{2n},Tx_{2n+1},t))+b(Sx_{2n},Tx_{2n+1})\psi(N(Fx_{2n},Sx_{2n},t))\\
&+c(Sx_{2n},Tx_{2n+1})\psi(N(Gx_{2n+1},Tx_{2n+1},t))+d(Sx_{2n},Tx_{2n+1})\psi(N(Fx_{2n},Tx_{2n+1},t))\\
&+e(Sx_{2n},Tx_{2n+1})\psi(N(Sx_{2n},Gx_{2n+1},t))\\
&\leq a(y_{2n-1},y_{2n})\psi(N(y_{2n-1},y_{2n},t))+b(y_{2n-1},y_{2n})\psi(N(y_{2n},y_{2n-1},t))\\
&+c(y_{2n-1},y_{2n})N(y_{2n+1},y_{2n},t)+d(y_{2n-1},y_{2n})N(y_{2n},y_{2n},t)\\
&+e(y_{2n-1},y_{2n})\psi(N(y_{2n-1},y_{2n+1},t)).
\end{aligned}\tag{3.2}$$

Due to the fact that the intuitionistic metric (M,N) is triangular, we can know that $(\frac{1}{M(y_{2n-1},y_{2n+1},t)}-1)\leq(\frac{1}{M(y_{2n-1},y_{2n},t)}-1)+(\frac{1}{M(y_{2n},y_{2n+1},t)}-1)$, $N(y_{2n-1},y_{2n+1},t)\leq N(y_{2n-1},y_{2n},t)+N(y_{2n},y_{2n+1},t)$.

From the equations (3.1) and (3.2), and the properties of the functions a,b,c,d,e, it's clear that

$$\begin{aligned}
(\frac{1}{M(y_{2n},y_{2n+1},t)}-1)&\leq\frac{a(y_{2n-1},y_{2n})+b(y_{2n-1},y_{2n})+e(y_{2n-1},y_{2n})}{1-c(y_{2n-1},y_{2n})-e(y_{2n-1},y_{2n})}\times\phi(\frac{1}{M(y_{2n-1},y_{2n},t)}-1)\\
&<\phi(\frac{1}{M(y_{2n-1},y_{2n},t)}-1)\\
&\leq(\frac{1}{M(y_{2n-1},y_{2n},t)}-1),
\end{aligned}$$

$$
\begin{aligned}
N(y_{2n},y_{2n+1},t) &\leq \frac{a(y_{2n-1},y_{2n})+b(y_{2n-1},y_{2n})+e(y_{2n-1},y_{2n})}{1-c(y_{2n-1},y_{2n})-e(y_{2n-1},y_{2n})}\times\psi(N(y_{2n-1},y_{2n},t))\\
&< \psi(N(y_{2n-1},y_{2n},t))\\
&\leq N(y_{2n-1},y_{2n},t).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{M(y_{2n},y_{2n-1},t)}-1=\frac{1}{M(Fx_{2n},Gx_{2n-1},t)}-1\\
&\leq a(Sx_{2n},Tx_{2n-1})\phi(\frac{1}{M(Sx_{2n},Tx_{2n-1},t)}-1)+b(Sx_{2n},Tx_{2n-1})\phi(\frac{1}{M(Fx_{2n},Sx_{2n},t)}-1)\\
&\quad+c(Sx_{2n},Tx_{2n-1})\phi(\frac{1}{M(Gx_{2n-1},Tx_{2n-1},t)}-1)+d(Sx_{2n},Tx_{2n-1})\phi(\frac{1}{M(Fx_{2n},Tx_{2n-1},t)}-1)\\
&\quad+e(Sx_{2n},Tx_{2n-1})\phi(\frac{1}{M(Sx_{2n},Gx_{2n-1},t)}-1)\\
&\leq a(y_{2n-1},y_{2n-2})\phi(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1)+b(y_{2n-1},y_{2n-2})(\frac{1}{M(y_{2n},y_{2n-1},t)}-1)\\
&\quad+c(y_{2n-1},y_{2n-2})\phi(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1)+d(y_{2n-1},y_{2n-2})\phi(\frac{1}{M(y_{2n},y_{2n-2},t)}-1)\\
&\quad+e(y_{2n-1},y_{2n-2})(\frac{1}{M(y_{2n-1},y_{2n-1},t)}-1),
\end{aligned}
$$

$$
\begin{aligned}
&N(y_{2n},y_{2n-1},t)=N(Fx_{2n},Gx_{2n-1},t)\\
&\leq a(Sx_{2n},Tx_{2n-1})\psi(N(Sx_{2n},Tx_{2n-1},t))+b(Sx_{2n},Tx_{2n-1})\psi(N(Fx_{2n},Sx_{2n},t))\\
&\quad+c(Sx_{2n},Tx_{2n-1})\psi(N(Gx_{2n-1},Tx_{2n-1},t))+d(Sx_{2n},Tx_{2n-1})\psi(N(Fx_{2n},Tx_{2n-1},t))\\
&\quad+e(Sx_{2n},Tx_{2n-1})\psi(N(Sx_{2n},Gx_{2n-1},t))\\
&\leq a(y_{2n-1},y_{2n-2})\psi(N(y_{2n-1},y_{2n-2},t))+b(y_{2n-1},y_{2n-2})N(y_{2n},y_{2n-1},t)\\
&\quad+c(y_{2n-1},y_{2n-2})\psi(N(y_{2n-1},y_{2n-2},t))+d(y_{2n-1},y_{2n-2})\psi(N(y_{2n},y_{2n-2},t))\\
&\quad+e(y_{2n-1},y_{2n-2})N(y_{2n-1},y_{2n-1},t).
\end{aligned}
$$

In the same way we can get

$$
\begin{aligned}
(\frac{1}{M(y_{2n},y_{2n-1},t)}-1) &\le \frac{a(y_{2n-1},y_{2n-2})+c(y_{2n-1},y_{2n-2})+d(y_{2n-1},y_{2n-2})}{1-b(y_{2n-1},y_{2n-2})-d(y_{2n-1},y_{2n-2})} \\
&\quad \times \phi(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1) \\
&< \phi(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1) \\
&\le (\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1),
\end{aligned}
$$

$$
\begin{aligned}
N(y_{2n},y_{2n-1},t) &\le \frac{a(y_{2n-1},y_{2n-2})+c(y_{2n-1},y_{2n-2})+d(y_{2n-1},y_{2n-2})}{1-b(y_{2n-1},y_{2n-2})-d(y_{2n-1},y_{2n-2})} \times \psi(N(y_{2n-1},y_{2n-2},t)) \\
&< \psi(N(y_{2n-1},y_{2n-2},t)) \\
&\le N(y_{2n-1},y_{2n-2},t).
\end{aligned}
$$

So we have $(\frac{1}{M(y_n,y_{n+1},t)}-1) < \phi(\frac{1}{M(y_{n-1},y_n,t)}-1) \le (\frac{1}{M(y_{n-1},y_n,t)}-1)$, $N(y_n,y_{n+1},t) < \psi(N(y_{n-1},y_n,t)) \le N(y_{n-1},y_n,t)$.

Suppose that $\lim_{n\to\infty} M(y_n,y_{n+1},t) = m$, $\lim_{n\to\infty} N(y_n,y_{n+1},t) = n$. We assert that $m = 1$, $n = 0$. If not, then we have

$$
\begin{aligned}
\lim_{n\to\infty}(\frac{1}{M(y_n,y_{n+1},t)}-1) &< \lim_{n\to\infty}\phi(\frac{1}{M(y_{n-1},y_n,t)}-1) \\
&= \phi(\lim_{n\to\infty}(\frac{1}{M(y_{n-1},y_n,t)}-1)),
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\lim_{n\to\infty} N(y_n,y_{n+1},t) &< \lim_{n\to\infty}\psi(N(y_{n-1},y_n,t)) \\
&= \psi(\lim_{n\to\infty}(N(y_{n-1},y_n,t))).
\end{aligned}
\tag{3.4}
$$

The equations (3.3) and (3.4) mean that $\frac{1}{m}-1 < \phi(\frac{1}{m}-1) \le \frac{1}{m}-1$, $n < \psi(n) \le n$, which are both contradictions. Hence $\lim_{n\to\infty} M(y_n,y_{n+1},t) = 1$, $\lim_{n\to\infty} N(y_n,y_{n+1},t) = 0$. For all $p \in \mathbb{N}^+$, $t > 0$,

$$
\begin{aligned}
\lim_{n\to\infty} M(y_{n+p},y_n,t) &\ge \lim_{n\to\infty}[M(y_{n+p},y_{n+p-1},\frac{t}{k}) * M(y_{n+p-1},y_{n+p-2},\frac{t}{k}) \\
&\quad * \ldots * M(y_{n+1},y_n,\frac{t}{k})] \\
&= 1 * 1 * \ldots * 1 = 1,
\end{aligned}
$$

$$\begin{aligned}\lim_{n\to\infty} N(y_{n+p},y_n,t) &\leq \lim_{n\to\infty}[N(y_{n+p},y_{n+p-1},\frac{t}{k})\diamond M(y_{n+p-1},y_{n+p-2},\frac{t}{k})\\ &\quad \diamond\ldots\diamond M(y_{n+1},y_n,\frac{t}{k})]\\ &= 0\diamond 0\diamond\ldots\diamond 0 = 0.\end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

If $S(X)$ is a complete subspace of X, then $\{y_{2n+1}\} = \{Sx_{2n+2}\}$ converges. Suppose that $\lim_{n\to\infty} y_{2n+1} = u$ and $v = S^{-1}u \in X$. Since $\{y_{2n+1}\}$ is a convergent subsequence of a Cauchy sequence $\{y_n\}$, hence $\{y_n\}$ also converges and $\lim_{n\to\infty} y_n = u$. For all $t > 0$,

$$\begin{aligned}&\lim_{n\to\infty}(\frac{1}{M(Fv,Gx_{2n+1},t)}-1)\\ &\leq \lim_{n\to\infty}[a(Sv,Tx_{2n+1})\phi(\frac{1}{M(Sv,Tx_{2n+1},t)}-1)+b(Sv,Tx_{2n+1})\phi(\frac{1}{M(Fv,Sv,t)}-1)\\ &\quad +c(Sv,Tx_{2n+1})\phi(\frac{1}{M(Gx_{2n+1},Tx_{2n+1},t)}-1)+d(Sv,Tx_{2n+1})\phi(\frac{1}{M(Fv,Tx_{2n+1},t)}-1)\\ &\quad +e(Sv,Tx_{2n+1})\phi(\frac{1}{M(Sv,Gx_{2n+1},t)}-1)],\end{aligned}$$

$$\begin{aligned}(\frac{1}{M(Fv,u,t)}-1) &\leq a(u,u)\phi(\frac{1}{M(u,u,t)}-1)+b(u,u)\phi(\frac{1}{M(Fv,u,t)}-1)\\ &\quad +c(u,u)\phi(\frac{1}{M(u,u,t)}-1)+d(u,u)\phi(\frac{1}{M(Fv,u,t)}-1)\\ &\quad +e(u,u)\phi(\frac{1}{M(u,u,t)}-1)\\ &= [b(u,u)+d(u,u)]\phi(\frac{1}{M(Fv,u,t)}-1)\\ &< \phi(\frac{1}{M(Fv,u,t)}-1) \leq (\frac{1}{M(Fv,u,t)}-1),\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty} N(Fv,Gx_{2n+1},t)\\ &\leq \lim_{n\to\infty}[a(Sv,Tx_{2n+1})\psi(N(Sv,Tx_{2n+1},t))+b(Sv,Tx_{2n+1})\psi(N(Fv,Sv,t))\\ &\quad +c(Sv,Tx_{2n+1})\psi(N(Gx_{2n+1},Tx_{2n+1},t))+d(Sv,Tx_{2n+1})\psi(N(Fv,Tx_{2n+1},t))\\ &\quad +e(Sv,Tx_{2n+1})\psi(N(Sv,Gx_{2n+1},t))],\end{aligned}$$

$$\begin{aligned}
N(Fv,u,t) &\leq a(u,u)\psi(N(u,u,t)) + b(u,u)\psi(N(Fv,u,t)) \\
&\quad + c(u,u)\psi(N(u,u,t)) + d(u,u)\psi(N(Fv,u,t)) \\
&\quad + e(u,u)\psi(N(u,u,t)) \\
&= [b(u,u)+d(u,u)]\psi(N(Fv,u,t)) \\
&< \psi(N(Fv,u,t)) \leq N(Fv,u,t)
\end{aligned}$$

which are both contradictions. Hence $Fv = Sv = u$ which mean that F and S have a point of coincidence. Since $F(X) \subseteq T(X)$, there exists $w \in X$ such that $u = Fv = Tw$. For all $t > 0$,

$$\begin{aligned}
&\lim_{n\to\infty}(\frac{1}{M(Fx_{2n},Gw,t)} - 1) \\
&\leq \lim_{n\to\infty}[a(Sx_{2n},Tw)\phi(\frac{1}{M(Sx_{2n},Tw,t)} - 1) + b(Sx_{2n},Tw)\phi(\frac{1}{M(Fx_{2n},Sx_{2n},t)} - 1) \\
&\quad + c(Sx_{2n},Tw)\phi(\frac{1}{M(Gw,Tw,t)} - 1) + d(Sx_{2n},Tw)\phi(\frac{1}{M(Fx_{2n},Tw,t)} - 1) \\
&\quad + e(Sx_{2n},Tw)\phi(\frac{1}{M(Sx_{2n},Gw,t)} - 1)],
\end{aligned}$$

$$\begin{aligned}
(\frac{1}{M(u,Gw,t)} - 1) &\leq a(u,u)\phi(\frac{1}{M(u,u,t)} - 1) + b(u,u)\phi(\frac{1}{M(u,u,t)} - 1) \\
&\quad + c(u,u)\phi(\frac{1}{M(Gw,u,t)} - 1) + d(u,u)\phi(\frac{1}{M(u,u,t)} - 1) \\
&\quad + e(u,u)\phi(\frac{1}{M(u,Gw,t)} - 1) \\
&= [c(u,u)+e(u,u)]\phi(\frac{1}{M(u,Gw,t)} - 1) \\
&< \phi(\frac{1}{M(u,Gw,t)} - 1) \\
&\leq (\frac{1}{M(u,Gw,t)} - 1).
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} N(Fx_{2n},Gw,t) &\leq \lim_{n\to\infty}[a(Sx_{2n},Tw)\psi(N(Sx_{2n},Tw,t)) + b(Sx_{2n},Tw)\psi(N(Fx_{2n},Sx_{2n},t)) \\
&\quad + c(Sx_{2n},Tw)\psi(N(Gw,Tw,t)) + d(Sx_{2n},Tw)\psi(N(Fx_{2n},Tw,t)) \\
&\quad + e(Sx_{2n},Tw)\psi(N(Sx_{2n},Gw,t))],
\end{aligned}$$

$$
\begin{aligned}
N(u,Gw,t) &\leq a(u,u)\psi(N(u,u,t)) + b(u,u)\psi(N(u,u,t)) \\
&\quad + c(u,u)\psi(N(Gw,u,t)) + d(u,u)\psi(N(u,u,t)) \\
&\quad + e(u,u)\psi(N(u,Gw,t)) \\
&= [c(u,u)+e(u,u)]\psi(N(u,Gw,t)) \\
&< \psi(N(u,Gw,t)) \\
&\leq N(u,Gw,t)
\end{aligned}
$$

which are both contradictions. Hence $Gw = Tw = u$ which mean that G and T have a point of coincidence.

If $T(X)$ is a complete subspace of X, then $\{y_{2n}\} = \{Tx_{2n+1}\}$ converges. Suppose that $\lim_{n\to\infty} y_{2n} = u$, $w = T^{-1}u \in X$. In the same manner $\{y_n\}$ also converges and $\lim_{n\to\infty} y_n = u$. For all $t > 0$,

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{M(Fx_{2n},Gw,t)} - 1\right) \\
&\leq \lim_{n\to\infty}[a(Sx_{2n},Tw)\phi\left(\frac{1}{M(Sx_{2n},Tw,t)} - 1\right) + b(Sx_{2n},Tw)\phi\left(\frac{1}{M(Fx_{2n},Sx_{2n},t)} - 1\right) \\
&\quad + c(Sx_{2n},Tw)\phi\left(\frac{1}{M(Gw,Tw,t)} - 1\right) + d(Sx_{2n},Tw)\phi\left(\frac{1}{M(Fx_{2n},Tw,t)} - 1\right) \\
&\quad + e(Sx_{2n},Tw)\phi\left(\frac{1}{M(Sx_{2n},Gw,t)} - 1\right)],
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{M(u,Gw,t)} - 1\right) &\leq a(u,u)\phi\left(\frac{1}{M(u,u,t)} - 1\right) + b(u,u)\phi\left(\frac{1}{M(u,u,t)} - 1\right) \\
&\quad + c(u,u)\phi\left(\frac{1}{M(Gw,u,t)} - 1\right) + d(u,u)\phi\left(\frac{1}{M(u,u,t)} - 1\right) \\
&\quad + e(u,u)\phi\left(\frac{1}{M(u,Gw,t)} - 1\right) \\
&= [c(u,u)+e(u,u)]\phi\left(\frac{1}{M(u,Gw,t)} - 1\right) \\
&< \phi\left(\frac{1}{M(u,Gw,t)} - 1\right) \\
&\leq \left(\frac{1}{M(u,Gw,t)} - 1\right).
\end{aligned}
$$

$$\begin{aligned}\lim_{n\to\infty} N(Fx_{2n},Gw,t) \leq \lim_{n\to\infty}[&a(Sx_{2n},Tw)\psi(N(Sx_{2n},Tw,t))+b(Sx_{2n},Tw)\psi(N(Fx_{2n},Sx_{2n},t))\\ &+c(Sx_{2n},Tw)\psi(N(Gw,Tw,t))+d(Sx_{2n},Tw)\psi(N(Fx_{2n},Tw,t))\\ &+e(Sx_{2n},Tw)\psi(N(Sx_{2n},Gw,t))],\end{aligned}$$

$$\begin{aligned}N(u,Gw,t) &\leq a(u,u)\psi(N(u,u,t))+b(u,u)\psi(N(u,u,t))\\ &\quad +c(u,u)\psi(N(Gw,u,t))+d(u,u)\psi(N(u,u,t))\\ &\quad +e(u,u)\psi(N(u,Gw,t))\\ &=[c(u,u)+e(u,u)]\psi(N(u,Gw,t))\\ &<\psi(N(u,Gw,t))\\ &\leq N(u,Gw,t)\end{aligned}$$

which are both contradictions. Hence $Gw = Tw = u$ which mean that G and T have a point of coincidence. Since $G(X) \subseteq S(X)$, there exists $v \in X$ such that $u = Gw = Sv$. For all $t > 0$,

$$\begin{aligned}\lim_{n\to\infty}(\frac{1}{M(Fv,Gx_{2n+1},t)}-1) \leq \lim_{n\to\infty}[&a(Sv,Tx_{2n+1})\phi(\frac{1}{M(Sv,Tx_{2n+1},t)}-1)\\ &+b(Sv,Tx_{2n+1})\phi(\frac{1}{M(Fv,Sv,t)}-1)\\ &+c(Sv,Tx_{2n+1})\phi(\frac{1}{M(Gx_{2n+1},Tx_{2n+1},t)}-1)\\ &+d(Sv,Tx_{2n+1})\phi(\frac{1}{M(Fv,Tx_{2n+1},t)}-1)\\ &+e(Sv,Tx_{2n+1})\phi(\frac{1}{M(Sv,Gx_{2n+1},t)}-1)],\end{aligned}$$

$$\begin{aligned}(\frac{1}{M(Fv,u,t)}-1) \leq\ &a(u,u)\phi(\frac{1}{M(u,u,t)}-1)+b(u,u)\phi(\frac{1}{M(Fv,u,t)}-1)\\ &+c(u,u)\phi(\frac{1}{M(u,u,t)}-1)+d(u,u)\phi(\frac{1}{M(Fv,u,t)}-1)\\ &+e(u,u)\phi(\frac{1}{M(u,u,t)}-1)\end{aligned}$$

$$
\begin{aligned}
&= [b(u,u)+d(u,u)]\phi(\frac{1}{M(Fv,u,t)}-1) \\
&< \phi(\frac{1}{M(Fv,u,t)}-1) \\
&\leq (\frac{1}{M(Fv,u,t)}-1),
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} N(Fv,Gx_{2n+1},t) \leq \lim_{n\to\infty}&[a(Sv,Tx_{2n+1})\psi(N(Sv,Tx_{2n+1},t)) \\
&+b(Sv,Tx_{2n+1})\psi(N(Fv,Sv,t)) \\
&+c(Sv,Tx_{2n+1})\psi(N(Gx_{2n+1},Tx_{2n+1},t)) \\
&+d(Sv,Tx_{2n+1})\psi(N(Fv,Tx_{2n+1},t)) \\
&+e(Sv,Tx_{2n+1})\psi(N(Sv,Gx_{2n+1},t))],
\end{aligned}
$$

$$
\begin{aligned}
N(Fv,u,t) &\leq a(u,u)\psi(N(u,u,t))+b(u,u)\psi(N(Fv,u,t)) \\
&\quad +c(u,u)\psi(N(u,u,t))+d(u,u)\psi(N(Fv,u,t)) \\
&\quad +e(u,u)\psi(N(u,u,t)) \\
&= [b(u,u)+d(u,u)]\psi(N(Fv,u,t)) \\
&< \psi(N(Fv,u,t)) \\
&\leq N(Fv,u,t)
\end{aligned}
$$

which are both contradictions. Hence $Fv = Sv = u$ which mean that F and S have a point of coincidence.

If $F(X)$ is a complete subspace of X, by $F(X) \subseteq T(X)$ we can get that there exists $w \in X$ such that $Tw = u$. The remaining proof is similar to the case of $T(X)$ is a complete subspace of X.

If $G(X)$ is a complete subspace of X, by $G(X) \subseteq S(X)$ we can get that there exists $v \in X$ such that $Sv = u$. The remaining proof is similar to the case of $S(X)$ is a complete subspace of X.

Summing up the above, $Fv = Sv = Gw = Tw = u$.

If (F,S), (G,T) satisfy any of the following conditions:

(i) If (F,S), (G,T) are weakly compatible, then $FSv = SFv = Fu = Su$, $GTw = TGw = Gu = Tu$.

(ii) If (F,S), (G,T) are compatible, then $FSv = SFv = Fu = Su$, $GTw = TGw = Gu = Tu$.

(iii) If (F,S), (G,T) are compatible of type (A), then $FSv = SSv = Fu = Su$, $GTw = TTw = Gu = Tu$.

(iv) If (F,S), (G,T) are compatible of type (P), then $FFv = SSv = Fu = Su$, $GGw = TTw = Gu = Tu$.

(v) If (F,S), (G,T) are compatible of type (K), then $FFv = Su = Fu$, $GGw = Tu = Gu$.

(vi) If (F,S), (G,T) are compatible of type (R), then $FSv = SFv = Fu = Su$, $GTw = TGw = Gu = Tu$.

(vii) If (F,S), (G,T) are compatible of type (E), then $FFv = FSv = Fu = Su$, $GGw = GTw = Gu = Tu$.

(viii) If (F,S) is F-compatible, (G,T) is G-compatible, then $FSv = SSv = Fu = Su$, $GTw = TTw = Gu = Tu$.

(ix) If (F,S) is S-compatible, (G,T) is T-compatible, then $SFv = FFv = Su = Fu$, $TGw = GGw = Tu = Gu$.

(x) If (F,S), (G,T) are semicompatible and reciprocally continuous, then $FSv = Su = Fu$, $GTw = Tu = Gu$.

According to the various compatible conditions mentioned above, it follows that $Fu = Su$, $Gu = Tu$. For all $t > 0$,

$$\begin{aligned}
&(\frac{1}{M(Fu,u,t)} - 1) = (\frac{1}{M(Fu,Gw,t)} - 1) \\
&\leq a(Su,Tw)\phi(\frac{1}{M(Su,Tw,t)} - 1) + b(Su,Tw)\phi(\frac{1}{M(Fu,Su,t)} - 1) \\
&\quad + c(Su,Tw)\phi(\frac{1}{M(Gw,Tw,t)} - 1) + d(Su,Tw)\phi(\frac{1}{M(Fu,Tw,t)} - 1) \\
&\quad + e(Su,Tw)\phi(\frac{1}{M(Su,Gw,t)} - 1),
\end{aligned}$$

$$
\begin{aligned}
(\frac{1}{M(Fu,u,t)}-1) &\leq a(Su,u)\phi(\frac{1}{M(Fu,u,t)}-1)+b(Su,u)\phi(\frac{1}{M(Fu,Su,t)}-1)\\
&\quad +c(Su,u)\phi(\frac{1}{M(u,u,t)}-1)+d(Su,u)\phi(\frac{1}{M(Fu,u,t)}-1)\\
&\quad +e(Su,u)\phi(\frac{1}{M(Fu,u,t)}-1)\\
&=[a(Su,u)+d(Su,u)+e(Su,u)]\phi(\frac{1}{M(Fu,u,t)}-1)\\
&<\phi(\frac{1}{M(Fu,u,t)}-1)\\
&\leq(\frac{1}{M(Fu,u,t)}-1),
\end{aligned}
$$

$$
\begin{aligned}
N(Fu,u,t) &= N(Fu,Gw,t)\\
&\leq a(Su,Tw)\psi(N(Su,Tw,t))+b(Su,Tw)\psi(N(Fu,Su,t))\\
&\quad +c(Su,Tw)\psi(N(Gw,Tw,t))+d(Su,Tw)\psi(N(Fu,Tw,t))\\
&\quad +e(Su,Tw)\psi(N(Su,Gw,t)),
\end{aligned}
$$

$$
\begin{aligned}
N(Fu,u,t) &\leq a(Su,u)\psi(N(Fu,u,t))+b(Su,u)\psi(N(Fu,Su,t))\\
&\quad +c(Su,u)\psi(N(u,u,t))+d(Su,u)\psi(N(Fu,u,t))\\
&\quad +e(Su,u)\psi(N(Fu,u,t))\\
&=[a(Su,u)+d(Su,u)+e(Su,u)]\psi(N(Fu,u,t))\\
&<\psi(N(Fu,u,t))\\
&\leq N(Fu,u,t)
\end{aligned}
$$

which are both contradictions. Hence, $Fu=u=Su$, which mean that u is a common fixed point of F and S. Suppose that $u_0 \neq u$ be another common fixed point of F and S. For all $t>0$,

$$
\begin{aligned}
&(\frac{1}{M(u_0,u,t)}-1)=(\frac{1}{M(Fu_0,Gw,t)}-1)\\
&\leq a(Su_0,Tw)\phi(\frac{1}{M(Su_0,Tw,t)}-1)+b(Su_0,Tw)\phi(\frac{1}{M(Fu_0,Su_0,t)}-1)
\end{aligned}
$$

$$+c(Su_0,Tw)\phi(\frac{1}{M(Gw,Tw,t)}-1)+d(Su_0,Tw)\phi(\frac{1}{M(Fu_0,Tw,t)}-1)$$
$$+e(Su_0,Tw)\phi(\frac{1}{M(Su_0,Gw,t)}-1),$$

$$\begin{aligned}(\frac{1}{M(u_0,u,t)}-1) &\leq a(u_0,u)\phi(\frac{1}{M(u_0,u,t)}-1)+b(u_0,u)\phi(\frac{1}{M(u_0,u_0,t)}-1)\\ &\quad +c(u_0,u)\phi(\frac{1}{M(u,u,t)}-1)+d(u_0,u)\phi(\frac{1}{M(u_0,u,t)}-1)\\ &\quad +e(u_0,u)\phi(\frac{1}{M(u_0,u,t)}-1)\\ &= [a(u_0,u)+d(u_0,u)+e(u_0,u)]\phi(\frac{1}{M(u_0,u,t)}-1)\\ &< \phi(\frac{1}{M(u_0,u,t)}-1)\\ &\leq (\frac{1}{M(u_0,u,t)}-1).\end{aligned}$$

$$\begin{aligned}N(u_0,u,t) &= N(Fu_0,Gw,t)\\ &\leq a(Su_0,Tw)\psi(N(Su_0,Tw,t))+b(Su_0,Tw)\psi(N(Fu_0,Su_0,t))\\ &\quad +c(Su_0,Tw)\psi(N(Gw,Tw,t))+d(Su_0,Tw)\psi(N(Fu_0,Tw,t))\\ &\quad +e(Su_0,Tw)\psi(N(Su_0,Gw,t)),\end{aligned}$$

$$\begin{aligned}N(u_0,u,t) &\leq a(u_0,u)\psi(N(u_0,u,t))+b(u_0,u)\psi(N(u_0,u_0,t))\\ &\quad +c(u_0,u)\psi(N(u,u,t))+d(u_0,u)\psi(N(u_0,u,t))\\ &\quad +e(u_0,u)\psi(N(u_0,u,t))\\ &= [a(u_0,u)+d(u_0,u)+e(u_0,u)]\psi(N(u_0,u,t))\\ &< \psi(N(u_0,u,t))\\ &\leq N(u_0,u,t)\end{aligned}$$

which are both contradictions. Hence u is the unique common fixed point of F and S. On the other hand, consider the case of G and T,

$$
\begin{aligned}
(\frac{1}{M(u,Gu,t)}-1) = (\frac{1}{M(Fv,Gu,t)}-1) \\
&\leq a(Sv,Tu)\phi(\frac{1}{M(Sv,Tu,t)}-1)+b(Sv,Tu)\phi(\frac{1}{M(Fv,Sv,t)}-1) \\
&\quad +c(Sv,Tu)\phi(\frac{1}{M(Gu,Tu,t)}-1)+d(Sv,Tu)\phi(\frac{1}{M(Fv,Tu,t)}-1) \\
&\quad +e(Sv,Tu)\phi(\frac{1}{M(Sv,Gu,t)}-1),
\end{aligned}
$$

$$
\begin{aligned}
N(u,Gu,t) &= N(Fv,Gu,t) \\
&\leq a(Sv,Tu)\psi(N(Sv,Tu,t))+b(Sv,Tu)\psi(N(Fv,Sv,t)) \\
&\quad +c(Sv,Tu)\psi(N(Gu,Tu,t))+d(Sv,Tu)\psi(N(Fv,Tu,t)) \\
&\quad +e(Sv,Tu)\psi(N(Sv,Gu,t)),
\end{aligned}
$$

$$
\begin{aligned}
(\frac{1}{M(u,Gu,t)}-1) &\leq a(u,Tu)\phi(\frac{1}{M(u,Tu,t)}-1)+b(u,Tu)\phi(\frac{1}{M(u,u,t)}-1) \\
&\quad +c(u,Tu)\phi(\frac{1}{M(u,u,t)}-1)+d(u,Tu)\phi(\frac{1}{M(Fu,u,t)}-1) \\
&\quad +e(u,Tu)\phi(\frac{1}{M(Su,u,t)}-1) \\
&= a(u,Tu)\phi(\frac{1}{M(u,Gu,t)}-1) \\
&< \phi(\frac{1}{M(u,Gu,t)}-1) \\
&\leq (\frac{1}{M(u,Gu,t)}-1).
\end{aligned}
$$

$$
\begin{aligned}
N(u,Gu,t) &\leq a(u,Tu)\psi(N(u,Tu,t))+b(u,Tu)\psi(N(u,u,t)) \\
&\quad +c(u,Tu)\psi(N(u,u,t))+d(u,Tu)\psi(N(Fu,u,t)) \\
&\quad +e(u,Tu)\psi(N(Su,u,t)) \\
&= a(u,Tu)\psi(N(u,Gu,t)) \\
&< \psi(N(u,Gu,t)) \leq N(u,Gu,t)
\end{aligned}
$$

which are both contradictions. Thus, $u = Fu = Gu = Su = Tu$, which mean that u is the common fixed point of F, G, S, T. Suppose that $u_1 \neq u$ be another common fixed point of G and T. For all $t > 0$,

$$\begin{aligned}
&(\frac{1}{M(u,u_1,t)} - 1) = (\frac{1}{M(Fv,Gu_1,t)} - 1) \\
&\leq a(Sv,Tu_1)\phi(\frac{1}{M(Sv,Tu_1,t)} - 1) + b(Sv,Tu_1)\phi(\frac{1}{M(Fv,Sv,t)} - 1) \\
&\quad + c(Sv,Tu_1)\phi(\frac{1}{M(Gu_1,Tu_1,t)} - 1) + d(Sv,Tu_1)\phi(\frac{1}{M(Fv,Tu_1,t)} - 1) \\
&\quad + e(Sv,Tu_1)\phi(\frac{1}{M(Sv,Gu_1,t)} - 1),
\end{aligned}$$

$$\begin{aligned}
(\frac{1}{M(u,u_1,t)} - 1) &\leq a(u,u_1)\phi(\frac{1}{M(u,u_1,t)} - 1) + b(u,u_1)\phi(\frac{1}{M(u,u,t)} - 1) \\
&\quad + c(u,u_1)\phi(\frac{1}{M(u_1,u_1,t)} - 1) + d(u,u_1)\phi(\frac{1}{M(u,u_1,t)} - 1) \\
&\quad + e(u,u_1)\phi(\frac{1}{M(u,u_1,t)} - 1) \\
&= [a(u,u_1) + d(u,u_1) + e(u,u_1)]\phi(\frac{1}{M(u,u_1,t)} - 1) \\
&< \phi(\frac{1}{M(u,u_1,t)} - 1) \leq (\frac{1}{M(u,u_1,t)} - 1).
\end{aligned}$$

$$\begin{aligned}
N(u,u_1,t) &= N(Fv,Gu_1,t) \\
&\leq a(Sv,Tu_1)\psi(N(Sv,Tu_1,t)) + b(Sv,Tu_1)\psi(N(Fv,Sv,t)) \\
&\quad + c(Sv,Tu_1)\psi(N(Gu_1,Tu_1,t)) + d(Sv,Tu_1)\psi(N(Fv,Tu_1,t)) \\
&\quad + e(Sv,Tu_1)\psi(N(Sv,Gu_1,t)),
\end{aligned}$$

$$\begin{aligned}
N(u,u_1,t) &\leq a(u,u_1)\psi(N(u,u_1,t)) + b(u,u_1)\psi(N(u,u,t)) \\
&\quad + c(u,u_1)\psi(N(u_1,u_1,t)) + d(u,u_1)\psi(N(u,u_1,t)) \\
&\quad + e(u,u_1)\psi(N(u,u_1,t)) \\
&= [a(u,u_1) + d(u,u_1) + e(u,u_1)]\psi(N(u,u_1,t)) \\
&< \psi(N(u,u_1,t)) < N(u,u_1,t)
\end{aligned}$$

which are both contradiction. Hence u is the unique common fixed point of G and T. In summary, u is the unique common fixed point of F, G, S, T. □

Example 4. *Let $X = [-1,1]$, $a*b = min\{a,b\}$ and $a \diamond b = max\{a,b\}$ for all $a,b \in [0,1]$. Denote $M(x,y,t) = \frac{t}{t+|x-y|}$, $N(x,y,t) = \frac{|x-y|}{t+|x-y|}$ for $x,y \in X$, $t > 0$. We define*

$$F(x) = \begin{cases} 0, if\, x = 0; \\ 1, if\, x > 0; \\ -1, (if) x < 0; \end{cases} \qquad T(x) = x^2;\ G(x) = \begin{cases} 0, if\, x = 0; \\ \frac{1}{2}, if\, x > 0; \\ -\frac{1}{2}, (if) x < 0; \end{cases} \qquad S(x) = x^3.\ \phi(r) = \psi(r) = r.$$

It's obvious that F, G, S, T, ϕ and ψ satisfy the condition of Theorem 3.1, so F, G, S, T have a unique common fixed point $z = 0$.

Corollary 3.2. *Let $(X,M,N,*,\diamond)$ be an triangular intuitionistic fuzzy metric space. F, G, S, T be the self-mappings of X satisfying the following conditions:*

(1) $F(X) \subseteq T(X)$, $G(X) \subseteq S(X)$.

(2) $\forall x \neq y \in X$, $t > 0$, $a,b,c,d,e \in (0,1)$ such that $a+b+c+2d+2e = k < 1$ and

$$\begin{aligned} \frac{1}{M(Fx,Gy,t)} - 1 \leq a(&\frac{1}{M(Sx,Ty,t)} - 1) + b(\frac{1}{M(Fx,Sx,t)} - 1) \\ &+ c(\frac{1}{M(Gy,Ty,t)} - 1) + d(\frac{1}{M(Fx,Ty,t)} - 1) \\ &+ e(\frac{1}{M(Sx,Gy,t)} - 1), \end{aligned}$$

$$\begin{aligned} N(Fx,Gy,t) \leq\ & a(N(Sx,Ty,t)) + b(N(Fx,Sx,t)) \\ &+ c(N(Gy,Ty,t)) + d(N(Fx,Ty,t)) \\ &+ e(N(Sx,Gy,t)). \end{aligned}$$

(3) One of $F(X)$, $G(X)$, $S(X)$ and $T(X)$ is a complete subspace of X.

Then F and S have a point of coincidence, G and T havea point of coincidence.

If (F,S), (G,T) are weakly compatible, then F, G, S, T have a unique common fixed point.

Remark 3.3. *According to Theorom 3.1, let $\phi,\psi = I$, that is $\phi(x) = x$, $\psi(x) = x$. And let a,b,c,d,e be some invariable constants. You can easily come to the conclusion.*

Corollary 3.4. [7] *Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space and $S,T:X\to X$ such that $S(X)$ is a complete subspace of X. If T is an $(IFC\Delta)$ w.r.t to S (there exists a mapping $\Delta:X\to[0,1)$ which $\Delta(Tx)\le\Delta(Sx)$ for all $x\in X$ such that $\frac{1}{M(Tx,Ty,t)}-1\le\Delta(Sx)(\frac{1}{M(Sx,Sy,t)}-1)$ and $N(Tx,Ty,t)\le\Delta(Sx)N(Sx,Sy,t)$ for all $x,y\in X$ and $t>0$) and the mappings S and T are weakly compatible, then S and T have a unique common fixed point.*

Remark 3.5. *According to Theorem 3.1, let $F=G=T$, $S=T=S$, $a(Sx,Ty)=\Delta(Sx)$, $b(Sx,Ty)=c(Sx,Ty)=d(Sx,Ty)=e(Sx,Ty)=0$, and $\phi,\psi=I$, that is $\phi(x)=x$, $\psi(x)=x$, then the results are easy to come by.*

Theorem 3.6 is a more basic form of Theorem 3.1.

Theorem 3.6. *Let $(X,M,N,*,\diamond)$ be an triangular intuitionistic fuzzy metric space, $\{F_1,F_2,...,F_m\}$, $\{G_1,G_2,...,G_l\}$, $\{S_1,S_2,...,S_p\}$, $\{T_1,T_2,...,T_q\}$ be four finite families of self-mappings of X such that $F=F_1F_2...F_m$ and $G=G_1G_2...G_l$, $S=S_1S_2...S_p$ and $T=T_1T_2...T_q$, satisfying the condition (1), (2), (3) and (4) of* Theorem 3.1. *Not only that, $\{F_1,F_2,...,F_m\}$ and $\{S_1,S_2,...,S_p\}$ are pairwise commuting, $\{G_1,G_2,...,G_l\}$ and $\{T_1,T_2,...,T_q\}$ are pairwise commuting, then for all $i_1\in\{1,2,...,m\}$, $i_2\in\{1,2,...,l\}$, $i_3\in\{1,2,...,p\}$, $i_4\in\{1,2,...,q\}$, F, G, S, T, F_{i_1}, G_{i_2}, S_{i_3}, T_{i_4} have a unique common fixed point.*

Proof. According to Theorem 3.1, it's already known that F, G, S, T have a unique common fixed point. Let's assume that the fixed point is u. Since $\{F_1,F_2,...,F_m\}$ and $\{S_1,S_2,...,S_p\}$ are pairwise commuting, then for all $i_1\in\{1,2,...,m\}$,

$$\begin{aligned} FF_{i_1}u &= F_1F_2...F_mF_{i_1}u=F_1F_2...F_{i_1}F_mu=F_1F_2...F_{i_1}F_{m-1}F_mu\\ &= ....=F_{i_1}F_1F_2...F_mu=F_{i_1}Fu=F_{i_1}u,\\ SF_{i_1}u &= S_1S_2...S_pF_{i_1}u=S_1S_2...F_{i_1}S_pu=S_1S_2...F_{i_1}S_{p-1}S_pu\\ &= ....=F_{i_1}S_1S_2...S_pu=F_{i_1}Su=F_{i_1}u. \end{aligned}$$

Thus $F_{i_1}u$ is a common fixed point of F, S. By the uniqueness of common fixed point of F and S we can know that $F_{i_1}u=u$. Similarly, for all $i_1\in\{1,2,...,m\}$, $i_2\in\{1,2,...,l\}$, $i_3\in\{1,2,...,p\}$, $i_4\in\{1,2,...,q\}$, $F_{i_1}u=G_{i_2}u=S_{i_3}u=T_{i_4}u=u=Fu=Gu=Su=Tu$ and u is the unique common fixed point of F, G, S, T, F_{i_1}, G_{i_2}, S_{i_3}, T_{i_4}. □

Corollary 3.7. *Let $(X,M,N,*,\diamond)$ be an triangular intuitionistic fuzzy metric space, F^m, G^l, S^p, T^q be the self-mappings of X satisfying the following conditions:*

(1) $F^m(X) \subseteq T^q(X)$, $G^l(X) \subseteq S^p(X)$.

(2) $\forall x \neq y \in X$, $t > 0$, $a,b,c,d,e : X \times X \to (0,1)$ such that $a+b+c+2d+2e < 1$ and

$$\begin{aligned}
\frac{1}{M(F^m x, G^l y, t)} - 1 \leq\ & a(S^p x, T^q y)\phi(\frac{1}{M(S^p x, T^q y, t)} - 1) \\
& + b(S^p x, T^q y)\phi(\frac{1}{M(F^m x, S^p x, t)} - 1) \\
& + c(S^p x, T^q y)\phi(\frac{1}{M(G^l y, T^q y, t)} - 1) \\
& + d(S^p x, T^q y)\phi(\frac{1}{M(F^m x, T^q y, t)} - 1) \\
& + e(S^p x, T^q y)\phi(\frac{1}{M(S^p x, G^l y, t)} - 1), \\
N(F^m x, G^l y, t) \leq\ & a(S^p x, T^q y)\psi(N(S^p x, T^q y, t)) \\
& + b(S^p x, T^q y)\psi(N(F^m x, S^p x, t)) \\
& + c(S^p x, T^q y)\psi(N(G^l y, T^q y, t)) \\
& + d(S^p x, T^q y)\psi(N(F^m x, T^q y, t)) \\
& + e(S^p x, T^q y)\psi(N(S^p x, G^l y, t)).
\end{aligned}$$

(3) One of $F^m(X)$, $G^l(X)$, $S^p(X)$ and $T^q(X)$ is a complete subspace of X.

then F^m and S^p have a point of coincidence, G^l and T^q have a point of coincidence.

If (F^m, S^p), (G^l, T^q) are weakly compatible, then F^m, G^l, S^p, T^q have a unique common fixed point.

Moreover, if F and S are commuting, G and T are commuting, then F, G, S, T, F^m, G^l, S^p, T^q have a unique common fixed point.

Remark 3.8. *According to Theorem 3.6, let $F_1 = F_2 = \ldots = F_m = F$, $G_1 = G_2 = \ldots = G_l = G$, $S_1 = S_2 = \ldots = S_m = S$, $T_1 = T_2 = \ldots = T_q = T$, and $\phi, \psi = I$, that is $\phi(x) = x$, $\psi(x) = x$, then the results are easy to come by.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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