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# A UNIQUE COMMON FIXED POINT THEOREM IN CONE METRIC SPACES

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**Abstract:** In this paper we prove a unique common fixed point theorem in cone metric spaces which generalize and extend metric space into cone metric spaces. Our result generalizes and extends some recent results.

**Keywords:** Cone metric space; fixed point; asymptotically regular.

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## 1. Introduction

In 2007 Huang and Zhang [5] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently , Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3,4] and the references mentioned therein). In this paper we extend the fixed point theorem of S.L.Singh et .al. [8] in metric space into cone metric spaces.

Throughout this paper, E is a real Banach space, N = {1, 2, 3,……} the set of all natural numbers. For the mappings f, g:X→X, let C(f ,g) denotes set of coincidence

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points of f, g , that is C(f, g):={z$\in$X : fz = gz }.

## 2. Preliminaries

We recall some definitions of cone metric spaces and some of their properties [5].

**Definition 1.1.** Let E be a real Banach Space and P a subset of E .The set P is Called a cone if and only if:

(a) P is closed, nonempty and P $\neq$ {0};

(b) a,b $\in R$ , a,b $\geq 0$ , x,y $\in P$ implies ax + by $\in P$ ;

(c) x $\in$ P and –x $\in$ P implies x = 0.

**Definition 1.2.** Let P be a cone in a Banach Space E , define partial ordering '$\leq$'on E with respect to P by x$\leq$ y if and only if y-x $\in P$ .We shall write x<y to indicate x $\leq y$ but x $\neq y$ while X<<y will stand for y-x$\in$Int P , where Int P denotes the interior of the set P. This Cone P is called an order cone.

**Definition 1.3.** Let E be a Banach Space and P$\subset$E be an order cone .The order cone P is called normal if there exists L>0 such that for all x, y$\in$E,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L \|y\| .$$

The least positive number L satisfying the above inequality is called the normal constant of P.

**Definition 1.4.** Let X be a nonempty set of E .Suppose that the map

d: X$\times$X$\rightarrow$E satisfies :

(d1) 0 $\leq$ d(x, y) for all x, y$\in X$ and

d(x, y) = 0 if and only if x = y ;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$ ;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ .

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

**Example 1.1.** ([5]). Let $E = R^2$ , $P = \{ (x , y) \in E$ such that : $x, y \geq 0\} \subset R^2$ ,

$X = R$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant .Then (X, d) is a cone metric space.

**Definition 1.5.** Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(a) a Cauchy sequence if for every c in E with $0 \ll c$ , there is N such that for all $n , m > N$, $d(x_n, x_m) \ll c$ ;

(b) a convergent sequence if for any $0 \ll c$ ,there is N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed $x \in X$.

A Cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

**Lemma 1.1.** ([5]) .Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L .Let $\{x_n\}$ be a sequence in X .Then

(i). $\{x_n\}$ converges to x if and only if $d(x_n ,x) \to 0$ $(n \to \infty)$.

(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n , x_m) \to 0$ $(n, m \to \infty)$.

**Definition 1.6. ([8]).** Let $f, g: X \to X$. Then the pair (f, g) is said to be (IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

## 3. Main results

In this section we obtain a unique common fixed point theorem in cone metric spaces, which extend a metric space into cone metric spaces.

The following theorem is extend and improves the theorem 2.3. [8]

**Theorem 3.1.** Let (X, d) be a cone metric space P be an order cone and f, g: X→X be self-maps. Let

(f, g) be asymptotically regular at $x_0 \in X$ and the following conditions are satisfied :

(C1): $f(X) \subseteq g(X)$;

(C2): $d(fx, gy) \leq \varphi(m(x, y))$ for all $x, y \in X$.

Where $m(x, y) = d(gx, gy) + \gamma[d(gx, fx) + d(gy, fy)]$, $o \leq \gamma \leq 1$ .

If f(X) or g(X) is a complete sub space of X. Then

(i). C (f, g) is non-empty. Further,

(ii). f and g have a unique common fixed point provided that f and g are (IT)-

Commuting at a point $u \in C(f, g)$.

**Proof.**

Let $x_0$ be an arbitrary point in X. Since if (f, g) is asymptotically regular at $x_0 \in X$,

Then there exists a sequence $\{x_n\}$ in X, such that

$$f x_n = gx_{n+1}, \quad n = 0, 1, 2, \ldots\ldots\text{and}$$

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.$$

First we shall show that $\{gx_n\}$ is a Cauchy sequence.

Suppose $\{gx_n\}$ is not a Cauchy sequence. Then there exists $\mu>0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k$ even and $n_k$ odd and for all k , $m_k < n_k$,

$$d(gx_{m_k}, gx_{n_k}) \geq \mu \text{ and } d(gx_{m_k}, gx_{n_k-1}) < \mu \tag{2.1.}$$

By the triangle inequality,

$d(gx_{m_k},\ gx_{n_k}) \le d(gx_{m_k},\ gx_{n_k-1}) + d(gx_{n_k-1},\ gx_{n_k})$. Letting k→∞, we get that

$\lim_{k\to\infty} d(gx_{m_k},\ gx_{n_k}) < \mu + 0.$

(Since, $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$, We get $\lim_{k\to\infty} d(gx_{n_k-1}, gx_{n_k}) = 0$ . )

Therefore there exists $k_0$ such that

$$d(gx_{m_k},\ gx_{n_k}) < \mu \ \forall\ k \ge k_0 \quad . \tag{2.2}$$

By (2.1) and (2.2), we get that

$\mu \le d(gx_{m_k},\ gx_{n_k}) < \mu \ \forall\ k \ge k_0$ .

Implies $\lim_{k\to\infty} d(fx_{m_k},\ fx_{n_k}) = \mu$ .

By (C2), we have

$$d(gx_{m_k+1},\ gx_{n_k+1}) = d(fx_{m_k},\ fx_{n_k}) \le \varphi(m(x_{m_k},\ x_{n_k}))$$

$$= \varphi(d(gx_{m_k}, gx_{n_k}) + \gamma\,[d(fx_{m_k}, gx_{m_k}) + d(gx_{n_k},\ fx_{n_k})]).$$

Letting k→∞, we get that

μ≤ φ(μ) and as per definition of φ-map, $\varphi(\mu) < \mu$.

Hence μ≤ φ (μ) < μ , a contradiction.

Thus $\{gx_n\}$ is Cauchy sequence. Suppose g(X) is a complete sub space of X. Then $\{gx_n\}$ being contained in g(X) has a limit in g(X) .Call it z. Let u = $g^{-1}z$.

Thus gu =z for some u ∈ X.

By using (C2), we have

$$d(fu,\ fx_{n_k}) \le \varphi(m(u,\ x_{n_k}))$$

$$\le \varphi(d(gu, gx_{n_k}) + \gamma\,[d(fu, gu) + d(fx_{n_k},\ gx_{n_k})])$$

Letting n→∞, we get that

$$d(fu, z) \le \varphi(\gamma[d(fu, z)]) < d(fu, z), \quad \text{a contradiction.}$$

Therefore, fu = z = gu. (2.3)

Thus C (f, g) is non-empty. This proves (i).

And the pair (f, g) is (IT) - Commuting at u , then

fgu = gfu and ffu = fgu = gfu = ggu. In view of (C2) it follows that

$$d(fu, ffu) \le \varphi(m(u, x_{n_k}))$$

$$\le \varphi(d(gu, gfu) + \gamma[d(fu, gu) + d(ffu, gfu)])$$

$$< d(fu, ffu), \quad \text{a contradiction .}$$

Therefore, ffu = fu and fgu = ffu ==fu = z.

Therefore, f and g have a common fixed point.

Uniqueness, let w be another fixed point of f and g.

Consider, $d(z, w) = d(fz, fw) \le \varphi(m(z, w))$

$$= \varphi(d(gz, gw) + \gamma[d(gz, fz) + d(gw, fw)])$$

$$\le \varphi(d(z, w) + \gamma[d(z, z) + d(w, w)])$$

$$\le \varphi(d(z, w) < d(z,w) \text{ (Since } \varphi\text{-map}, \varphi(\omega) < \omega),$$

a contradiction.

Therefore, f and g have a unique common fixed point.

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