Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:2
https://doi.org/10.28919/afpt/6778
ISSN: 1927-6303

AN ITERATIVE METHOD FOR SOLUTIONS OF A GENERALIZED VARIATIONAL INEQUALITY IN REAL HILBERT SPACES

J. ZHANG[1,*], Q. YUAN[2]

[1]School of Civil Engineering and Architecture, Linyi University, Shandong, China

[2]School of Mathematics and Statistics, Linyi University, Shandong, China

Abstract. In this article, we investigate an iterative algorithm for solutions of generalized variational inequalities. A strong convergence theorem is established in the framework of Hilbert spaces.

Keywords: iterative algorithm; generalized variational inequality; monotone operator.

2020 AMS Subject Classification: 47H05, 47H09, 47H10

1. INTRODUCTION

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed and convex subset of H and $A : C \to H$ a nonlinear mapping. Recall the following definitions:

(1) A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) A is said to be ρ-strongly monotone if there exists a positive real number $\rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho\|x - y\|^2, \quad \forall x, y \in C.$$

*Corresponding author

E-mail address: 1513309941@qq.com

Received November 25, 2022

(3) A is said to be η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(4) A is said to be relaxed η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(5) A is said to be relaxed (η, ρ)-cocoercive if there exist positive real numbers $\eta, \rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)\|Ax - Ay\|^2 + \rho\|x - y\|^2, \quad \forall x, y \in C.$$

Given nonlinear mappings $A : C \to H$ and $B : C \to H$, find an $u \in C$ such that

$$\langle u - \tau Bu + \lambda Au, v - u\rangle \geq 0, \quad \forall v \in C,$$

where λ and τ are two positive constants. In this paper, we use $GVI(C,B,A)$ to denote the set of solutions of the generalized variational inequality.

It is easy to see that an element $u \in C$ is a solution to the variational inequality if and only if $u \in C$ is a fixed point of the mapping $P_C(\tau B - \lambda A)$, where P_C denotes the metric projection from H onto C. Indeed, we have the following relations:

$$u = P_C(\tau B - \lambda A)u \Longleftrightarrow \langle u - \tau Bu + \lambda Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

Next, we consider a special case of the inequality. If $B = I$, the identity mapping and $\tau = 1$, then the generalized variational inequality is reduced to the following. Find $u \in C$ such that

$$\langle \lambda Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

The variational inequality emerging as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences was introduced by Stampacchia [1] in 1964. In this paper, we use $VI(C,A)$ to denote the set of solutions of the variational inequality.

Let $S : C \to C$ be a mapping. We use $F(S)$ to denote the set of fixed points of the mapping S. Recall that S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recall that S is said to be demi-closed at the origin if for each sequence $\{x_n\}$ in C, $x_n \rightharpoonup x_0$ and $Sx_n \to 0$ imply $Sx_0 = 0$, where $\rightharpoonup$ and $\to$ stand for weak convergence and strong convergence.

Iterative methods recently have been investigated for treating fixed point problems; which include variational inequalities, saddle problems and optimization problems as special case; see [2-5] and the references therein. In this article, we investigate an viscosity iteration for solutions of generalized variational inequalities. Strong convergence theorems are established in the framework of Hilbert spaces.

Lemma 1.1 *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and $\{\beta_n\}$ a sequence in $(0,1)$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2 *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1 : C \to C$ and $S_2 : C \to C$ be nonexpansive mappings on C. Suppose that $F(S_1) \cap F(S_2)$ is nonempty. Define a mapping $S : C \to C$ by*

$$Sx = aS_1x + (1-a)S_2x, \quad \forall x \in C.$$

Then S is nonexpansive with $F(S) = F(S_1) \cap F(S_2)$.

Lemma 1.3 *Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 1.4 *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main Results

Theorem 2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and μ_m-Lipschitz continuous mapping and $B_m : C \to H$ a relaxed $(\widehat{\eta}_m, \widehat{\rho}_m)$-cocoercive and $\widehat{\mu}_m$-Lipschitz continuous mapping for each $m \ge 1$. Assume that $\cap_{m=1}^{\infty} GVI(C, B_m, A_m) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{m=1}^{\infty} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n), \quad n \ge 1,$$

where $f : C \to C$ is a fixed point, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{(1,n)}\}, \ldots,$ and $\{\delta_{(r,n)}\}$ are sequences in $(0,1)$ satisfying the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{m=1}^{r} \delta_{(m,n)} = 1, \forall n \ge 1$;

(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha = \infty$;

(d) $\lim_{n\to\infty} \delta_{(m,n)} = \delta_m \in (0,1)$,

and $\{\tau_m\}_{m=1}^{\infty}$, $\{\lambda_m\}_{m=1}^{\infty}$ are two positive sequences such that

(e) $\sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\mu_m^2 + 2\lambda_m\eta_m\mu_m^2} + \sqrt{1 - 2\widehat{\lambda}_m\widehat{\rho}_m + \widehat{\lambda}_m^2\widehat{\mu}_m^2 + 2\widehat{\lambda}_m\widehat{\eta}_m\widehat{\mu}_m^2} \le 1$.

Then the sequence $\{x_n\}$ converges strongly to a common element $\bar{x} \in \cap_{m=1}^{\infty} GVI(C, B_m, A_m)$.

Proof. First, we prove that the mapping $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \le m \le r$. For each $x, y \in C$, we have

$$\begin{aligned} &\|P_C(\tau_m B_m - \lambda_m A_m)x - P_C(\tau_m B_m - \lambda_m A_m)y\| \\ &\le \|(\tau_m B_m - \lambda_m A_m)x - (\tau_m B_m - \lambda_m A_m)y\| \\ &\le \|(x-y) - \lambda_m(A_m x - A_m y)\| + \|(x-y) - \tau_m(B_m x - B_m y)\|. \end{aligned} \tag{2.1}$$

It follows from the assumption that each A_m is relaxed (η_m, ρ_m)-cocoercive and μ_m-Lipschitz continuous that

$$\begin{aligned}
&\|x-y-\lambda_m(A_m x - A_m y)\|^2 \\
&= \|x-y\|^2 - 2\lambda_m\langle A_m x - A_m y, x-y\rangle + \lambda_m^2\|A_m x - A_m y\|^2 \\
&\leq \|x-y\|^2 - 2\lambda_m[(-\eta_m)\|A_m x - A_m y\|^2 + \rho_m\|x-y\|^2] + \lambda_m^2\mu_m^2\|x-y\|^2 \\
&= (1-2\lambda_m\rho_m + \lambda_m^2\mu_m^2)\|x-y\|^2 + 2\lambda_m\eta_m\|A_m x - A_m y\|^2 \\
&\leq \xi_m^2\|x-y\|^2,
\end{aligned}$$

where $\xi_m = \sqrt{1-2\lambda_m\rho_m + \lambda_m^2\mu_m^2 + 2\lambda_m\eta_m\mu_m^2}$. This shows that

$$\|x-y-\lambda_m(A_m x - A_m y)\| \leq \xi_m\|x-y\|. \tag{2.2}$$

In a similar way, we can obtain that

$$\|x-y-\tau_m(B_m x - B_m y)\| \leq \zeta_m\|x-y\|, \tag{2.3}$$

where $\zeta_m = \sqrt{1-2\widehat{\lambda}_m\widehat{\rho}_m + \widehat{\lambda}_m^2\widehat{\mu}_m^2 + 2\widehat{\lambda}_m\widehat{\eta}_m\widehat{\mu}_m^2}$. Substituting (2.2) and (2.3) into (2.1), we from the condition (e) see that $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \leq m \leq r$. Put $y_n = \sum_{m=1}^r \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n), \quad \forall n \geq 1$. Fixing $p \in \cap_{m=1}^r GVI(C, B_m, A_m)$, we see that $\|y_n - p\| \leq \|x_n - p\|$. It follows that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\
&\leq (1-\alpha_n(1-\alpha))\|f(p)-p\| + (1-\alpha_n)\|x_n - p\|.
\end{aligned}$$

By mathematical inductions, we find that $\{x_n\}$ is bounded. Since the mapping $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \leq m \leq r$, we see that

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&= \|\sum_{m=1}^r \delta_{(m,(n+1))} P_C(\tau_m B_m x_{n+1} - \lambda_m A_m x_{n+1}) - \sum_{m=1}^r \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n)\| \\
&\leq \|x_{n+1}-x_n\| + M\sum_{m=1}^r |\delta_{(m,(n+1))} - \delta_{(m,n)}|,
\end{aligned} \tag{2.4}$$

where M is an appropriate constant such that

$$M = \max\{\sup_{n\geq 1}\|P_C(\tau_m B_m x_n - \lambda_m A_m x_n)\|, \forall 1 \leq m \leq r\}.$$

Put $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, for all $n \geq 1$. That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 1$. Now, we estimate $\|l_{n+1} - l_n\|$. Note that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - y_{n+1}) + \frac{\alpha_n}{1-\beta_n}(y_n - f(x_n)) + y_{n+1} - y_n, \end{aligned}$$

which yields that

$$\begin{aligned} &\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - Sy_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Sy_n - f(x_n)\| + M\sum_{m=1}^{r}|\delta_{(m,(n+1))} - \delta_{(m,n)}|. \end{aligned}$$

It follows from the conditions (b), (c) and (d) that

$$\limsup_{n\to\infty}\left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_{n+1}\|\right) \leq 0.$$

It follows from Lemma 1.1 that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. We see that $x_{n+1} - x_n = (1-\beta_n)(l_n - x_n)$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.5}$$

On the other hand, from the iterative algorithm (Υ), we see that $x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(y_n - x_n)$. It follows from (2.5) and the conditions (b), (c) that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.6}$$

Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle u - \bar{x}, x_{n_i} - \bar{x}\rangle. \tag{2.7}$$

Since $\{x_{n_i}\}$ is bounded, we obtain that there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to q. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$. Next, we show that $q \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$. Define a mapping $R : C \to C$ by

$$Rx = \sum_{m=1}^{\infty}\delta_m P_C(\tau_m B_m - \lambda_m A_m)x, \quad \forall x \in C,$$

where $\delta_m = \lim_{n\to\infty} \delta_{(m,n)}$. From Lemma 1.2, we see that R is nonexpansive with

$$F(R) = \bigcap_{m=1}^{\infty} F(P_C(\tau_m B_m - \lambda_m A_m)) = \bigcap_{m=1}^{\infty} GVI(C, B_m, A_m).$$

Now, we show that $Rx_n - x_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} &\|Rx_n - x_n\| \\ &= \|\sum_{m=1}^{\infty} \delta_m P_C(\tau_m B_m - \lambda_m A_m)x_n - \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n)\| + \|y_n - x_n\| \\ &\le M \sum_{m=1}^{\infty} |\delta_{(m,n)} - \delta_m| + \|y_n - x_n\|. \end{aligned}$$

From the condition (d) and (2.7), we obtain that $\lim_{n\to\infty} \|Rx_n - x_n\| = 0$. From Lemma 1.3, we see that

$$q \in F(R) = \bigcap_{m=1}^{\infty} F(P_C(\tau_m B_m - \lambda_m A_m)) = \bigcap_{m=1}^{\infty} GVI(C, B_m, A_m).$$

It follows that

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x} \rangle = \langle u - \bar{x}, q - \bar{x} \rangle \le 0.$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned} &\|x_{n+1} - \bar{x}\|^2 \\ &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle, \end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \le (1-\alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle.$$

It follows that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

This completes the proof.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C.R. Acad. Sci. Paris. 258 (1964), 4413-4416.

[2] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967), 197-228. https://core.ac.uk/display/81135918.

[3] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability, Springer-Verlag, New York, 1989.

[4] S.P. Han, G. Lou, A parallel algorithm for a class of convex programs, SIAM J. Control Optim. 26 (1988), 345-355. https://doi.org/10.1137/0326019.

[5] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, SIAM J. Control Optim. 29 (1991), 119-138. https://doi.org/10.1137/0329006.