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ON BANACH CONTRACTION PRINCIPLE AND SEHGAL FIXED POINT THEOREM IN EXTENDED b-METRIC SPACE

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Abstract. In this paper, we show that Banach contraction principle and other known fixed point results, in the frame of extended b-metric space, are immediate consequences of the analogous theorems in b-metric space. We establish a more general and extended version of Banach contraction principle, we also prove a Sehgal-Guseman type theorem for mappings with contractive iterate at each point in extended b-metric space.

Keywords: fixed point; extended b-metric; Banach contraction; contractive iterate.

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1. Introduction and Preliminaries

Fixed point theory provides a large set of theorems, that plays a very important role to proof existence and to determine uniqueness of solutions of numerous problems in mathematics, physics and applied sciences.

In 1922, Banach [1] initiated the study of fixed point theory by proving the contraction principle (BCP for short), which states that any contraction on a complete metric space has a unique fixed point. As generalization of BCP, Sehgal [2] introduced a new type of mappings with contractive iterate at each point. This result was extended by many authors in different settings and various

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generalized metric spaces (see [3–9]).

On the other side, recent trends in fixed point theory are focused on generalizing the metric structure of the space, trying to solve the problem of limitation and unsatisfactory of classical metric. In 2017, Kamran [10] et al. introduced a very interesting generalization of the notion of b-metric space which they called extended b-metric space. Since then fixed point theory in this new settings has been widely investigated by many authors, interested reader is referred to [11–14] for further details.

Kamran et al. introduced the concept of extended b-metric space as follows:

Definition 1. Let X be a nonempty set and $\theta : X \times X \longrightarrow [1,+\infty[$ a real valued mapping. A function $d_\theta : X \times X \to [0,\infty)$ is said to be an extended b-metric if and only if for all $x,y,z \in X$ the following conditions are satisfied:

($d_{\theta 1}$) $d(x,y) = 0$ if and only if $x = y$,

($d_{\theta 2}$) $d(x,y) = d(y,x)$,

($d_{\theta 3}$) $d(x,z) \leq \theta(x,z)[d(x,y) + d(y,z)]$.

The pair (X,d_θ) is called an extended b-metric space.

Remark 1. If we take $\theta(x,z) = s$ ($s \geq 1$ a positive real), we obtain the definition of a b-metric space (see [15]).

We notice that a number of results (see [10], [13]) assume the continuity of the extended b metric d_θ, this assumption is a direct consequence of Lemma 1.1 in [10] stated as follows:

Lemma 1.1. *Let (X,d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.*

This Lemma is given without proof, it seems that it is not necessary to assume continuity of the metric d_θ, to assure uniqueness of the limit of convergent sequences in extended b metric spaces. Therefore we give a correct form of this Lemma as follows.

Lemma 1.2. *In every extended b-metric space (X,d_θ) every convergent sequence has a unique limit.*

Proof. Let $\{x_n\}$ be a sequence that converges to both x and y in X.
We have

$$\lim_{n\to\infty} d_\theta(x_n,x) = \lim_{n\to\infty} d_\theta(x_n,y) = 0.$$

By extended relaxed triangular inequality, we have

$$d_\theta(x,y) \le \theta(x,y)(d_\theta(x,x_n)+d_\theta(x_n,y)).$$

By taking the limit as $n \to +\infty$, we get $d_\theta(x,y) = 0$ and by $(d_{\theta 1})$ we have $x = y$. $\square$

However, by assuming the continuity of the extended b-metric, we can show that Banach contraction principle in this new setting (see [10]) is not a real generalization, in fact, we establish that this result and many others, are direct consequences of their counterparts in b-metric space.
The obtained BCP version in [10] can be summarized as follows.

Theorem 1.1. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is a continuous functional. Let $T : X \longrightarrow X$ satisfy:*

$$d_\theta(Tx,Ty) \le kd_\theta(x,y), \text{ for all } x,y \in X, \tag{1}$$

where $k \in [0,1)$ such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n,x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1,2,\ldots$. Then, T has precisely one fixed point u. Moreover for each $y \in X$, $T^n y \to u$.

This theorem is a generalization of the following result due to M. Jovanovic et al. [see [16] Theorem 3.3] in the frame of b-metric space.

Theorem 1.2. *Let (X,d) be a complete b-metric space with coefficient $s \ge 1$ and $T : X \to X$ be a mapping satisfying: $d(Tx,Ty) \le kd(x,y)$ for all $x,y \in X$, and some $k \in \left[0,\frac{1}{s}\right)$. Then, T has a unique fixed point u and $\lim_{n\to\infty} T^n x_0 = u \in X$ for every $x_0 \in X$.*

The above Theorem has been extended for value of constant k in the range $[0,1)$, indeed in [17] N.V. Dung and V.T.L. Hang prove that the conclusion of Theorem 1.2 remain true for $k \in [1/s,1)$.

Theorem 1.3 (Theorem 2.1 [17]). *Let (X,d) be a complete b-metric space and let $T : X \to X$ be a map such that for all $x,y \in X$, and some $k \in [0,1)$*

$$d(Tx,Ty) \leq kd(x,y).$$

Then, T has a unique fixed point u and $\lim_{n\to\infty} T^n x_0 = u \in X$ for every $x_0 \in X$.

Definition 2. Let (X,d_θ) be an extended b-metric space and let $T : X \to X$ be a map. We said that T is continuous, if for every $x \in X$ such that $x_n \longrightarrow x$ we have $Tx_n \longrightarrow Tx$.

Remark 2. Note that a contraction mapping in extended b-metric space is continuous.

2. Main Results

2.1. Fixed point theorems for some contractions are not real generalization under continuity of d_θ.

In the next, we show that under the continuity condition of the extended b-metric d_θ, some fixed point theorems obtained in this new settings are consequences of their correspondings results in b-metric space.

Theorem 2.1. *If d_θ is continuous then Theorem 1.2 $\Longrightarrow$ Theorem 1.1.*

Proof. For any $x_0 \in X$ define the iterative sequence (x_n) by

$$x_n = T^n x_0, n = 1,2,\ldots$$

By conditions of Theorem 1.1, we have $\lim_{n,m\to\infty} \theta(x_n,x_m) = r \in \mathbb{R}$ exists and $r < \dfrac{1}{k}$.
Then for $s = \dfrac{1}{2}(\frac{1}{k}+r)$, there exists $N \in \mathbb{N}$ such that

$$\theta(x_n,x_m) < s \text{ for all } n,m \geq N. \tag{2}$$

Let us consider the subset $\mathscr{O}_{T,N}(x_0)$ of X defined by

$$\mathscr{O}_{T,N}(x_0) = \{T^n x_0 / n \geq N\}.$$

Let $\mathscr{C} = \overline{\mathscr{O}_{T,N}(x_0)}$ be the closure of $\mathscr{O}_{T,N}(x_0)$ with respect to the extended metric d_θ
We claim that :

(1) d_θ is b-metric over $\mathscr{C}$ with coefficient s,

(2) and $T(\mathscr{C}) \subset \mathscr{C}$.

1- Let $x, y \in \mathscr{C}$. By definition of $\mathscr{C}$, there exists two sequence $(T^{n_k}x_0)_k \subset \mathscr{O}_{T,N}(x_0)$ and $(T^{n_p}x_0)_p \subset \mathscr{O}_{T,N}(x_0)$ that converges respectively to x and y with respect to the metric d_θ. We have

$$d_\theta(T^{n_k}x_0, T^{n_p}x_0) \leq \theta(T^{n_k}x_0, T^{n_p}x_0)(d_\theta(T^{n_k}x_0, z) + d_\theta(z, T^{n_p}x_0)), \text{ for all } z \in \mathscr{C}. \tag{3}$$

Then by (2) we get

$$d_\theta(T^{n_k}x_0, T^{n_p}x_0) \leq s(d_\theta(T^{n_k}x_0, z) + d_\theta(z, T^{n_p}x_0)).$$

Taking the limit as $k, p \to \infty$ and considering the continuity of d_θ it follows that

$$d_\theta(x, y) \leq s(d_\theta(x, z) + d_\theta(z, y)).$$

Which implies that $(\mathscr{C}, d_\theta)$ is a complete b-metric space with coefficient s.
2- Let $x \in \mathscr{C}$. From the definition of $\mathscr{C}$, there exists a sequence $(T^{n_k}x_0)_k$ that converges to x with respect to d_θ.
Since the mapping T is continuous, the sequence $(T^{n_k+1}x_0)_k$ converges to Tx. We have $(T^{n_k+1}x_0)_k \subseteq \mathscr{C}$ and $\mathscr{C}$ is closed, then $Tx \in \mathscr{C}$ and This finishes the proof.
Now, we have $T : \mathscr{C} \to \mathscr{C}$ a contraction mapping with constant k in the complete b-metric space $(\mathscr{C}, d_\theta)$ and by taking into account that $kr < 1$ we get $ks < 1$.
Hence T satisfies all the hypotheses of Theorem 1.2, so T has a unique fixed point. □

It is natural to ask if the result of Theorem 1.1 could be extended to all values of $k \in [0, 1)$. The next Theorem give a positive answer to this question, first we need the following preliminary result.

Proposition 2.2. *Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be a mapping. Then*

(1) If T^n has a unique fixed point $\omega \in X$, then ω is a unique fixed point for T.
(2) If T is a Banach contraction mapping with lipschitz constant k, then T^n $(n \in \mathbb{N}^*)$ is Banach contraction with lipschitz constant k^n.

Theorem 2.3. *Let (X, d_θ) be a complete extended b-metric space. Let $T : X \longrightarrow X$ a mapping satisfies (1), where $k \in [0, 1)$, such that for some $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m)$ exists and finite,* *here* $x_n = T^n x_0$, $n = 1,2,\dots$.

Then T *has precisely one fixed point* u*. Moreover the sequnece* $(T^n x_0)$ *converges to* u*.*

Proof. Since $k \in [0,1)$, we obtain $\lim_{n\to\infty} k^n = 0$. We have $\lim_{n,m\to\infty} \theta(x_n, x_m)$ exists and finite, so there exists a natural number p such that

$$k^p \lim_{n,m\to\infty} \theta(x_n, x_m) < 1.$$

Then, we have $d_\theta(T^p x, T^p y) \le \lambda d_\theta(x,y)$ for all $x,y \in X$, where $\lambda = k^p$ and $0 \le \lambda \lim_{n,m\to\infty} \theta(x_n^{'}, x_m^{'}) < 1$, with $(x_n^{'})$ is the subsequence of (x_n) defined by $x_n^{'} = x_{pn}$, for all $n \in \mathbb{N}$.

From Theorem 1.1, it follows that T^p has a unique fixed point $u \in X$ and by proposition 2.2, we deduce that u is a fixed point of the mapping T and $T^{np} x_0 \to u$ as n tends to infinity.

Now, we show that $T^n x_0 \to u$, so for n a sufficiently large integer, there exists $(r,q) \in \mathbb{N} \times \mathbb{N}$, such that $n = pr + q$ with $0 \le q < p$.

Set

$$\delta^* = \max\left\{ d_\theta(T^i x_0, u) / i = 0, 1, \dots p-1 \right\}.$$

We obtain by successive iterations

$$\begin{aligned}
d_\theta(T^n x_0, u) = d_\theta(T^{rp+q} x_0, T^p u) &\le \lambda d_\theta(T^{p(r-1)+q} x_0, u) \\
&\le \lambda d_\theta(T^{p(r-1)+q} x_0, T^p u) \\
&\le \lambda^2 d_\theta(T^{p(r-2)+q} x_0, u) \\
&\le \dots \\
&\le \lambda^r d_\theta(T^q x_0, u) \le \lambda^r \delta^*.
\end{aligned}$$

Since $n \to +\infty$ implies $r \to +\infty$, we obtain $d_\theta(T^n x_0, u) \to 0$ as $n \to +\infty$. □

In [18] A.K. Dubey et al. proved a generalization of Hardy-Rogers fixed point theorem stated as follows.

Theorem 2.4. *[18] Let* (X,d) *be a complete b-metric space with constant* $s \ge 1$*. If* $T : X \to X$ *satisfies the inequality:*

$$d(Tu,Tv) \le k_1 d(u,v) + k_2 d(u,Tu) + k_3 d(v,Tv) + k_4[d(v,Tu) + d(u,Tv)], \textit{ for all } u,v \in X.$$

where $k_i \geq 0$, for $i = 1, \dots, 4$ and $k_1 + k_2 + k_3 + 2sk_4 < 1$, then T has a fixed point.

Using the same arguments, assuming continuity of the mapping T and the extended metric d_θ, we deduce the following result from Theorem 2.4.

Theorem 2.5. *Let (X, d_θ) be a complete extended b-metric space with $\theta : X \times X \to [1, \infty)$ and d_θ is continuous. If $T : X \to X$ is continuous and satisfies the inequality:*

$$\begin{aligned} d_\theta(Tu, Tv) \leq & k_1 d_\theta(u, v) + k_2 d_\theta(u, Tu) + k_3 d_\theta(v, Tv) \\ & + k_4[d_\theta(v, Tu) + d_\theta(u, Tv)], \text{ for all } u, v \in X. \end{aligned}$$

Where $k_i \geq 0$, for $i = 1, \dots, 4$ and

$$k_1 + k_2 + k_3 + 2k_4 \lim_{n,m\to\infty} \theta(u_n, u_m) < 1.$$

Then T has a fixed point.

2.2. Sehgal–Guseman fixed point theorem in extended b metric space.

Now, we state our fixed point result for mapping with a contractive iterate at each point.

Theorem 2.6. *Let T be a self-mapping on a complete extended b-metric space (X, d_θ). Suppose that the following conditions hold:*

(i) There exists a $k \in [0, 1)$, and for all $x \in X$, there exists a positive integer $n(x)$ such that:

$$d_\theta(T^{n(x)}x, T^{n(x)}y) \leq k d_\theta(x, y), \text{ for all } y \in X, \tag{4}$$

(ii) there exists $x_0 \in X$ such that $\theta^*(x_0) < \frac{1}{k}$ where $\theta^*(x) = \sup_{n\in\mathbb{N}^*} \theta(T^n(x), x)$,

(iii) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, where $x_{n+1} = T^{n(x_n)}x_n$, for all $n \in \mathbb{N}$.

Then $(x_n)_{n\geq 0}$ converges to some $u \in X$. Moreover, if $\limsup_{n\to\infty} \theta(T^{n(u)}x_n, u) < \infty$, then u is the unique fixed point of T and $T^n x_0 \to u$.

Lemma 2.1. *Let $T : X \to X$ be a mapping satisfying the condition (14), then $r(x) = \sup_{n\in\mathbb{N}^*} d_\theta(T^n(x), x)$ is finite for each $x \in X$ such that $\theta^*(x) < \frac{1}{k}$.*

Proof of the Lemma. Let $x \in X$ and $l(x) = \max\left\{d_\theta(T^k(x),x) : k = 1,2,\dots,n(x)\right\}$, for each positive integer n, there exists an integer p such that $pn(x) \leq n < (p+1)n(x)$, and we have

$$\begin{aligned}
d_\theta(T^n(x),x) &\leq \theta(T^n(x),x)\left[d_\theta(T^{n(x)}T^{n-n(x)}(x),T^{n(x)}x) + d_\theta(T^{n(x)}x,x)\right] \\
&\leq \theta(T^n(x),x)\left[d_\theta(T^{n(x)}T^{n-n(x)}(x),T^{n(x)}x) + l(x)\right] \\
&\leq \theta(T^n(x),x)\left[kd_\theta(T^{n-n(x)}(x),x) + l(x)\right] \\
&\leq \theta(T^n(x),x)l(x) + \theta(T^n(x),x)kd_\theta(T^{n-n(x)}(x),x) \\
&\leq \theta(T^n(x),x)l(x) + \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)kl(x) \\
&+ \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)k^2d_\theta(T^{n-2n(x)}(x),x) \\
&\leq \theta(T^n(x),x)l(x) + \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)kl(x) + \dots \\
&+ \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)\dots\theta(T^{n-pn(x)}(x),x)k^p d_\theta(T^{n-pn(x)}(x),x) \\
&\leq \theta(T^n(x),x)l(x) + \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)kl(x) + \dots \\
&+ \theta(T^n(x),x)\theta(T^{n-n(x)}(x),x)\dots\theta(T^{n-pn(x)}(x),x)k^p l(x).
\end{aligned} \tag{5}$$

Thus, we obtain

$$\begin{aligned}
d_\theta(T^n(x),x) &\leq l(x)\sum_{i=0}^{i=p} k^i \prod_{j=0}^{j=i}\theta(T^{n-jn(x)}(x),x) \\
&\leq l(x)\sum_{i=0}^{i=p} k^i \theta^*(x)^{i+1} \\
&\leq l(x)\theta^*(x)\sum_{i=0}^{i=p}(k\theta^*(x))^i
\end{aligned} \tag{6}$$

We have $0 \leq k\theta^*(x) < 1$, so we get

$$d_\theta(T^n(x),x) \leq \frac{l(x)\theta^*(x)}{1-k\theta^*(x)} \text{ for all } n \in \mathbb{N}^*.$$

□

Now, we give a proof for Theorem 2.6.

Proof. Starting with an arbitrary point $x_0 \in X$, with $\theta^*(x_0) < \frac{1}{k}$ and define $x_{n+1} = T^{n(x_n)}x_n$, for all $n \in \mathbb{N}$. We show that (x_n) is convergent. We have

$$\begin{aligned} d_\theta(x_{n+1},x_n) &= d_\theta(T^{n(x_n)}x_n, T^{n(x_{n-1})}x_{n-1}) \\ &\leq d_\theta(T^{n(x_{n-1})}T^{n(x_n)}x_{n-1}, T^{n(x_{n-1})}x_{n-1}) \\ &\leq k d_\theta(T^{n(x_n)}x_{n-1}, x_{n-1}) \\ &\leq \dots \\ &\leq k^n d_\theta(T^{n(x_n)}x_0, x_0) \\ &\leq k^n r(x_0). \end{aligned} \tag{7}$$

We have $\lim_{n,m\to\infty} \theta(x_n,x_m) = r \in \mathbb{R}$ exists and $r < \dfrac{1}{k}$.
Then for $s = \dfrac{1}{2}(\frac{1}{k} + r)$, there exists $N_0 \in \mathbb{N}$ such that

$$\theta(x_n,x_m) < s \text{ for all } n,m \geq N_0. \tag{8}$$

Hence, by the extended triangular inequality for $m > n \geq N_0$, we obtain

$$\begin{aligned} d_\theta(x_n,x_m) &\leq \theta(x_n,x_m)k^n r(x_0) + \theta(x_n,x_m)\theta(x_{n+1},x_m)k^{n+1}r(x_0) \\ &\quad + \dots + \theta(x_n,x_m)\theta(x_{n+1},x_m)\theta(x_{n+2},x_m)\dots\theta(x_{m-2},x_m)\theta(x_{m-1},x_m)k^{m-1}r(x_0) \\ &\leq r(x_0)\sum_{i=n}^{i=m-1} k^i \prod_{j=n}^{j=i} \theta(x_j,x_m) \\ &\leq r(x_0)\sum_{i=n}^{i=m-1} k^i s^{i-n+1} \\ &\leq r(x_0)s^{1-n}\sum_{i=n}^{i=m-1} (ks)^i \\ &\leq r(x_0)s^{1-n}\frac{[(ks)^n - (ks)^m]}{1-ks}. \end{aligned} \tag{9}$$

We have $0 \leq ks < 1$ and by letting $n,m \to \infty$, we get $\lim_{n\to\infty} d_\theta(x_n,x_m) = 0$, this proves that $(x_n)_n$ is a Cauchy sequence.

Since the extended b-metric space (X,d_θ) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d_\theta(x_n,u) = 0. \tag{10}$$

From (14), we have $d_\theta(T^{n(u)}x_n, T^{n(u)}u) \leq k d_\theta(x_n, u)$.

Which yields

$$\lim_{n\to\infty} d_\theta(T^{n(u)}x_n, T^{n(u)}u) = 0. \tag{11}$$

Also from (14), we have

$$\begin{aligned} d_\theta(T^{n(u)}x_n, x_n) &= d_\theta(T^{n(x_{n-1})}T^{n(u)}x_{n-1}, T^{n(x_{n-1})}x_{n-1}) \\ &\leq k d_\theta(T^{n(u)}x_{n-1}, x_{n-1}) \\ &\leq \dots \\ &\leq k^n d_\theta(T^{n(u)}x_0, x_0). \end{aligned}$$

As $k \in [0,1[$, we obtain

$$\lim_{n\to\infty} d_\theta(T^{n(u)}x_n, x_n) = 0. \tag{12}$$

On the other hand, by applying the extended triangle inequality twice, we obtain

(13)

$$\begin{aligned} & d_\theta(T^{n(u)}u, u) \\ & \leq \theta(T^{n(u)}u, u) d_\theta(T^{n(u)}u, T^{n(u)}x_n) + \theta(T^{n(u)}u, u) d_\theta(T^{n(u)}x_n, u) \\ & \leq \theta(T^{n(u)}u, u) d_\theta(T^{n(u)}u, T^{n(u)}x_n) + \theta(T^{n(u)}u, u)\theta(T^{n(u)}x_n, u)\left[d_\theta(T^{n(u)}x_n, x_n) + d_\theta(x_n, u)\right]. \end{aligned}$$

Since $\limsup_{n\to\infty} \theta(T^{n(u)}x_n, u) < \infty$, using (10), (11), (12) and (13), we get

$$d_\theta(T^{n(u)}u, u) = 0.$$

Hence, we have $T^{n(u)}u = u$.

Fom inequality (14), it follows that $T^{n(u)}$ has a unique fixed point and by Proposition 2.2, u is the unique fixed point of T.

Following the same lines as in the last part of the proof of Theorem 2.3, we show that the sequence $(T^n x_0)$ converges to the fixed point u. □

Corollary 2.6.1. *Let (X, d_θ) be a complete extended b-metric space with θ continuous and T be a self-mapping on X. Suppose that the following conditions hold:*

(i) There exists a $k \in [0,1)$, and for all $x \in X$, there exists a positive integer $n(x)$ such that:

$$d_\theta(T^{n(x)}x, T^{n(x)}y) \leq k d_\theta(x,y), \text{ for all } y \in X, \tag{14}$$

(ii) there exists $x_0 \in X$ such that $\theta^*(x_0) < \frac{1}{k}$ where $\theta^*(x) = \sup_{n \in \mathbb{N}^*} \theta(T^n(x), x)$,

(iii) $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, where $x_{n+1} = T^{n(x_n)}x_n$, for all $n \in \mathbb{N}$.

Then, T has a unique fixed point $u \in X$ and the sequence $(T^n x_0)$ converges to u.

Proof. By (11), we have $T^{n(u)}x_n \to T^{n(u)}u$, from continuity of the function θ, it follows that $\limsup_{n\to\infty} \theta(T^{n(u)}x_n, u) = \theta(T^{n(u)}u, u) < \infty$ and the conclusion follows by using Theorem 2.6. □

Corollary 2.6.2 (Theorem 3.2 [9]). *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and T be a self-mapping on X. Suppose that for all $x \in X$, there exists a positive integer $n(x)$ such that:*

$$d(T^{n(x)}x, T^{n(x)}y) \leq k d(x,y), \text{ for all } y \in X.$$

If $k < \frac{1}{s}$, then the sequence $(T^n x_0)_{n\geq 0}$ converges to $u \in X$, the unique fixed point of T.

For further research, it is interesting to study the following questions.

Question 1: Can we deduce Theorem 1.1 from Theorem 1.2 without assuming continuity of the metric d_θ?

Question 2: Does the conclusion of Theorem 2.6 remains true, if we replace the condition ”$\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$” by this one ”$\lim_{n,m\to\infty} \theta(x_n, x_m)$ exists” ?

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181.

[2] M. Sehgal, A fixed point theorem for mappings with a contractive iterate, Proc. Amer. Math. Soc. 23 (1969) 631–634.

[3] Lj. Gajić, Z. Lozanov-Crvenković, On mappings with contractive iterate at a point in generalized metric spaces, Fixed Point Theory Appl. 2010 (2010), Art. ID 458086.

[4] Lj. Gajić, M. Stojaković, On Ćirić generalization of mappings with a contractive iterate at a point in G-metric space, Appl. Math. Comput. 219 (2012), 435–441.

[5] Lj. Gajić, M. Stojaković, Sehgal-Thomas type fixed point theorems in generalized metric spaces, Filomat, 31 (2017), 3335–3346.

[6] Lj. Gajić, M. Stojaković, On mappings with ϕ-contractive iterate at a point on generalized metric spaces, Fixed Point Theory Appl. 2014 (2014), 46.

[7] L. F. Guseman, Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 26 (1970), 615–618.

[8] B. Alqahtani, A. Fulga, E. Karapinar, A fixed point result with a contractive iterate at a point, Mathematics, 7 (2019), 606.

[9] M.F. Bota, Fixed point theorems for operators with a contractive iterate in b-metric spaces, Stud. Univ. Babeş-Bolyai Math. 61 (2016), 435–442.

[10] T. Kamran, M. Samreen, Q. Ul. Ain, A generalization of b-metric space and some fixed point theorems, Mathematics, 5 (2017), 19.

[11] Q. Kiran, N. Alamgir, N. Mlaiki, H. Aydi, On some new fixed point results in complete extended b-metric spaces, Mathematics, 7 (2019), 476.

[12] B. Nurwahyu, Fixed point theorems for cyclic weakly contraction mappings in dislocated quasi extended b-metric space, J. Funct. Spaces, 2019 (2019), Article ID 1367879.

[13] B. Alqahtani, A. Fulga, E. Karapinar, Non-unique fixed point results in extended b-metric space, Mathematics, 6 (2018), 68.

[14] M. Samreen, W. Ullah, E. Karapinar, Multivalued ϕ-Contractions on Extended b-Metric Spaces, J. Funct. Spaces, 2020 (2020), Article ID 5989652.

[15] S. Czerwik, Contraction mappings in b-metric spaces. Acta Math. Univ. Ostrav. 1 (1993), 5-11

[16] M. Jovanovic, Z. Kadelburg, S. Radenovic, Common fixed point results in metric-type spaces, Fixed Point Theory Appl. 2010 (2010), 978121.

[17] N.V. Dung, V.T.L. Hang, On relaxations of contraction constants and Caristi's theorem in b-metric spaces, J. Fixed Point Theory Appl. 18 (2016), 267–284.

[18] A.K. Dubey, R. Shukla, R.P. Dubey, Some fixed point results in b-metric spaces, Asian J. Math. Appl. 2014 (2014), Article ID ama0147.