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THE PHASE-ISOMETRIES BETWEEN THE UNIT SPHERE OF $\ell_p(\Gamma, H)$-TYPE SPACES

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Abstract. In this paper, Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry between the unit spheres of two real $\ell_p(\Gamma, H)$-type spaces X and Y. We prove that the mapping f is phase equivalent to an isometry. Otherwise, this isometry is the restriction of a linear isometry between the whole spaces, i.e., this isometry on the unit sphere can be linearly extended into isometry in the whole space.

Keywords: phase-isometry; Winger's theorem; Tingley's problem; $\ell_p(\Gamma, H)$-type spaces.

2010 AMS Subject Classification: 46B04, 46B20.

1. INTRODUCTION

Let X and Y be real normed spaces. A mapping $f : X \to Y$ is called a phase-isometry if it satisfies the functional equation

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in X).$$

We say that the mapping f is a plus-minus linear isometry if and only if there exists a phase function $\varepsilon : X \to \{1, -1\}$ such that $\varepsilon f(\cdot)$ is a linear isometry. Then we called the mapping f is phase equivalent to a linear isometry. We can say that linear isometry is g, $g = \varepsilon f$.

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The famous Wigner's theorem plays a very important role in quantum mechanics and in representation theory in physics. We refer the reader to the papers [1, 2, 3, 4, 5, 6] for more information and background on Wigner's theorem. Rätz[5, Corollary 8(a)] presented the real version of Wigner's theorem, which implies that any phase-isometry between two real inner product spaces is a plus-minus linear isometry. Recently, Zeng and Huang[7] showed that every surjective phase-isometry between real $\ell_p(\Gamma, H)$-type spaces for $p \geqslant 1$ is equivalent to a linear isometry, which generalizes Wigner's theorem to real $\ell_p(\Gamma, H)$-type spaces for $p \leqslant 1$.

The relationship between the metric structure and linear structure of normed space had been a problem that many scholars in the space theory field pay attention to. In 1987, Tingley proposed the following question in [8]: Let X and Y be normed spaces, whose unit spheres are denoted by S_X and S_Y, respectively. Suppose $f : S_X \to S_Y$ is a surjective isometry. Whether or not exist F, the extend of f, is a real linear (bijective) isometry from X onto Y? This problem is known as the Tingly's problem or isometric extension problem. We refer the reader to the introduction of [9, 11] for more information and recent development on this problem. The survey of Ding[10] is one of the good reference for understanding the history of the problem. We could consider the natural positive homogeneous extension F of f from X to Y defined by

$$F(x) = \begin{cases} \|x\| f(\frac{x}{\|x\|}) & , \quad x \neq 0, \\ 0 & , \quad x = 0. \end{cases}$$

is the desired extension of f on the whole space X. For this we need to present a property of F. This property that holds for general normed spaces may be of independent interest.

Problem 1.1 *Let f be a surjective phase-isometry between the unit spheres S_X and S_Y of real normed spaces X and Y respectively. Is it true that the natural positive homogeneous extension F is a phase-isometry?*

In this paper, we answer Problem 1.1 in positive for real $\ell^p(\Gamma, H)$-type spaces for $p \geqslant 1$. That is for every phase-isometry from the unit sphere $S_{\ell^p\Gamma,H}$ onto $S_{\ell^p\Delta,K}$ of real $\ell^p(\Gamma, H)$-type spaces for $p \geqslant 1$, the natural positive homogeneous extension is phase equivalent to a linear isometry, and therefore actually a phase-isometry. We also show that the Problem 1.1 is solved in positive for real inner product spaces.

2. Main Results

Throughout this section, we consider the spaces all over the real field and denote by $\mathbb{R}$ the set of reals. The spaces X and Y are used to denote real normed spaces. We use S_X and S_Y to denote the unit spheres of X and Y, respectively. This paper mainly discusses the $\ell_p(\Gamma,H)$-type spaces with $1 \leq p < \infty, p \neq 2$, where Γ is a nonempty index set and H is a real inner product space. Let's describe the $\ell_p(\Gamma,H)$ space.

$$\ell_p(\Gamma,H) = \{x = \sum_{\gamma\in\Gamma} x_\gamma \otimes e_\gamma | \quad x_\gamma \in H, \|x\|^p = \sum_{\gamma\in\Gamma} \|x_\gamma\|^p < +\infty\}$$

For the elements on the unit sphere $S_{\ell^p\Gamma,H}$, a restriction $\|x\| = 1$ is added. For each $x = \sum_{\gamma\in\Gamma} x_\gamma \otimes e_\gamma \in S_{\ell_p(\Gamma,H)}$, we denote the support of x by Γ_x, i.e.,

$$\text{supp}(x) = \Gamma_x = \{\gamma \in \Gamma : x_\gamma \neq 0\}.$$

So we can write $x = \sum_{\gamma\in\Gamma} x_\gamma \otimes e_\gamma = \sum_{\gamma\in\Gamma_x} x_\gamma \otimes e_\gamma$. For any $x, y \in S_{\ell_p(\Gamma,H)}$, we say x is $p-$ *orthogonal* to y if

$$\Gamma_x \cap \Gamma_y = \varnothing,$$

we also can write by $x \perp_p y$.

In first lemma, we simply explain the relationship between orthogonality in $S_{\ell_p(\Gamma,H)}$ and orthogonality in $S_{\ell_p(\Delta,K)}$.

Lemma 2.1 *Let $X = \ell_p(\Gamma,H), Y = \ell_p(\Delta,K)$, $1 \leq p < \infty, p \neq 2$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then f is a norm-preserving map. Moreover, we have $x \perp_p y \Leftrightarrow f(x) \perp_p f(y)$ for any two elements $x, y \in S_X$.*

Proof. An important conclusion had been proved in the [7, Lemma2.1], which is applicable to the whole $\ell_p(\Gamma,H)$ space

$$x \perp_p y \Leftrightarrow \|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p).$$

Obviously, it is also true in unit spherical space. Next, we prove that f is a norm-preserving map. It only needs to make $x = y$ in the definition of phase-isometry and they are non-zero elements.

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\}$$

$$\{\|f(x)+f(x)\|,\|f(x)-f(x)\|\}=\{\|x+x\|,\|x-x\|\}$$

$$\{2\|f(x)\|,0\}=\{2\|x\|,0\}$$

Then f is a norm-preserving map, $\|f(x)\|=\|x\|$. Finally,

$$\{\|f(x)+f(y)\|,\|f(x)-f(y)\|\}=\{\|x+y\|,\|x-y\|\} \quad (phase-isometry)$$

$$2(\|f(x)\|^p+\|f(y)\|^p)=2(\|x\|^p+\|y\|^p) \quad (norm-preserving)$$

If $x\perp_p y$,then the above two equations are equal, imply that $f(x)\perp_p f(y)$.

□

Lemma 2.2 *Let $X=\ell_p(\Gamma,H), Y=\ell_p(\Delta,K)$, $1\le p<\infty, p\neq 2$. Suppose that $f:S_X\to S_Y$ is a surjective phase-isometry. Let $\gamma\in\Gamma$ and denote by $\Delta_{f(e_\gamma)}$ the support of $f(e_\gamma)$. For every Γ_x is a singleton, $x\in S_X$, then $\Delta_{f(x)}$ is a singleton.*

Proof. We defined $x:=u\otimes e_{\gamma_0}$, $u\in S_H$. If $\Delta_{f(x)}$ is not a singleton, set $\delta_1,\delta_2\in\Delta_{f(x)}$ and $\delta_1\neq\delta_2$. There is exist four nonzero elements $y,z\in S_X$, $u_1,u_2\in S_K$, with Γ_y and Γ_z is a singleton, such that $f(y)=\frac{u_1}{\|u_1\|}\otimes e_{\delta_1}$ and $f(z)=\frac{u_2}{\|u_2\|}\otimes e_{\delta_2}$. Obvious, $f(y)\perp_p f(z)$, by Lemma 2.1, we have $y\perp_p z$, then $x\perp_p y$ or $x\perp_p z$. By Lemma 2.1 again, we get $f(x)\perp_p f(y)$ or $f(x)\perp_p f(z)$, it is a contradiction, so we get the result. □

Theorem 2.3 *Let $X=\ell_p(\Gamma,H), Y=\ell_p(\Delta,K)$, $1\le p<\infty, p\neq 2$. Suppose that $f:S_X\to S_Y$ is a surjective phase-isometry. Then there is a bijection $\pi:\Gamma\to\Delta$ such that for each $x=\sum_{\gamma\in\Gamma}x_\gamma\otimes e_\gamma\in S_X$, $f(x)=\sum_{\gamma\in\Gamma}x'_{\pi(\gamma)}\otimes e_{\pi(\gamma)}\in S_Y$. Where $x'_{\pi(\gamma)}\in K$ with $\|x_\gamma\|=\|x'_{\pi(\gamma)}\|$ for all $\gamma\in\Gamma$.*

Proof. The proof of this theorem is divided into two aspects. On one hand we need show the $\pi:\Gamma\to\Delta$ is a bijective mapping, on the other hand x and $f(x)$ have the same norm values of elements on the corresponding indexes. First, we show the one point. Defined the mapping $\pi:\Gamma\to\Delta$ by $\pi(\gamma)=\Delta_{f(u\otimes e_\gamma)}$, $u\in S_H$. If $\gamma_1\neq\gamma_2\in\Gamma$, $u\otimes e_{\gamma_1}\perp_p u\otimes e_{\gamma_2}$, by Lemma 2.1, $f(u\otimes e_{\gamma_1})\perp_p f(u\otimes e_{\gamma_2})$, thus π is a injective mapping. Next, we prove its surjective property. We can set up $\delta\in\Delta/\pi(\Gamma), v\in S_K$. Because f is a surjective phase-isometry, there is exist $x\in S_X$, such that $f(x)=v\otimes e_\delta$. For each $\gamma\in\Gamma$ and $u\in S_H$, $f(x)\perp_p f(u\otimes e_\gamma)$, by Lemma 2.1, we get $x \perp_p u\otimes e_\gamma$. By the arbitrariness of u and γ, we only reach $x=0$, it is a contradiction. Then π is a bijective mapping.

For all $x=\sum_{\gamma\in\Gamma} x_\gamma\otimes e_\gamma \in S_X$, by the first part of the proof, we have $f(x)=\sum_{\gamma\in\Gamma} x'_{\pi(\gamma)}\otimes e_{\pi(\gamma)}\in S_Y$. For each $\gamma\in\Gamma_x$, there exist $v_{\pi(\gamma)}\in S_K$ such that $f(\frac{x_\gamma\otimes e_\gamma}{\|x_\gamma\|}) = v_{\pi(\gamma)}\otimes e_{\pi(\gamma)}$. Then

$$\begin{aligned}
& 1-\|x_\gamma\|^p+(1+\|x_\gamma\|)^p \\
= \; & \{1-\|x_\gamma\|^p+\|x_\gamma+\frac{x_\gamma}{\|x_{x_\gamma}\|}\|^p\}\vee\{1-\|x_\gamma\|^p+\|x_\gamma-\frac{x_\gamma}{\|x_{x_\gamma}\|}\|^p\} \\
= \; & \{\|x+\frac{x_\gamma\otimes e_\gamma}{\|x_\gamma\|}\|^p\}\vee\{\|x-\frac{x_\gamma\otimes e_\gamma}{\|x_\gamma\|}\|^p\} \\
= \; & \{\|f(x)+f(\frac{x_\gamma\otimes e_\gamma}{\|x_\gamma\|})\|^p\}\vee\{\|f(x)-f(\frac{x_\gamma\otimes e_\gamma}{\|x_\gamma\|})\|^p\} \\
= \; & \{1-\|x'_{\pi(\gamma)}\|^p+\|x'_{\pi(\gamma)}+v_{\pi(\gamma)}\|^p\}\vee\{1-\|x'_{\pi(\gamma)}\|^p+\|x'_{\pi(\gamma)}-v_{\pi(\gamma)}\|^p\} \\
\leqslant \; & 1-\|x'_{\pi(\gamma)}\|^p+(1+\|x'_{\pi(\gamma)}\|)^p
\end{aligned}$$

We note that the function $\varphi(t)=(1+t)^p-t^p$ is strictly increasing on $(0,+\infty)$ when $p>1$, we have $\|x_\gamma\|\leqslant\|x'_{\pi(\gamma)}\|$ for each $\gamma\in\Gamma_x$. Then the equation $\|f(x)\|=\|x\|=1$ implies that $\|x_\gamma\|=\|x'_{\pi(\gamma)}\|$ for each $\gamma\in\Gamma_x$.

□

Remark 2.4 *From Theorem 2.3, we know for every* $x\in S_X, x_\gamma\in H, x'_{\pi(\gamma)}\in K$, $x=\sum_{\gamma\in\Gamma} x_\gamma\otimes e_\gamma\in S_X$, $f(x)=\sum_{\gamma\in\Gamma} x'_{\pi(\gamma)}\otimes e_{\pi(\gamma)}\in S_Y$, *there have* $\|x_\gamma\|=\|x'_{\pi(\gamma)}\|$. *We can take any* $y\in S_X$ *with* $\Gamma_x\cap\Gamma_y=\emptyset, y_\gamma\in H, y'_{\pi(\gamma)}\in K$, *it is clear that* $\|y_\gamma\|=\|y'_{\pi(\gamma)}\|$. *For* $\lambda\in\mathbb{R}$, *we can structure*

$$\frac{x+\lambda y}{\|x+\lambda y\|}=\sum_{\gamma\in\Gamma_x}\frac{x_\gamma}{\|x+\lambda y\|}\otimes e_\gamma+\sum_{\gamma\in\Gamma_y}\frac{\lambda y_\gamma}{\|x+\lambda y\|}\otimes e_\gamma$$

We can get $z=\frac{1}{\|x+\lambda y\|}x+\frac{\lambda}{\|x+\lambda y\|}y$, *so we have*

$$f(\frac{x+\lambda y}{\|x+\lambda y\|}):=\sum_{\gamma\in\Gamma_x}\frac{x''_\gamma}{\|x+\lambda y\|}\otimes e_\gamma+\sum_{\gamma\in\Gamma_y}\frac{\lambda y''_\gamma}{\|x+\lambda y\|}\otimes e_\gamma.$$

$$By \quad Theorem 2.3, f(z)=\sum_{\gamma\in\Gamma_x}\frac{x'_\gamma}{\|x+\lambda y\|}\otimes e_\gamma+\sum_{\gamma\in\Gamma_y}\frac{\lambda y'_\gamma}{\|x+\lambda y\|}\otimes e_\gamma.$$

This means $\|x'_\gamma\|=\|x''_\gamma\|, \|y'_\gamma\|=\|y''_\gamma\|$, *for every* $\gamma\in\Gamma_x\cup\Gamma_y$.

Lemma 2.5 *Let $X=\ell_p(\Gamma,H), Y=\ell_p(\Delta,K)$, $1\le p<\infty, p\neq 2$. Suppose that $f:S_X\to S_Y$ is a surjective phase-isometry. Then for all $x,y\in S_X$, $\Gamma_x\cap\Gamma_y=\emptyset$, there exist two real numbers $\alpha(x,\lambda y)$ and $\beta(x,\lambda y)$ in $\mathbb{R}$ with $|\alpha(x,\lambda y)|=|\beta(x,\lambda y)|=1$ such that*

$$\|x+\lambda y\|f(\frac{x+\lambda y}{\|x+\lambda y\|})=\alpha(x,\lambda y)f(x)+\beta(x,\lambda y)\lambda f(y),$$

for all $\lambda\in\mathbb{R}$ with $\lambda y\in X$. Otherwise, $\alpha(x,y)\beta(x,y)=\alpha(x,\lambda y)\beta(x,\lambda y)$

Proof. Suppose that $x=\sum_{\gamma\in\Gamma_x}x_\gamma\otimes e_\gamma$ and $y=\sum_{\gamma\in\Gamma_y}y_\gamma\otimes e_\gamma$. By Theorem 2.3 and Remark 2.4, set $\lambda\in\mathbb{R}$, we can write that

$$f(x)=\sum_{\gamma\in\Gamma_x}x'_{\pi(\gamma)}\otimes e_{\pi(\gamma)}, f(y)=\sum_{\gamma\in\Gamma_y}y'_{\pi(\gamma)}\otimes e_{\pi(\gamma)}$$

$$\|x+\lambda y\|f(\frac{x+\lambda y}{\|x+\lambda y\|})=\sum_{\gamma\in\Gamma_x}x''_\gamma\otimes e_\gamma+\lambda\sum_{\gamma\in\Gamma_y}y''_\gamma\otimes e_\gamma$$

with $\|x_\gamma\|=\|x'_\gamma\|=\|x''_\gamma\|$ and $\|y_\gamma\|=\|y'_\gamma\|=\|y''_\gamma\|$ for all $\gamma\in\Gamma_x\cup\Gamma_y$. We can analyze this constant $t=\frac{1}{\|x+\lambda y\|}=\frac{1}{(\|x\|^p+|\lambda|^p\|y\|^p)^{\frac{1}{p}}}\le 1$, obvious $t>0$. Because f is a phase-isometry, we have

$$\begin{aligned}&\{(t+1)^p,(1-t)^p\}\\ =&\{\|\frac{x+\lambda y}{\|x+\lambda y\|}+x\|^p-|\lambda t|^p,\|\frac{x+\lambda y}{\|x+\lambda y\|}-x\|^p-|\lambda t|^p\}\\ =&\{\|f(\frac{x+\lambda y}{\|x+\lambda y\|})+f(x)\|^p-|\lambda t|^p,\|f(\frac{x+\lambda y}{\|x+\lambda y\|})-f(x)\|^p-|\lambda t|^p\}\\ =&\{\sum_{\gamma\in\Gamma_x}\|x'_\gamma+tx''_\gamma\|^p,\sum_{\gamma\in\Gamma_x}\|x'_\gamma-tx''_\gamma\|^p\}.\end{aligned}$$

So $(t+1)^p=\sum_{\gamma\in\Gamma_x}\|x'_\gamma+x''_\gamma\|^p$ or $\sum_{\gamma\in\Gamma_x}\|x'_\gamma-tx''_\gamma\|^p$. By norm triangle inequality, we have $\sum_{\gamma\in\Gamma_x}\|x'_\gamma\pm tx''_\gamma\|^p\le\sum_{\gamma\in\Gamma_x}(\|x'_\gamma\|+\|tx''_\gamma\|)^p=(t+1)^p$. It follows that $\sum_{\gamma\in\Gamma_x}x''_\gamma\otimes e_\gamma=\pm f(x)$. The same reason is available

$$\sum_{\gamma\in\Gamma_y}y''_\gamma\otimes e_\gamma=\pm f(y).$$

Next, we will show that $\alpha(x,y)\beta(x,y)=\alpha(x,\lambda y)\beta(x,\lambda y)$. Using the first conclusion, we get

$$\|x+y\|f(\frac{x+y}{\|x+y\|})=\alpha(x,y)f(x)+\beta(x,y)f(y), |\alpha(x,y)|=|\beta(x,y)|=1$$

$$\|x+\lambda y\|f(\frac{x+\lambda y}{\|x+\lambda y\|})=\alpha(x,\lambda y)f(x)+\beta(x,\lambda y)\lambda f(y), |\alpha(x,\lambda y)|=|\beta(x,\lambda y)|=1$$

Reference t, we set $t_0 = \frac{1}{\|x+y\|} = \frac{1}{2}$. By Theorem 2.3 we have

$$\begin{aligned}
& \{|t_0+t|^p+|t_0+\lambda t|^p, |t_0-t|^p+|t_0-\lambda t|^p\} \\
= & \{\|\frac{x+y}{\|x+y\|}+\frac{x+\lambda y}{\|x+\lambda y\|}\|^p, \frac{x+y}{\|x+y\|}-\frac{x+\lambda y}{\|x+\lambda y\|}\|^p\} \\
= & \{\|f(\frac{x+y}{\|x+y\|})+f(\frac{x+\lambda y}{\|x+\lambda y\|})\|^p, \|f(\frac{x+y}{\|x+y\|})-f(\frac{x+\lambda y}{\|x+\lambda y\|})\|^p\} \\
= & \{\|\alpha(x,y)\beta(x,y)t_0f(x)+t_0f(y)+\alpha(x,\lambda y)\beta(x,\lambda y)tf(x)+\lambda tf(y)\|^p, \\
& \|\alpha(x,y)\beta(x,y)t_0f(x)+t_0f(y)-\alpha(x,\lambda y)\beta(x,\lambda y)tf(x)-\lambda tf(y)\|^p\} \\
= & \{|\alpha(x,y)\beta(x,y)t_0+\alpha(x,\lambda y)\beta(x,\lambda y)t|^p+|t_0+\lambda t|^p, \\
& |\alpha(x,y)\beta(x,y)t_0-\alpha(x,\lambda y)\beta(x,\lambda y)t|^p+|t_0-\lambda t|^p\}.
\end{aligned}$$

It follows that $\alpha(x,y)\beta(x,y) = \alpha(x,\lambda y)\beta(x,\lambda y)$ and the proof is complete.

□

Theorem 2.6 *Let $X = \ell_p(\Gamma,H), Y = \ell_p(\Delta,K)$, $1 \le p < \infty, p \ne 2$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then it is phase equivalent to an isometry of the unit sphere which is the restriction of a surjective linear isometry between the whole spaces.*

Proof. When $p = 2$, X and Y are real inner product spaces. Though the famous Wigner's theorem, we can show F is a plus-minus linear isometry. we only consider the case $p \ge 1, p \ne 2$. By the theorem 2.3, we can define a bijection $\pi : \Gamma \to \Delta$, for fixed $\gamma_0 \in \Gamma$ and corresponding the $\pi(\gamma_0) = \delta_0 \in \Delta$. Thus we can define two proper subsets of S_X and S_Y, which are also unit spheres. $S_U = \{x \in S_X : \gamma_0 \notin \Gamma_x\}$, $S_V = \{f(x) \in S_Y : \delta_0 \notin \Delta_{f(x)}\}$. Then we know $S_X = \frac{S_U \oplus_p H \otimes e_{\gamma_0}}{\|S_U \oplus_p H \otimes e_{\gamma_0}\|}$, $S_Y = \frac{S_V \oplus_p K \otimes e_{\delta_0}}{\|S_V \oplus_p K \otimes e_{\delta_0}\|}$. From the theorem 2.3, we obtain $f(S_U) = S_V$. For any $h \in H, u \in S_U$, exist $v \in S_v, k \in K$, such that $\|\frac{h}{u \oplus_p h \otimes e_{\gamma_0}}\| = \|\frac{h}{v \oplus_p k \otimes e_{\delta_0}}\|$. By $\|u\| = \|v\| = 1$, we can get $\|h\| = \|k\|$. According to the definition of $S_{\ell_p(\Gamma,H)}$ type-spaces, when only γ_0 position has elements, we can see $x_0 = \frac{h \otimes e_{\gamma_0}}{\|h \otimes e_{\gamma_0}\|}$, $\lambda x_0 \in H \otimes e_{\gamma_0}$, $\lambda \in R$. Then $f(x_0) = \frac{k \otimes e_{\delta_0}}{\|k \otimes e_{\delta_0}\|}$. By $\|h\| = \|k\| \Longrightarrow \|h \otimes e_{\gamma_0}\| = \|k \otimes e_{\delta_0}\|$, then $f(\frac{h \otimes e_{\gamma_0}}{\|h \otimes e_{\gamma_0}\|}) = f(x_0) = \frac{k \otimes e_{\delta_0}}{\|k \otimes e_{\delta_0}\|} = \frac{k \otimes e_{\delta_0}}{\|h \otimes e_{\gamma_0}\|}$. So $\|h \otimes e_{\gamma_0}\| f(\frac{h \otimes e_{\gamma_0}}{\|h \otimes e_{\gamma_0}\|}) = k \otimes e_{\delta_0}$. We apply Wigner's Theorem to mapping $f : \frac{H \otimes e_{\gamma_0}}{\|H \otimes e_{\gamma_0}\|} \to \frac{K \otimes e_{\delta_0}}{\|K \otimes e_{\delta_0}\|}$ to obtain a phase function $\varepsilon : \frac{H \otimes e_{\gamma_0}}{\|H \otimes e_{\gamma_0}\|} \to \{1,-1\}$ with $\varepsilon(x_0) = 1$ such that $\varepsilon f : \frac{H \otimes e_{\gamma_0}}{\|H \otimes e_{\gamma_0}\|} \to \frac{K \otimes e_{\delta_0}}{\|K \otimes e_{\delta_0}\|}$

is a linear isometry.

By Lemma 2.5 for each $u\in S_U$, we have

$$\|u+\lambda x_0\| f(\frac{u+\lambda x_0}{\|u+\lambda x_0\|}) = \alpha(u,\lambda x_0)f(u)+\beta(u,\lambda x_0)\lambda f(x_0), \quad |\alpha(u,\lambda x_0)| = |\beta(u,\lambda x_0)| = 1.$$

Define a mapping $g: S_U\to S_V$ given by

$$g(u) = \alpha(u,\lambda x_0)\beta(u,\lambda x_0)f(u)$$

for all $u\in S_U$ and $x_0\in H\otimes e_{\gamma_0}$. Obviously, g is a phase-isometry. Through proving that g is a surjective isometry, we can get that $g: S_U\to S_V$ is a linear isometry by Mazur-Ulam Theorem. From the defined of g, we can get $g(u) = \pm f(u)$ for each $u\in S_U$. Let u_1,u_2 be in S_U and $\lambda x_0\in H\otimes e_{\gamma_0}$, we have

$$\begin{aligned}
& \{\|u_1+u_2\|^p+(2\lambda)^p, \|u_1-u_2\|^p\} \\
= \quad & (1+\lambda^p)\{\|\frac{u_1+\lambda x_0}{\|u_1+\lambda x_0\|}+\frac{u_2+\lambda x_0}{\|u_2+\lambda x_0\|}\|^p, \|\frac{u_1+\lambda x_0}{u_2+\lambda x_0}-\frac{u_2+\lambda x_0}{\|u_2+\lambda x_0\|}\|^p\} \\
= \quad & (1+\lambda^p)\{\|g(\frac{u_1+\lambda x_0}{\|u_1+\lambda x_0\|})+g(\frac{u_2+\lambda x_0}{\|u_2+\lambda x_0\|})\|^p, \|g(\frac{u_1+\lambda x_0}{u_2+\lambda x_0})-g(\frac{u_2+\lambda x_0}{\|u_2+\lambda x_0\|})\|^p\} \\
= \quad & \{\|g(u_1)+g(u_2)\|^p+(2\lambda)^p, \|g(u_1)-g(u_2)\|^p\}
\end{aligned}$$

This implies $\|g(u_1)-g(u_2)\| = \|u_1-u_2\|$ for all $u_1,u_2\in S_U$. Otherwise, we just need to let $u_1=-u_2, \lambda=1$, then $g(-u)=-g(u)$ for all $u\in S_U$. So g is a surjective isometry.
Next we define a linear isometry $\tilde{f}: S_X\to S_Y$ by $\|u+\lambda x_0\|\tilde{f}(\frac{u+\lambda x_0}{\|u+\lambda x_0\|}) = g(u)+\varepsilon(x_0)\lambda f(x_0)$ for each $u\in S_U$ and $x_0\in H\otimes e_{\gamma_0}$. We only need to show that $\|u+\lambda x_0\|\tilde{f}(\frac{u+\lambda x_0}{\|u+\lambda x_0\|}) = \pm\|u+\lambda x\| f(\frac{u+\lambda x}{\|u+\lambda x\|})$ for each $0\neq u\in S_U$ and $0\neq \lambda x\in H\otimes e_{\gamma_0}$. From the definition of $\tilde{f}$ and Lemma 2.5, we can have

$$\begin{aligned}
&\|u+\lambda x_0\|\tilde{f}(\frac{u+\lambda x_0}{\|u+\lambda x_0\|}) = \alpha(u,\lambda x_0)\beta(u,\lambda x_0)f(u)+\varepsilon(x_0)\lambda f(x_0), \\
&\|u+\lambda x\| f(\frac{u+\lambda x}{\|u+\lambda x\|}) = \alpha(u,\lambda x)f(u)+\beta(u,\lambda x)\lambda f(x),
\end{aligned}$$

where $|\alpha(u,\lambda x_0)| = |\beta(u,\lambda x_0)| = |\alpha(u,\lambda x)| = |\beta(u,\lambda x)| = |\varepsilon(x_0)| = 1$. We need to show that

$$\varepsilon(x_0)\alpha(u,\lambda x_0)\beta(u,\lambda x_0) = \alpha(u,\lambda x)\beta(u,\lambda x)$$

The proof of the equation can be referred to [7, Theorem2.8]. Then we get the $\tilde{f}$ is phase equivalent to f. Finally, by[12] we can know that $\tilde{f}$ can be extended from the unit sphere to the isometric operator of the whole space.

This completes the proof.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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