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# SOLUTIONS OF SYSTEM OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS IN LOCALLY $FC$-UNIFORM SPACES

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**Abstract.** In this paper, we establish a collectively fixed point theorem and an equilibrium existence theorem for generalized games in product locally $FC$-uniform spaces. As applications, some new existence theorems of solutions for the system of generalized vector quasi-equilibrium problems are derived in product locally $FC$-uniform spaces. These theorems are new and generalize some known results in the literature.

**Keywords:** collectively fixed point; generalized game; system of generalized vector quasi-equilibrium problems; locally $FC$-uniform space.

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## 1. Introduction

Giannessi [1] first introduced the vector variational inequality problem in finite dimensional Eudidean spaces. Since then, such problem was extended and generalized by many authors in various different directions. Motivation for this comes from the fact that vector variational inequality and its various generalizations have extensive and important applications in vector optimation, optimal control, mathematical programming, operations research and equilibrium problems of economics. Inspired and motivated by the above applications, various generalized

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vector (quasi) equilibrium problems, system of generalized vector (quasi) variational inequality problems, and system of generalized vector (quasi) equilibrium problems have become important developed directions of vector variational inequality theory, for example, see [2-5].

Following the trend of the above research fields, we will introduce and study new classes of system of generalized vector quasi-equilibrium problems on a product space of $FC$-spaces. In this paper, By applying a Himmelbergerg type fixed point theorem in locally $FC$-uniform spaces due to Ding [6], we will establish a collectively fixed point theorem and an equilibrium existence theorem for generalized games in product locally $FC$-uniform spaces. As applications, some new existence theorems of solutions for several classes of systems of generalized vector quasi-equilibrium problems are obtained in locally $FC$-uniform spaces. These results are new and generalize some known results from the literature. Let us first recall the following preliminaries which will be needed in the sequel.

## 2. PRELIMINARIES

Let $X$ and $Y$ be two nonempty sets. We denote by $2^Y$ and $\langle X\rangle$ the family of all subsets of $Y$ and the family of all nonempty finite subsets of $X$, respectively. For each $A\in\langle X\rangle$, we denote by $|A|$ the cardinality of $A$. Let $\Delta_n$ denote the standard n-dimensional simplex with the vertices $\{e_0,\ldots,e_n\}$. If $J$ is a nonempty subset of $\{0,1,\ldots,n\}$, we shall denote by $\triangle_J$ the convex hull of the vertices $\{e_j : j\in J\}$.

The following notions were introduced by Ding [6].

**Definition 2.1.** An $FC$-space $(X,\{\varphi_N\})$ is said to be a finitely continuous topological space (for short, $FC$-space) if $X$ is a topological space such that for each $N=\{x_0,\ldots,x_n\}\in\langle X\rangle$ where some elements in $N$ may be the same, there exists a continuous mapping $\varphi_N:\Delta_n\to X$. A subset $D$ of $(X,\{\varphi_N\})$ is said to be an $FC$-subspace of $X$ if for each $N=\{x_0,\ldots,x_n\}\in\langle X\rangle$ and for any $\{x_{i_0},\cdots,x_{i_k}\}\subset D\cap N$, $\varphi_N(\Delta_k)\subset D$.

By the definition of $FC$-subspace of an $FC$-space,it is easy to see that each $FC$-subspace of $(X,\{\varphi_N\})$ is also an $FC$-space and if $\{B_i\}_{i\in I}$ is a family of $FC$-subspace of $FC$-space $(X,\{\varphi_N\})$ and $\cap_{i\in I}B_i\neq\emptyset$, then $\cap_{i\in I}B_i$ is also an $FC$-subspace of $(X,\{\varphi_N\})$ where $I$ is any index set.

A nonempty subset $M$ of a topological space $X$ is said to be compactly open (resp., compactly closed) in $X$ if for each compact subset $K$ of $X$, $M\bigcap K$ is open (resp., closed) in $K$. Clearly, each open (resp., closed) set in $X$ is compactly open (resp., compactly closed) in $X$.

**Definition 2.2.** A uniformity for a set $X$ is a nonempty family $\mathscr{U}$ of subsets of $X\times X$ satisfying the following conditions:
(i) each member of $\mathscr{U}$ contains the diagonal $\Delta$,
(ii) for each $U\in\mathscr{U}$, $U^{-1}\in\mathscr{U}$,
(iii) for each $U\in\mathscr{U}$, there exists $V\in\mathscr{U}$ such that $V\circ V\subset U$,
(iv) if $U\in\mathscr{U}$ and $U\subset V\subset X\times X$, then $V\in\mathscr{U}$.

Every member in $\mathscr{U}$ is called an entourage. An entourage $V$ is said to be symmetric if $(x,y)\in V$ whenever $(y,x)\in V$.

The $(X,\mathscr{U})$ is called a uniform space if $X$ has a topological $\tau$ derived from the uniformity $\mathscr{U}$ which takes the family $\{V[x]:V\in\mathscr{U},x\in X\}$ as a basis where $V[x]=\{y\in X:(x,y)\in V\}$.

The uniformity $\mathscr{U}$ is called separating if $\bigcap\{U\in X\times X:U\in\mathscr{U}\}=\Delta$. The uniform space $(X,\mathscr{U})$ is Hausdorff if and only if $\mathscr{U}$ is separating.

In the following, all uniform spaces $(X,\mathscr{U})$ are assumed to be Hausdorff.

**Definition 2.3.** $(X,\mathscr{U},\{\varphi_N\})$ is said to be a locally $FC$-uniform space if $(X,\mathscr{U})$ is a uniform space and $(X,\{\varphi_N\})$ is an $FC$-space such that $\mathscr{U}$ has a basis $\mathscr{B}$ consisting of entourages satisfying that for each $V\in\mathscr{B}$, the set $\{x\in X:M\cap V[x]\neq\emptyset\}$ is an $FC$-subspace of $X$ whenever $M\subset X$ is an $FC$-subspace of $X$.

We observe that the class of locally $FC$-uniform space in Definition 2.3 includes locally $H$-convex uniform space of Tarafdar [7] and locally $G$-convex space of Park [8] as true subclasses.

In order to obtain our main results, we need the following Lemmas. The following result is Theorem 2.1 of Ding [6].

**Lemma 2.1** Let $(X,\mathscr{U},\{\varphi_N\})$ be a locally $FC$-uniform space, and $F:X\to 2^X$ be a compact upper semicontinuous set-valued mapping with closed values such that for each $x\in X, F(x)$ is an $FC$-subspace of $X$. Then $F$ has a fixed point $x_0\in X$,i.e.,$x_0\in F(x_0)$.

The following result is Theorem 2.2 of Ding [6].

**Lemma 2.2** Let $I$ be any index set. For each $i\in I$, let $(X_i,\mathscr{U}_i,\{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i,\mathscr{U}_i)$ having a basis $\mathscr{B}_i$ consisting of symmetric entourages, and $Y=\prod_{i\in I}Y_i$, $\mathscr{U}=\prod_{i\in I}\mathscr{U}_i$ and $\varphi_N=\prod_{i\in I}\varphi_{Ni}$ for any $N\in\langle X\rangle$. Then $(X,\mathscr{U},\{\varphi_N\})$ is also a locally $FC$-uniform space.

The following result is Theorem 14.18 in [9].

**Lemma 2.3** Let $X$, $Y$ be topological spaces and $\varphi:X\to 2^Y$ be a set-valued mapping. Then the following statements are equivalent:
(i) $\varphi$ is lower semicontinuous at a point $x\in X$,
(ii) if $x_\alpha\to x$, then for each $y\in\varphi(x)$ there exists a subnet $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of the index set $\{\alpha\}$ and elements $y_\lambda\in\varphi(x_{\alpha_\lambda})$ for each $\lambda\in\Lambda$ such that $y_\lambda\to y$.

The following result is Lemma 4.7.3 in [10].

**Lemma 2.4** Let $X$, $Y$ be topological spaces and $A$ be a closed (resp., open) subset of $X$. Suppose that $F_1:X\to 2^Y$ and $F_2:A\to 2^Y$ are both lower semicontinuous (resp., upper semicontinuous) such that $F_2(x)\subseteq F_1(x)$ for each $x\in A$. Then the mapping $F:X\to 2^Y$ defined by

$$F(x)=\begin{cases} F_2(x),\ if\ x\in A,\\ \\ F_1(x),\ if\ x\in X\setminus A.\end{cases} \tag{2.1}$$

is also lower semicontinuous (resp., upper semicontinuous).

## 3. Collectively Fixed Point Theorem and Equilibrium Existence Theorem

**Theorem 3.1.** Let $(X_i,\mathscr{U}_i,\{\varphi_{N_i}\})_{i\in I}$ be a family of locally $FC$-uniform spaces with each $(X_i,\mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages. For each $i\in I$, let $G_i: X=\prod_{i\in I}X_i\to 2^{X_i}$ be an upper semicontinuous compact set-valued mapping with nonempty closed values and for each $x\in X, G_i(x)$ is an $FC$-subspace of $X_i$ . Then there exists a point $\hat{x}=(\hat{x}_i)_{i\in I}\in X$ such that $\hat{x}_i\in G_i(\hat{x})$ for each $i\in I$.

**Proof.** Let $X=\prod_{i\in I}X_i$, $\mathscr{U}=\prod_{i\in I}\mathscr{U}_i$ and $\varphi_N=\prod_{i\in I}\varphi_{Ni}$. By Lemma 2.2, $(X,\mathscr{U},\{\varphi_N\})$ is a locally $FC$-uniform space. Define a set-valued mapping $G:X\to 2^X$ by

$$G(x)=\prod_{i\in I}G_i(x),\ \forall x\in X.$$

Since for each $i\in I$, $G_i$ is an upper semicontinuous compact mapping with nonempty closed values and for each $x\in X, G_i(x)$ is an $FC$-subspace of $X_i$, it follows from Lemma 3 of Ky Fan [11] and Lemma 2.2 of Ding [6] that $G$ is also an upper semicontinuous compact mapping with nonempty closed values and for each $x\in X, G(x)$ is an $FC$-subspace of $X$. By Lemma 2.1, there exists a point $\hat{x}\in X$ such that $\hat{x}\in G(\hat{x})$, i.e. $\hat{x}_i\in G_i(\hat{x})$ for each $i\in I$. This completes the proof.

Now we describe a generalized game $\varepsilon=(X_i,A_i,P_i)_{i\in I}$ where $I$ is a finite or infinite set of agents; for each $i\in I$, $X_i$ is a strategy set (or commodity space) of $ith$ agent; $A_i: X=\prod_{i\in I}X_i\to 2^{X_i}$ is the constrained correspondence (set-valued mapping) and $P_i: X\to 2^{X_i}$ is the preference correspondence. A point $\hat{x}\in X$ is called an equilibrium point of the generalized game $\varepsilon$ if $\hat{x}_i\in A_i(\hat{x})$ and $A_i(\hat{x})\bigcap P_i(\hat{x})=\emptyset$ for each $i\in I$.

**Theorem 3.2.** Let $(X_i,A_i,P_i)_{i\in I}$ be a generalized game, $X=\prod_{i\in I}X_i$ such that for each $i\in I$,
(i)$(X_i,\mathscr{U}_i,\{\varphi_{N_i}\})_{i\in I}$ be a locally $FC$-uniform space with each $(X_i,\mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages,
(ii)for each $x\in X,A_i(x)$ is nonempty $FC$-subspace of $X_i$,
(iii) $A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iv) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$ is open in $X$,

(v) $P_i(x)$ is an upper semicontinuous closed mapping such that for each $x \in X, P_i(x)$ is an $FC$-subspace of $X_i$,

(vi) for each $x \in X, x_i \notin A_i(x) \bigcap P_i(x)$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \ and \ A_i(\hat{x}) \bigcap P_i(\hat{x}) = \emptyset.$$

**Proof.** Let $\mathscr{U} = \prod_{i \in I} \mathscr{U}_i$ and $\varphi_N = \prod_{i \in I} \varphi_{Ni}$. By Lemma 2.2 and condition (i), $(X, \mathscr{U}, \{\varphi_N\})$ is a locally $FC$-uniform space. For each $i \in I$, define a set-valued mapping $T_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} A_i(x) \bigcap P_i(x), \ if \ x \in E_i, \\ A_i(x), \ if \ x \notin E_i. \end{cases} \tag{3.1}$$

By condition (ii) and (v), for each $x \in X, T_i(x)$ is a nonempty $FC$-subspace of $X_i$. By condition (v) and Theorem 3.18 in [12], $A_i(x) \bigcap P_i(x)$ is also an upper semicontinuous compact mapping with nonempty closed values. By condition (iii) and (iv), Lemma 3 of Ky Fan [11] and Lemma 2.4, $T_i$ is also an upper semicontinuous compact mapping with nonempty closed values. So all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a point $\hat{x} \in X$ such that for each $i \in I, \hat{x}_i \in T_i(\hat{x})$. If for some $i \in I, \hat{x}_i \in E_i$, then we have $\hat{x}_i \in A_i(\hat{x}) \bigcap P_i(\hat{x})$ which contradicts the condition (vi). Hence we conclude that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \ and \ A_i(\hat{x}) \bigcap P_i(\hat{x}) = \emptyset,$$

i.e., $\hat{x}$ is an equilibrium point of the generalized game $\varepsilon$.

**Remark 3.1.** Theorem 3.1-3.2 are new results which are different from the corresponding results in [11] and [13].

## 4. Existence of Solutions for $SGVQEP$

**Definition 4.1.** Let $(X,\{\varphi_N\})$ be $FC$-space and $Z$ be a nonempty set. Let $F: X \to 2^Z$ and $C: X \to 2^Z$ be set-valued mappings. $F$ is said to be $FC$-quasiconvex (resp., $FC$-quasiconcave) with respect to $C$ if the set $\{x \in X : F(x) \subseteq C(x)\}$ (resp., $\{x \in X : F(x) \nsubseteq C(x)\}$) is an $FC$-subspace of $X$.

Let $I$ be a finite or infinite index set, $\{X_i\}_{i\in I}$ be a family of topological space, and $\{Z_i\}_{i\in I}$ be a family of nonempty sets. Let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $A_i : X \to 2^{X_i}, F_i : X \times X_i \to 2^{Z_i}, C_i : X \to 2^{Z_i}$ and $\varphi_i : X \times X_i \to 2^{X_i}$ be set-valued mappings. In this section, we shall consider the following systems of generalized vector quasi-equilibrium problems:

(I) Find $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \; and \; F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}). \; (SGVQEP(I))$$

(II)Find $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \; and \; F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}). \; (SGVQEP(II))$$

In this section, we shall derive some new existence theorems of solutions for SGVQEP(I) and SGVQEP(II) in product locally $FC$-uniform spaces by using Theorem 3.2.

**Theorem 4.1.** For each $i \in I$, let $(X_i, \mathscr{U}_i, \{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i, \mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages and $\{Z_i\}$ be a nonempty set. For each $i \in I$, let $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}, F_i : X \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be set-valued mappings such that for each $i \in I$,

(i)for each $x \in X, A_i(x)$ is nonempty $FC$-subspace of $X_i$,

(ii)$A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iii) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \phi\}$ is open in $X$ where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i : F_i(x, z_i) \subseteq C_i(x)\}$,

(iv) $P_i(x)$ is an upper semicontinuous closed mapping such that for each $x \in X, P_i(x)$ is an $FC$-subspace of $X_i$,

(v) for each $x \in X, x_i \notin A_i(x) \bigcap P_i(x)$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}).$$

**Proof.** It is easy to check that all conditions of Theorem 3.2 are satisfied. By Theorem 3.2, there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } A_i(\hat{x}) \bigcap P_i(\hat{x}) = \emptyset.$$

It follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}),$$

i.e., $\hat{x}$ is a solution of the SGVQEP(I).

**Theorem 4.2.** For each $i \in I$, let $(X_i, \mathscr{U}_i, \{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i, \mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages and $\{Z_i\}$ be a nonempty set. For each $i \in I$, let $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}, F_i : X \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be set-valued mappings such that for each $i \in I$,

(i)for each $x \in X, A_i(x)$ is nonempty $FC$-subspace of $X_i$,

(ii)$A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iii) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$ is open in $X$ where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i : F_i(x, z_i) \subseteq C_i(x)\}$,

(iv) $F_i(x, z_i)$ is lower semicontinuous on $X \times X_i$,

(v) the mapping $C_i$ has closed graph,

(vi) for each $x \in X, z_i \mapsto F_i(x, z_i)$ is $FC$-quasiconvex with respect to $C_i$,

(vii) for each $x \in X, F_i(x, x_i) \nsubseteq C_i(x)$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}).$$

**Proof.** It is easy to follow that $P_i$ has closed graph. Indeed, let $\{(x_\alpha, z_{i,\alpha})\}$ be a net in $Gr(P_i)$ and $(x_\alpha, z_{i,\alpha}) \to (x_0, z_{i,0})$. Then we have that $F_i(x_\alpha, z_{i,\alpha}) \subseteq C_i(x_\alpha)$ for each $\alpha$. If $F_i(x_0, z_{i,0}) \not\subseteq C_i(x_0)$, then there exists a point $u_{i,0} \in F_i(x_0, z_{i,0})$ such that $u_{i,0} \notin C_i(x_0)$. By condition (iv) and Lemma 2.3, there exists a subnet $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of $\{\alpha\}$ and $u_{i,\alpha_\lambda} \in F_i(x_{\alpha_\lambda}, z_{i,\alpha_\lambda})$ such that $u_{i,\alpha_\lambda} \to u_{i,0}$. Since $u_{i,\alpha_\lambda} \in F_i(x_{\alpha_\lambda}, z_{i,\alpha_\lambda}) \subseteq C_i(x_{\alpha_\lambda})$ for each $\lambda \in \Lambda$ and $C_i$ has closed graph, we must have $u_{i,0} \in C_i(x_0)$ which is a contradiction. Hence we have $F_i(x_0, z_{i,0}) \subseteq C_i(x_0)$. So the mapping $P_i$ has closed graph. By (iv) and Theorem 3.18.in [12], $P_i(x)$ is also an upper semicontinuous compact mapping with nonempty closed values. By (vi) for each $x \in X$, $P_i(x) = \{z_i \in X_i : F_i(x, z_i) \subseteq C_i(x)\}$ is an $FC$-subspace of $X_i$ and the condition (iv) of Theorem 4.1 is also satisfied. By (vii) for each $x \in X, x_i \notin P_i(x)$. Hence $x_i \notin A_i(x) \bigcap P_i(x)$ and the condition (v) of Theorem 4.1 is also satisfied. So all conditions of Theorem 4.1 are satisfied. By Theorem 4.1, there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \text{ and } F_i(\hat{x}, z_i) \not\subseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}).$$

**Corollary 4.1.** For each $i \in I$, let $(X_i, \mathscr{U}_i, \{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i, \mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages. Let $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ be a set-valued mapping and $\varphi_i : X \times X_i \to [-\infty, +\infty]$ be a single-valued continuous function such that for each $i \in I$,

(i)for each $x \in X, A_i(x)$ is nonempty $FC$-subspace of $X_i$,

(ii)$A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iii) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$ is open in $X$ where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i : \varphi_i(x, z_i) \leq 0\}$,

(iv) for each $x \in X$, the set $\{z_i \in X_i : \varphi_i(x, z_i) \leq 0\}$ is an $FC$-subspace of $X_i$,

(v) for each $x \in X, \varphi_i(x, x_i) > 0$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \text{ and } \varphi_i(\hat{x}, z_i) > 0, \forall z_i \in A_i(\hat{x}).$$

**Proof.** Let $Z_i = [-\infty, +\infty], C_i(x) = [-\infty, 0]$ for each $x \in X$ and $F_i(x, z_i) = \{\varphi_i(x, z_i)\}$ for all $(x, z_i) \in X \times X_i$. Noting that $\varphi_i$ is continuous, we have $F_i(x, z_i)$ is lower semicontinuous on $X \times X_i$. It is easy to check that the mapping $C_i$ has closed graph and for each $x \in X, F_i(x, x_i) \nsubseteq C_i(x)$. From (iv), we follow that for each $x \in X, z_i \mapsto F_i(x, z_i)$ is $FC$-quasiconvex with respect to $C_i$. So all conditions of Theorem 4.2 are satisfied. The conclusion of Corollary 4.1 follows from Theorem 4.2.

**Theorem 4.3.** For each $i \in I$, let $(X_i, \mathscr{U}_i, \{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i, \mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages and $\{Z_i\}$ be a nonempty set. For each $i \in I$, let $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}, F_i : X \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be set-valued mappings such that for each $i \in I$,

(i)for each $x \in X, A_i(x)$ is nonempty $FC$-subspace of $X_i$,

(ii)$A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iii) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$ is open in $X$ where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i : F_i(x, z_i) \nsubseteq C_i(x)\}$,

(iv) $F_i(x, z_i)$ is an upper semicontinuous compact mapping with closed values,

(v) the mapping $C_i$ has open graph,

(vi) for each $x \in X, z_i \mapsto F_i(x, z_i)$ is $FC$-quasiconcave with respect to $C_i$,

(vii) for each $x \in X, F_i(x, x_i) \subseteq C_i(x)$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \text{ and } F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}).$$

**Proof.** For each $i \in I$, define a set-valued mapping $T_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} A_i(x) \bigcap P_i(x), \ if \ x \in E_i, \\ \\ A_i(x), \ if \ x \notin E_i. \end{cases} \tag{4.1}$$

For each $x \in E_i$, let $H_i(x) = A_i(x) \bigcap P_i(x) = \{z_i \in A_i(x) : F_i(x,z_i) \bigcap (Z_i \setminus C_i(x)) \neq \emptyset\}$. Since $A_i$ is a compact mapping, $H_i$ is also a compact mapping. We claim that $H_i$ has closed graph. Indeed, let $\{(x_\alpha, z_{i,\alpha})\}_{\alpha \in I}$ be a net in $Gr(H_i)$ and $(x_\alpha, z_{i,\alpha}) \to (x_0, z_{i,0})$. Then we have $z_{i,\alpha} \in A_i(x_\alpha)$ and $F_i(x_\alpha, z_{i,\alpha}) \bigcap (Z_i \setminus C_i(x_\alpha)) \neq \emptyset$ for each $\alpha \in I$. Hence there exists $u_{i,\alpha} \in F_i(x_\alpha, z_{i,\alpha})$ such that $u_{i,\alpha} \in Z_i \setminus C_i(x_\alpha)$ for each $\alpha \in I$. Without loss of generality, by (iv) we can assume that $u_{i,\alpha} \to u_{i,0}$ and so $u_{i,0} \in F_i(x_0, z_{i,0})$. By (v) the mapping $W_i : X \to 2^{X_i}$ defined by $W_i(x) = Z_i \setminus C_i(x)$ has closed graph. It follows that $u_{i,0} \in W_i(x_0) = Z_i \setminus C_i(x_0)$ and $F_i(x_0, z_{i,0}) \bigcap (Z_i \setminus C_i(x_0)) \neq \emptyset$. By (i) we have $z_{i,0} \in A_i(x_0)$. Therefore $(x_0, z_{i,0}) \in Gr(H_i)$ and the graph $Gr(H_i)$ of $H_i$ is closed. Hence $H_i = A_i \bigcap P_i$ is an upper semicontinuous compact mapping with nonempty closed values. By conditions (i) and (vi), for each $x \in X, T_i(x)$ is a nonempty $FC$-subspace of $X_i$. By the condition (v), Lemma 3 of Ky Fan [11] and Lemma 2.4, $T_i$ is also an upper semicontinuous compact mapping with nonempty closed values. By (vii), for each $x \in X, x_i \notin A_i(x) \bigcap P_i(x)$. It is easy to check that all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a point $\hat{x} \in X$ such that for each $i \in I, \hat{x}_i \in T_i(\hat{x})$. If for some $i \in I, \hat{x}_i \in E_i$, then we have $\hat{x}_i \in A_i(\hat{x}) \bigcap P_i(\hat{x})$ which contradicts the condition (vii). Hence we conclude that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } A_i(\hat{x}) \bigcap P_i(\hat{x}) = \emptyset.$$

It follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}), \forall z_i \in A_i(\hat{x}).$$

**Corollary 4.2.** For each $i \in I$, let $(X_i, \mathscr{U}_i, \{\varphi_{N_i}\})$ be a locally $FC$-uniform space with each $(X_i, \mathscr{U}_i)$ having the basis $\mathscr{B}_i$ consisting of symmetric entourages. Let $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ be a set-valued mapping and $\varphi_i : X \times X_i \to [-\infty, +\infty]$ be a single-valued continuous function such that for each $i \in I$,

(i)for each $x \in X, A_i(x)$ is nonempty $FC$-subspace of $X_i$,

(ii)$A_i$ is an upper semicontinuous compact closed set-valued mapping,

(iii) the set $E_i = \{x \in X : A_i(x) \bigcap P_i(x) \neq \emptyset\}$ is open in $X$ where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i : \varphi_i(x, z_i) \geq 0\}$,

(iv) $\varphi_i(x, y)$ is a continuous bounded function,

(v) for each $x \in X$, the set $\{z_i \in X_i : \varphi_i(x, z_i) \geq 0\}$ is an $FC$-subspace of $X_i$,

(vi) for each $x \in X, \varphi_i(x, x_i) < 0$.

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \textit{ and } \varphi_i(\hat{x}, z_i) < 0, \forall z_i \in A_i(\hat{x}).$$

**Proof.** Let $Z_i = [-\infty, +\infty], C_i(x) = [-\infty, 0]$ for each $x \in X$ and $F_i(x, z_i) = \{\varphi_i(x, z_i)\}$ for all $(x, z_i) \in X \times X_i$. It is easy to check that all conditions of Theorem 4.3 are satisfied. The conclusion of Corollary 4.2 follows from Theorem 4.3.

**Remark 4.1.** Theorem 4.1-Theorem 4.3 and Corollary 4.1-Corollary 4.2 are new results which are different from the corresponding results in [[1]-[5], [14]-[21]].

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## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCES

[1] F. Giannessi, Theorems of alternative, quadratic programs and complementarity problems, in: R.W.Cottle, F. Giannessi, J.L.Lions(Eds.), Variational Inequalities and Complementarity Problems, Wiley, New York, 1980, pp.151-186.

[2] X.P. Ding, J.Y. Park, Fixed points and generalized vector equilibrium problems in $G$-convex spaces, Indian J. Pure. Appl. Math. 34(6) (2003) 973-990.

[3] X.P. Ding, J.Y. Park, Generalized vector equilibrium problems in generalized convex spaces, J. Optim. Theory Appl. 120(2) (2004) 937-990.

[4] F. Giannessi, Vector Variational Inequalities and Vector Equilibria, Kluwer Academic, London 2000.

[5] L.J. Lin, Z.T. Yu, G. Kassay, Existence of equilibria for multivalued mappings and its applications to vector equilibria, J. Optim. Theory Appl. 114 (2002) 189-208.

[6] X.P. Ding, The generalized games and the system of generalized vector quasi-equilibrium problems in locally $FC$-uniform spaces, Nonlinear Anal. 68 (2008) 1028-1036.

[7] C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis, Springer-Verlag, New York, 1994.

[8] G.X.-Z. Yuan, *KKM* Theory and Application in Nonlinear Analysis, Marcel Dekker, Inc., New York, 1999.

[9] K. Fan, Fixed-points and minimax theorems in locally convex topological linear spaces, Proc. Nat. Acad. Sci. USA 38 (1952) 131-136.

[10] J.P. Aubin, I. Ekland, Applied Nonlinear Analysis, Wiley, New York, 1984.

[11] X.P. Ding, J.C. Yao, L.J. Lin, Solutions of system of generalized vector quasi-equilibrium problems in locally $G$-convex uniform spaces, J. Math. Anal. Appl. 292 (2004) 398-410.

[12] X.P. Ding, Collectively fixed points and equilibria of generalized games with $\mathscr{U}$-majorized correspondences in locally $G$-convex uniform spaces, J. Sichuan Normal Univ. (N.S.) 25 (2001) 551-556.

[13] Q.H. Ansari, J.C. Yao, System of generalized variational inequalities and their applications, Appl. Anal. 76 (2000) 203-217.

[14] Q.H. Ansari, S. Schaible, J.C. Yao, System of vector equilibrium problems and their applications, J. Optim. Theory Appl. 107 (2000) 547-557.

[15] M.P. Chen, L.J. Lin, S. Park, Remarks on generalized quasi-equilibrium problems, Nonlinear Anal. 52 (2003) 433-444.

[16] X.P. Ding, Quasi-equilibrium problems with applications to infinite optimization and constrained games in general topological spaces, Appl. Math. Lett. 13 (2000) 21-26.

[17] X.P. Ding, Quasi-equilibrium problems and constrained mult-objective games in general convex spaces, Appl. Math. Mech. 22 (2001) 160-172.

[18] X.P. Ding, Maximal element principles on generalized convex spaces and their application, in: R.P. Argawal(Ed.), Set Valued Mappings with Applications in Nonlinear Analysis(SIMMA). 4 (2002) 149-174.

[19] X.P. Ding, J.Y. Park, Existence theorems of solutions for generalized quasi-variational inequalities in non-compact $G$-convex spaces, in: Y.J. Yao, J.K. Kim, S.M. Kang(Eds.), Fixed Point Theory and Application, Nova Science, New York, 2002, pp. 53-62.

[20] L.J. Lin, Z.T. Yu, On some equilibrium problems for multmaps, J. Comput. Appl. Math. 129 (2001) 171-183.

[21] S. Park, Fixed-points and quasi-equilibrium problems, Math. Comput. Model. 32 (2000) 1297-1340.