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COMMON FIXED POINT THEOREMS USING "E.A LIKE" PROPERTY AND IMPLICIT RELATION IN FUZZY METRIC SPACES

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Abstract: The present paper deals with "E.A. Like" property and its application in proving common fixed point results in a fuzzy metric space.

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1. Introduction

In 1965 Zadeh [15] introduced the concept of fuzzy sets. In the next decade Kramosil and Michalek [6] introduced the concept of fuzzy metric space. Then George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. Consequently in due course of time some metric fixed points results were generalized to fuzzy metric spaces by various authors. Sessa [12] improved commutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space. Vasuki [13] proved fixed point theorems for R-weakly commuting mapping Pant [9] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [6] in fuzzy metric space is generalized by [5] by introducing concept of weakly compatible maps. Aamri and

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Moutawakil [1] generalized the notion of non compatible mapping in metric space by E. A. property. It was pointed out in [4], that property E. A. buys containment of ranges without any continuity requirements besides minimizes the commutativity conditions of the maps to the commutativity at their points of coincidence. More-over, E. A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Some common fixed point theorems in probabilistic or fuzzy metric spaces by E. A. property under weak compatibility have been recently obtained in[10]

Recently, implicit relations are used as a tool for finding common fixed point of contraction maps([2], [7]). These implicit relations guarantee coincidence point of pair of maps that ultimately leads to the existence of common fixed points of a quadruple of maps satisfying weak compatibility criterion.

In 2011, S.Kumar and B. Fisher[8] proved common fixed point in fuzzy metric space using E.A property and implicit function.Our objective is to prove a common fixed point by removing the assumption $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$,If range of one of the maps A, B, S or T is a complete subspace of X.

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2 Preliminaries

Definition 2.1 [15] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2 [11] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norms if $*$ is satisfying conditions:

(i) $*$ is an commutative and associative

(ii) $*$ is continuous

(iii) $a * 1 = a$ for all $a \in [0, 1]$

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Example 2.3 $a*b = \min. \{a,b\}$, $a*b = a.b$

Definition 2.4 [3] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set of $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$

(f1) $M(x, y, t) > 0$

(f2) $M(x, y, t) = 1$ if and only if $x = y$

(f3) $M(x, y, t) = M(y, x, t)$

(f4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(f5) $M(x, y, \cdot): (0, \infty) \rightarrow (0, 1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.5 (Induced fuzzy metric [3]) Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.6 Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$,for some x in X.

Lemma 2.7 Let $(X, M, *)$ be fuzzy metric space. If there exist $q \in (0, 1)$ such that $M(x,y,qt) \geq M(x,y,t)$ for all $x,y \in \mathbf{X}$ and $t > 0$, then $x = y$.

Definition 2.8 Let X be a set, f and g selfmaps of X. A point $x \in X$ is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.9[5] A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

Definition 2.10 Let f and g be two self-maps of a fuzzy metric space $(X, M, *)$. We say that f and g satisfy the property E. A. if there exists a sequence $\{x_n\}$ such that,

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.11[14] Let f and g be two self-maps of a fuzzy metric space (X, M,$*$). We say that f and g satisfy the property E. A. Like property if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$

for some $z \in f(X)$ or $z \in g(X)$, i.e. $z \in f(X) \cup g(X)$

Definition 2.12(Common E. A. Property) Let A, B, S, T : X → X where X is a fuzzy metric space, then the pair {A, S} and {B, T } said to satisfy common E. A. property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$, for some $z \in X$.

Definition 2.13[14](Common E. A. like Property) Let A, B, S and T be self maps of a fuzzy metric space(X, M,$*$) then the pairs (A, S) and (B, T) said to satisfy common E. A. Like property if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$,

Where $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$

Example 2.14[14] Let X = [0, 2) and M(x,y,t) = $\frac{t}{t+d(x,y)}$ for all $x,y \in X$ then (X, M,$*$)is a fuzzy metric space. Where a $*$ b = min. {a,b}

A(x) = $\begin{cases} 0.25, 0 \le x \le .52 \\ \frac{x}{2}, x > .52 \end{cases}$, S(X) = $\begin{cases} 0.25, 0 \le x \le .6 \\ x - 0.25, x > .6 \end{cases}$

T(X) = $\begin{cases} 0.25, 0 \le x \le .6 \\ \frac{x}{4}, x > .6 \end{cases}$, B(X) = $\begin{cases} 0.25, 0 \le x \le .95 \\ x - 0.75, x > .95 \end{cases}$

We define $x_n = 0.5 + \frac{1}{n}$ and $y_n = 1 + \frac{1}{n}$. we have A(X) $= \{0.25\} \cup (.26,1]$,

S(X) $= \{0.25\} \cup (0.35,1.75]$, T(X) = $(0.15,0.5]$ and B(X) $= \{0.25\} \cup (0.20,1.25]$.

Role of E.A. property in proving common fixed point theorems can be concluded by following,

(1) It buys containment of ranges without any continuity requirements.
(2) It minimizes the commutativity conditions of the maps to the commutativityat their points of coincidence.
(3) It allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

Of course, if two mappings satisfy E. A. like property then they satisfy E. A. property also, but, on the other hand, E. A. like property relaxes the condition of containment of ranges and closeness of the ranges to prove common fixed point theorems, which are necessary with E. A. property.

3 Implicit Relation

In our result, we deal with implicit relation used in [8]. In [8], S.Kumar and B. Fisher used the following implicit relation: Let I = [0, 1], $*$ be a continuous t-norm and F be the set of all real continuous functions F**:** $I^6 \to R$ satisfying the following conditions:

(F-1) F is non increasing in the fifth and sixth variables,
(F-2) if, for some constant k ∈(0, 1) we have
(F-a) $F(u(kt), v(t), v(t), u(t), 1, u(t/2) * v(t/2)) \geq 1), or$
(F-b)$F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1)$
for any fixed t > 0 and any non decreasing functions u, v **:** $(0,\infty) \to I$ with
$0 \leq u(t), v(t) \leq 1$ then there exists h $\in$ (0, 1) with $u(ht) \geq v(t) * u(t)$,
(F-3) if, for some constant k $\in$ (0, 1) we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$
for any fixed t > 0 and any non decreasing function u **:** $(0,\infty) \to I$ then
$u(kt) \geq u(t)$.

4 Main Result

In 2011, S. Kumar and B .Fisher (8) proved following result.

Theorem 4.1: Let (X, M,$*$) be a fuzzy metric space with continuous t-norm. Let A, B, S, T be self mappings of X satisfying:

(4.1) A(X)$\subseteq$T(X) and B(X)$\subseteq$S(X),

(4.2) pairs (A, S) and (B, T) are weakly compatible,

(4.3) there exist k$\in$(0,1) and F$\in\mathbf{F}$ such that

$$F\begin{pmatrix} M(Ax,By,kt), M(Sx,Ty,t), M(Ax,Sx,t) \\ M(By,Ty,t), M(Ax,Ty,t), M(By,Sx,t) \end{pmatrix} \geq 1$$

For all x,y$\in$X, t > 0.

(4.4) pair (A, S) or (B , T) satisfies the property (E.A.),

(4.5) If range of one of the maps A, B, S or T is a complete subspace of X.

Then A, B, S, T have a unique common fixed point in X.

Now, we prove the following.

Theorem 4.2: Let (X, M,$*$) be a fuzzy metric space with continuous t-norm. Let A, B, S, T be self mappings of X satisfying:

(4.1) there exist k $\in$(0, 1) and F$\in\mathbf{F}$ such that

$$F\begin{pmatrix} M(Ax,By,kt), M(Sx,Ty,t), M(Ax,Sx,t) \\ M(By,Ty,t), M(Ax,Ty,t), M(By,Sx,t) \end{pmatrix} \geq 1$$

For all x,y$\in$X, t > 0.

(4.2) pair (A, S) or (B , T) satisfy common E.A. Like property

(4.3) pairs (A, S) and (B, T) are weakly compatible

Then A, B, S, T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E. A. Like property therefore there exist two sequences $\{x_n\}$ and$\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$

Where z $\in$ S(X) $\cap$T(X) or z $\in$ A(X) $\cap$B(X).

Suppose that z $\in$S(X) $\cap$T(X), now we have

$\lim_{n\to\infty} Ax_n = z \in$ S(X) then z = Su for some u $\in$ X.

Now, we claim that Au = Su, from (I) we have

$$F\begin{pmatrix} M(Au,By_n,kt),M(Su,Ty_n,t),M(Au,Su,t) \\ M(By_n,Ty_n,t),M(Au,Ty_n,t),M(By_n,Su,t) \end{pmatrix} \geq 1$$

Taking limit $n \to \infty$, we get

$$F\begin{pmatrix} M(Au,Su,kt),M(z,z,t),M(Au,Su,t) \\ M(z,z,t),M(Au,Su,t),M(z,z,t) \end{pmatrix} \geq 1$$

$$F\left(M(Au,Su,kt),1,M(Au,Su,t),1,M(Au,Su,t),1\right) \geq 1$$

On the other hand, since

M (Au, Su, t) $\geq$ M (Au, Su, $\frac{t}{2}$) = M (Au, Su, $\frac{t}{2}$) $*$ 1

And F is non increasing in the fifth variable, we have, for any t > 0

$$F\left(M(Au,Su,kt),1,M(Au,Su,t),1,M(Au,Su,\frac{t}{2})*1,1 \right)$$

$$\geq F\left(M(Au,Su,kt),1,M(Au,Su,t),1,M(Au,Su,t),1\right)$$

$$\geq 1$$

Which implies, by (F-2) that Au = z =Su. The weak compatibility of A and S implies that ASu = SAu and then AAu = ASu = SAu = SSu.

Again $\lim_{n\to\infty} By_n = z \in T(X)$ then z = Tv for some v $\in$ X.

Now, we claim that Tv = Bv, from (I) we have

$$F\begin{pmatrix} M(Au,Bv,kt),M(Su,Tv,t),M(Au,Su,t) \\ M(Bv,Tv,t),M(Au,Tv,t),M(Bv,Su,t) \end{pmatrix} \geq 1$$

That is,

$$F\left(M(Tv,Bv,kt),1,1,M(Bv,Tv,t),1,M(Bv,Tv,t)\right) \geq 1$$

On the other hand, since

M (Bv, Tv, t) $\geq$ M (Bv, Tv, $\frac{t}{2}$) = M (Bv, Tv, $\frac{t}{2}$) $*$ 1

And F is non increasing in the sixth variable, we have,for any t > 0.

$$F\left(M(Tv,Bv,kt),1,1,M(Bv,Tv,t),1,M(Bv,Tv,\frac{t}{2})*1 \right)$$

$$\geq F\left(M(Tv,Bv,kt),1,1,M(Bv,Tv,t),1,M(Bv,Tv,t)\right)$$
$$\geq 1$$

Which implies, by (F-2) that Bv = Tv.

This implies that Au = Su = Tv = Bv. The weak compatibility of B and T implies that BTv = TBv and then TTv = TBv = BTv = BBv.

Let us show that Au is a common fixed point of A, B, S and T. In view of (4.1), it follows

$$F\begin{pmatrix} M(AAu,Bv,kt),M(SAu,Tv,t),M(AAu,SAu,t) \\ M(Bv,Tv,t),M(AAu,Tv,t),M(Bv,SAu,t) \end{pmatrix} \geq 1$$

That is,

$$F\left(M(AAu,Au,kt),M(SAu,Au,t),M(AAu,Au,t),M(Au,AAu,t)\right) \geq 1$$

Thus, from (F-3), we have M (AAu, Au, kt) ≥ M (AAu, Au, t). By the Lemma 2.7, we have, AAu = Au.

Therefore, Au = AAu = SAu and Au is a common fixed point of A and S. Similarly, we can prove that Bv is a common fixed point of B and T. Since Au = Bv we conclude that Au is a common fixed point of A, B, S and T. The proof is similar when $z \in$ T (X). The cases in which $z \in$ A(X) $\cap$ B(X).The proof is similar to the cases in which $z \in$ S(X) $\cap$ T(X). respectively. If Au = Bu = Tu = Su = u and Av = Bv = Sv = Tv = v then (4.1) gives

$$F\begin{pmatrix} M(Au,Bv,kt),M(Su,Tv,t),M(Au,Su,t) \\ M(Bv,Tv,t),M(Au,Tv,t),M(Bv,Su,t) \end{pmatrix} \geq 1$$

That is,

$$F\left(M(u,v,kt),M(u,v,t),1,1,M(u,v,t),M(v,u,t)\right) \geq 1$$

Thus, from (F-3), we have M (u, v, kt) ≥ M (u, v, t). By Lemma 2.7,

we have u = v. Therefore,u = v and the common fixed point is unique

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