Available online at http://scik.org
Adv. Fixed Point Theory, 2022, 12:4
https://doi.org/10.28919/afpt/6935
ISSN: 1927-6303

APPROXIMATE FIXED POINTS OF OPERATORS ON MODULAR G-METRIC SPACES

S. A. M. MOHSENIALHOSSEINI*

Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Abstract. In this paper, we will first introduce the approximate fixed point property and a new class of operators and contraction mapping for a cyclic map T on modular G-metric spaces. Also, we prove two general lemmas regarding approximate fixed Point of cyclic maps on modular G-metric spaces. Using these results we prove several approximate fixed point theorems for a new class of operators and contraction mapping on modular G-metric spaces.

Keywords: modular G-metric spaces, modular G-MN operator, modular G-MNC operator, Approximate fixed points, Diameter approximate fixed point.

2010 AMS Subject Classification: 47H10, 54H25, 46B20.

1. Introduction

Fixed point theory is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the study of existence and uniqueness of solutions for a class of nonlinear integral equations. Some recent work on fixed point theorems of integral type in G-metric spaces, stability of functional difference equation can be found in [24, 25] and the references therein.

*Corresponding author
E-mail addresses: amah@vru.ac.ir, mohsenialhosseini@gmail.com

Received October 27, 2021

In 1968, Kannan (see [11]) proved a fixed point theorem for operators which need not be continuous. Further, Chatterjea (see[6]), in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. In 1972, by combining the above three independent contraction conditions above, Zamfirescu (see [22]) obtained another fixed point result for operators which satisfy the following. In 2001, Rus (see [26]) defined $\alpha-$contraction. In [3], the author obtained a different contraction condition, also he formulated a corresponding fixed point theorem. In 2006, Berinde (see [4]) obtained some result on $\alpha-$contraction for approximate fixed point in metric space. Miandaragh et al. [13, 14] obtained some result on approximate fixed points in metric space. In [5], Chandok et al. obtained some result on Coupled common fixed point in partially ordered G-metric spaces,

On the other hand, in 2006, Mustafa and Sims [20, 21] introduced the notion of generalized metric spaces or simply G-metric spaces. Many researchers have obtained fixd point, coupled fixed point, coupled common fixed point results on G-metric spaces (see [1, 5, 24]).

In 2011, Mohsenalhosseini et al [15], introduced the approximate best proximity pairs and proved the property of approximate best proximity pairs. Also, In 2012 , Mohsenalhosseini et al [16], introduced the approximate fixed point for complete norm spaces and map T_α and proved the property of approximate fixed point. In 2014 Mohsenalhosseini and Ahmadi [17] introduced approximate fixed point in $G-$metric spaces for various types of operators. Also, Mohsenalhosseini in [18] introduced the approximate fixed points of operators on G-metric spaces.

The theory of modular spaces was initiated by Nakano [19] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [23] in 1959. In 2008, Chistyakov [7] introduced the notion of modular metric spaces generated by F-modular and developed the theory of this spaces , on the same idea he defined the notion of a modular on an arbitrary set and developed the theory of metric spaces generated by modular such that called the modular metric spaces in 2010 [8].

The main idea behind this newconcept is the physical interpretation of the modular. One of the most interesting problems in this setting is the famous Dirichlet energy problem

[9, 10].The classical technique used so far in studying thisproblemis to convert the energy functional,naturally defined by a modular, to a convoluted and complicated problem which involves a norm (the Luxemburg norm).

In 2013, Azadifar et al [2] introduced the notion of modular G-metric spaces and proved some known fixed point theorems on the modular G-metric spaces. The aim of this paper is to introduce the new classes of operators and contraction maps regarding approximate fixed point and diameter approximate fixed point for a cyclic map $T : A \cup B \cup C \to A \cup B \cup C$ i.e. $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$ on modular G-metric spaces. Also, we give some illustrative example of our main results.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows. Throughout the paper X is a nonempty set, $\lambda > 0$ is understood in the sense that $\lambda \in (0,\infty)$ and, due to the disparity of the arguments, functions $\omega : (0,\infty) \times X \times X \times X \longrightarrow [0,\infty]$ will be written as $\omega_\lambda(x,y,z) = \omega(\lambda,x,y,z)$ for all $\lambda > 0$ and $x,y,z \in X$.

Definition 2.1. [2] Let X be a nonempty set and let $\omega : (0,\infty) \times X \times X \times X \longrightarrow [0,\infty]$ be a function satisfying the following properties:

(G1) $\omega_\lambda(x,y,z) = 0 \; for \; all \; x,y \in X \; and \; \lambda > 0 \; if \; and \; only \; if \; x = y = z$;

(G2) $\omega_\lambda(x,x,y) > 0 for \; all \; x,y \in X \; and \; \lambda > 0 \; with \; x \neq y$;

(G3) $\omega_\lambda(x,x,y) \leq \omega(x,y,z) for \; all \; x,y,z \in X \; and \; \lambda > 0 \; with \; z \neq y$;

(G4) $\omega_\lambda(x,y,z) = \omega_\lambda(x,z,y) = \omega_\lambda(y,z,x) = \cdots \; and \; \lambda > 0 \; (symmetry \; in \; all \; three \; variables)$;

(G5) $\omega_{\lambda+\mu}(x,y,z) \leq \omega_\lambda(x,a,a) + \omega_\mu(a,y,z) for \; all \; x,y,z,a \in X \; and \; \lambda,\mu > 0$.

Then, the function ω_λ is called modular G-metric on X.

Given $x_0 \in X$, the set $X_\omega = \{x \in X : lim_{\lambda\to\infty}\omega(\lambda,x,x_0) = 0\}$ is a metric space with metric

$$d_\omega(x,y) = inf\{\lambda > 0 : \omega(\lambda,x,y) \leq \lambda\},$$

called modular space.

Let us fix an element $x_0 \in X$ arbitrarily and set $X_\omega = X^\circ_\omega(x_0)$. The set X_ω is call a modular set.

Theorem 2.2. [2] *If ω is modular G-metric on X, then the modular set X_{ω} is a G-metric space with G-metric given by*

$$G^{\circ}_{\omega} = inf\{\lambda > 0 : \omega_{\lambda}(x,y,z) \leq \lambda\},$$

for all $x,y,z \in X$.

Proposition 2.3. [20] *Every G-metric (X,G) defines a metric space (X,d_G) by*
1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.
if (X,G) is a symmetric G- metric space. Then
2) $d_G(x,y) = 2G(x,y,y)$.

Proposition 2.4. *Every modular G-metric (X,ω_{λ}) defines a metric space (X,d_{ω}) by*
1) $d_{\omega}(x,y) = \omega_{\lambda}(x,y,y) + \omega_{\lambda}(y,x,x)$.
if (X,ω_{λ}) is a symmetric modular G- metric space. Then
2) $d_{\omega}(x,y) = 2\omega_{\lambda}(x,y,y)$.

Proof: since ω_{λ} is modular G-metric on X by Theorem 2.2 X_{ω} is a $G-$metrice space with G-metric given by

$$G^{\circ}_{\omega} = inf\{\lambda > 0 : \omega_{\lambda}(x,y,z) \leq \lambda\},$$

hence by proposition 2.3 (X,d_{ω}) with $d_{\omega}(x,y) = \omega_{\lambda}(x,y,y) + \omega_{\lambda}(y,x,x)$ is a metric space.

Definition 2.5. [16] Let $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is an $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.

Remark 2.6. [16] *In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :*

$$AF(T) = \{x \in X \mid x \; is \; an \; \varepsilon - fixed \; point \; of \; T\}.$$

Definition 2.7. [16] Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \varepsilon > 0, \; AF(T) \neq \varnothing.$$

Theorem 2.8. [16] *Let $(X,\|.\|)$ be a complete norm space, $T : X \to X$, $x_0 \in X$ and $\varepsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

3. Approximate Fixed Point on Modular G-Metric Space

We begin with two lemmas which will be used in order to prove all the results given in third section. Let (X, ω_λ) be a modular G-metric space.

Definition 3.1. Let A, B, C are closed subsets of a modular G-metric space X and $T : A \cup B \cup C \to A \cup B \cup C$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A \cup B \cup C$. Then x_0 is an ε- fixed point of T if

$$[\omega_\lambda(x_0, Tx_0, Tx_0) + \omega_\lambda(Tx_0, x_0, x_0)] < \varepsilon.$$

Remark 3.2. In this paper we will denote the set of all ε-fixed points of T, for a given ε, by:

$$F^{\varepsilon}_{\omega_\lambda}(T) = \{x \in A \cup B \cup C \mid x \; is \; an \; \varepsilon - fixed \; point \; of \; T\}.$$

Definition 3.3. [18] Let A, B, C are closed subsets of a $G-metric$ space X, $T : A \cup B \cup C \to A \cup B \cup C$ be a cyclic map and $\varepsilon > 0$. We define diameter of the set $F^{\varepsilon}_{G}(T)$, i.e.,

$$\delta(F^{\varepsilon}_{G}(T)) = \sup\{G(x, y, z) : \; x, y, z \in F^{\varepsilon}_{G}(T)\}.$$

Definition 3.4. Let A, B, C are closed subsets of a modular $G-metric$ space X, $T : A \cup B \cup C \to A \cup B \cup C$ be a cyclic map and $\varepsilon > 0$. We define diameter of the set $F^{\varepsilon}_{\omega_\lambda}(T)$, i.e.,

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) = \sup\{\omega_\lambda(x, y, z) : \; x, y, z \in F^{\varepsilon}_{\omega_\lambda}(T)\}.$$

Definition 3.5. Let A, B, C are closed subsets of a modular $G-metric$ space X and $T : A \cup B \cup C \to A \cup B \cup C$ be a cyclic map. Then T has the approximate fixed point property (a.f.p.p) if $\forall \varepsilon > 0$,

$$F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset.$$

Definition 3.6. Let A, B, C are closed subsets of a modular $G-metric$ space X. A cyclic map $T : A \cup B \cup C \to A \cup B \cup C$ is said to be asymptotically regular at a point $x \in A \cup B \cup C$, if

$$\lim_{n \to \infty} \{\omega_\lambda(T^n x, T^{n+1} x, T^{n+1} x) + \omega_\lambda(T^{n+1} x, T^n x, T^n x)\} = 0,$$

where T^n denotes the nth iterate of T at x.

Lemma 3.7. [17] *Let A,B,C are closed subsets of a $G-metric$ space X. If $T: A\cup B\cup C \to A\cup B\cup C$ is asymptotically regular at a point $x\in A\cup B\cup C$, Then T has an approximate fixed point.*

Lemma 3.8. *Let A,B,C are closed subsets of a modular $G-metric$ space X. If $T: A\cup B\cup C \to A\cup B\cup C$ is asymptotically regular at a point $x\in A\cup B\cup C$, Then T has an approximate fixed point.*

Proof: Using Proposition 2.4 and Lemma 3.7, we find that T has an approximate fixed point

Lemma 3.9. *Let A,B,C are closed subsets of a modular $G-metric$ space X, $T: A\cup B\cup C \to A\cup B\cup C$ a cyclic map and $\varepsilon > 0$. We assume that:*
a) $F^{\varepsilon}_{\omega_\lambda}(T)\neq\emptyset$;
b) $\forall\xi>0\ \exists\psi(\xi)>0$ *such that*

$$[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]<\xi \Longrightarrow$$

$$\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)\leq\psi(\xi),\quad \forall x,y\in F^{\varepsilon}_{\omega_\lambda}(T).$$

Then:

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T))\leq\psi(2\varepsilon).$$

Proof. Let $\varepsilon>0$ and $x,y\in F^{\varepsilon}_{\omega_\lambda}(T)$. Then

$$[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]<\varepsilon,\ [\omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y)]<\varepsilon.$$

By $G5$ of Definition 2.1 we can write:

$$\begin{aligned}\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x) &\leq \omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)\\ &+\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)\\ &+\omega_\lambda(Ty,y,y)+\omega_\lambda(y,Ty,Ty)\\ &\leq 2\varepsilon+\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx). \Longrightarrow\end{aligned}$$

$$\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]<2\varepsilon.$$

Now by (b) it follow that

$$\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x) \leq \psi(2\varepsilon),$$

So

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \psi(2\varepsilon).$$

□

4. Diameter Approximate Fixed Point on Modular G-Metric Spaces

In this section a series of qualitative and quantitative results will be obtained regarding the diameter approximate fixed point. Also, we prove diameter approximate fixed point theorems for a new class of operators on modular G-metric spaces.

Definition 4.1. Let A,B and C be non-empty subsets of a modular $G-$metric space X. The cyclic mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a modular $G-$ MN operator if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$\begin{aligned}\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq \alpha[&\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)\\ &+\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)].\end{aligned}$$

Theorem 4.2. *Let A,B and C be non-empty subsets of a modular G-metric space X. Suppose that the cyclic mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a modular $G-MN$ operatore. Then for every $\varepsilon > 0$,*

$$F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ and $x \in A\cup B\cup C$.

$$\begin{aligned}\omega_\lambda(T^n x,T^{n+k}x,T^{n+k}x)+\omega_\lambda(T^{n+k}x,T^n x,T^n x) &= \omega_\lambda(T(T^{n-1}x),T(T^{n+k-1}x),T(T^{n+k-1}x))\\ &\quad+\omega_\lambda(T(T^{n+k-1}x),T(T^{n-1}x),T(T^{n-1}x))\\ &\leq \alpha[\omega_\lambda(T^{n-1}x,T^{n+k-1}x,T^{n+k-1}x)\\ &\quad+\omega_\lambda(T^{n+k-1}x,T^{n-1}x,T^{n-1}x)\\ &\quad+\omega_\lambda(T^n x,T^{n+k}x,T^{n+k}x)+\omega_\lambda(T^{n+k}x,T^n x,T^n x)].\end{aligned}$$

Therefore,

$$(1-\alpha)\omega_\lambda(T^nx,T^{n+k}x,T^{n+k}x)+\omega_\lambda(T^{n+k}x,T^nx,T^nx)]\leq$$
$$\alpha[\omega_\lambda(T^{n-1}x,T^{n+k-1}x,T^{n+k-1}x)$$
$$+\omega_\lambda(T^{n+k-1}x,T^{n-1}x,T^{n-1}x)].$$

So,

$$\begin{aligned}\omega_\lambda(T^nx,T^{n+k}x,T^{n+k}x)+\omega_\lambda(T^{n+k}x,T^nx,T^nx)]\leq&\\ &\frac{\alpha}{1-\alpha}[\omega_\lambda(T^{n-1}x,T^{n+k-1}x,T^{n+k-1}x)\\ &+\omega_\lambda(T^{n+k-1}x,T^{n-1}x,T^{n-1}x)]\\ &\vdots\\ &\leq(\frac{\alpha}{1-\alpha})^n[\omega_\lambda(x,T^kx,T^kx)+\omega_\lambda(T^kx,x,x)].\end{aligned}$$

But $\alpha\in(0,\frac{1}{2})$, therefore $(\frac{\alpha}{1-\alpha})\in(0,1)$. Hence

$$\lim_{n\longrightarrow\infty}[\omega_\lambda(T^nx,T^{n+k}x,T^{n+k}x)+\omega_\lambda(T^{n+k}x,T^nx,T^nx)]=0,\forall x\in A\cup B\cup C.$$

Using proposition 2.4 and Theorem 2.8, we find that $F^{\varepsilon}_{\omega_\lambda}(T)\neq\emptyset$ for all $\varepsilon>0$. $\square$

Theorem 4.3. *Let (X,G) be a modular G-metric space. Suppose that the cyclic mapping $T:A\cup B\cup C\to A\cup B\cup C$ is a modular $G-$ MN operator. Then for every $\varepsilon>0$,*

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T))\leq\frac{2\varepsilon(1+\alpha)}{1-2\alpha}.$$

Proof. Let $\varepsilon>0$. Condition i) in Lemma 3.9 is satisfied, as one can see in the proof of Theorem 4.2 we only verify that condition ii) in Lemma 3.9 holds. Let $\theta>0$ and $x,y\in F^{\varepsilon}_{\omega_\lambda}(T)$ and assume that Then:

$$[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]<\theta.$$

Then:

$$[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]\quad\leq\quad\alpha[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]+[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]+\theta.$$

Therefore As $x,y \in F^{\varepsilon}_{\omega_\lambda}(T)$, we know that

$$\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x) \leq \varepsilon, \omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y) \leq \varepsilon.$$

Therfore, $\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x) \leq \frac{2\alpha\varepsilon+\theta}{1-2\alpha}$. So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha\varepsilon+\theta}{1-2\alpha} > 0$ such that

$$[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] < \theta \Rightarrow \omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)] \leq \phi(\theta).$$

Now by Lemma 3.9, it follows that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \phi(2\varepsilon), \forall\varepsilon > 0,$$

which means exactly that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}.$$

□

Example 4.4. Let $X = \{0,1,2,...,18\}$, $\lambda \in (0,\infty)$ and $\omega_\lambda(x,y,y) = \frac{G(x,y,z)}{\lambda}$ such taht $G : X \times X \times X \to \mathrm{R}^+$ be defind as follows:

$$G(x,y,z) = \begin{cases} x+y+z & if\ x \neq y \neq z \neq 0, \\ x+y & if\ x = y \neq z\ x,y,z \neq 0, \\ y+z+1 & if\ x = 0,\ y \neq z,\ y,z \neq 0, \\ y+2 & if\ x = 0,\ y = z \neq 0, \\ z+1 & if\ x = 0,\ y = 0,\ z \neq 0, \\ 0 & if\ x = y = z. \end{cases}$$

Let $A = \{4,18\}$, $B = \{3,7,17\}$ and $C = \{0\}$. Obviously A,B,C are closed subsets of modular $G-metric$ space X. Define the mapping $T : A\cup B\cup C \to A\cup B\cup C$ by

$$Tx = \begin{cases} x-1 & if\ x \in \{4,18\} \\ 0 & if\ x \in \{3,7,17\} \\ 4 & if\ x = 0. \end{cases}$$

It is easily to be checked that $T(A) \subseteq B$, $T(B) \subseteq A$ and $T(C) \subseteq A$. For any $x, y \in A \cup B \cup C$ we have the chain of inequalities

$$\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)] \leq \alpha[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x) + \omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)].$$

So T satisfies all the conditions of Theorems 4.2 and 4.3 and thus for every $\varepsilon > 0$, $F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset$ and $\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}$ respectively.

Definition 4.5. Let A, B and C be non-empty subsets of a modular G-metric space X. The cyclic mapping $T : A \cup B \cup C \to A \cup B \cup C$ is a modular $G_\alpha-$contraction if for all $\lambda > 0$ there exists $\alpha \in (0,1)$ such that

$$\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx) \leq \alpha[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x)], \forall x,y \in A \cup B \cup C.$$

Definition 4.6. Let A, B and C be non-empty subsets of a modular $G-$metric space X. The cyclic mapping $T : A \cup B \cup C \to A \cup B \cup C$ is a modular G-Chatterjea operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)] \leq \alpha[\omega_\lambda(x,Ty,Ty) + \omega_\lambda(Ty,x,x) + \omega_\lambda(y,Tx,Tx) + \omega_\lambda(Tx,y,y)], \forall x,y \in A \cup B \cup C.$$

By combining the three independent contraction conditions: modular $G_\alpha-$contraction, MN, and Chatterjea operators we will be obtained another approximate fixed point result for operators which satisfy the followings:

Definition 4.7. Let A, B and C be non-empty subsets of a modular $G-$metric space X. The cyclic mapping $T : A \cup B \cup C \to A \cup B \cup C$ is a modular $G-$MNC operator if there exists $\alpha \in [0,1)$, $\beta \in [0, \frac{1}{2})$, $\gamma \in [0, \frac{1}{2})$ such that for all $x, y \in A \cup B \cup C$ at least one of the following is true.

$i)[\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)] \leq \alpha[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x)];$

$ii)\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx) \leq \beta[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x) + \omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)];$

$iii)\omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx)] \leq \gamma[\omega_\lambda(x,Ty,Ty) + \omega_\lambda(Ty,x,x) + \omega_\lambda(y,Tx,Tx) + \omega_\lambda(Tx,y,y)].$

Theorem 4.8. *Let A,B and C be non-empty subsets of a modular G-metric space X. Suppose that the cyclic mapping $T: A\cup B\cup C \to A\cup B\cup C$ is a modular $G-MNC$ operator. Then for every $\varepsilon>0$, $F^{\varepsilon}_{\omega_\lambda}(T)\neq\emptyset$.*

Proof. Let $\varepsilon>0$ and $x\in A\cup B\cup C$. Supposing (ii) holds, we have that:

$$\begin{aligned}
[G(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] &\leq \beta[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)+\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]\\
&\leq \beta[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)+\omega_\lambda(Tx,y,y)+\omega_\lambda(y,Tx,Tx)\\
&+ \omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]\\
&= 2\beta[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]+\beta[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]\\
&+ \beta[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)].
\end{aligned}$$

Thus,

$$\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] &\leq 2\beta[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]\\
&+ (\frac{\beta}{1-\beta})[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]. \qquad (3.1)
\end{aligned}$$

Supposing (iii) holds, we have that:

$$\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] &\leq \gamma[\omega_\lambda(x,Ty,Ty)+\omega_\lambda(Ty,x,x)+\omega_\lambda(y,Tx,Tx)+\omega_\lambda(Tx,y,y)].\\
&\leq \gamma[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)+\omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y)]\\
&+\gamma[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]\\
&=\gamma[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]+2\gamma[\omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y)]\\
&+\gamma[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)].
\end{aligned}$$

Thus,

$$\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] &\leq \frac{2\gamma}{1-\gamma}[\omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y)]\\
&+ (\frac{\gamma}{1-\gamma})[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]. \qquad (3.2)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]\\
&\leq \gamma[\omega_\lambda(x,Ty,Ty)+G(Ty,x,x)+\omega_\lambda(y,Tx,Tx)+\omega_\lambda(Tx,y,y)].\\
&\leq \gamma[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)+\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)]\\
&+\gamma[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)+\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]\\
&=2\gamma[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]+\gamma[\omega_\lambda(Tx,Ty,Ty)\\
&+\omega_\lambda(Ty,Tx,Tx)]+\gamma[G(x,y,y)+\omega_\lambda(y,x,x)].
\end{aligned}
$$

Then

$$
\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] &\leq (\frac{2\gamma}{1-\gamma})[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]\\
&+ (\frac{\gamma}{1-\gamma})[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]. \qquad (3.3)
\end{aligned}
$$

In view of (i), (3.1),(3.1), (3.2) and (3.3), we have, $\xi = max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and it is easy to see that $\xi \in [0,1)$ for T satisfying at least one of the condition (i), (ii) and (iii) we have that.

$$
\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq 2\xi[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)]\\
+\xi[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)] \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq 2\xi[\omega_\lambda(y,Ty,Ty)+\omega_\lambda(Ty,y,y)]\\
+\xi[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]
\end{aligned}
$$

hold. Using these conditions implied by i) -iii) and taking $x \in A\cup B\cup C$, we have

$$
\begin{aligned}
& \omega_\lambda(T^nx,T^{n+1}x,T^{n+1}x)+\omega_\lambda(T^{n+1}x,T^nx,T^nx)\\
= \quad & \omega_\lambda(T(T^{n-1}x),T(T^nx),T(T^nx))\\
+ \quad & \omega_\lambda(T(T^nx),T(T^{n-1}x),T(T^{n-1}x))\\
\leq^{(3.4)} \quad & 2\xi[\omega_\lambda(T^{n-1}x,T(T^{n-1}x),T(T^{n-1}x))\\
+ \quad & \omega_\lambda(T(T^{n-1}x),T^{n-1}x,T^{n-1}x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \xi\omega_\lambda(T^{n-1}x, T^n x, T^n x) + \omega_\lambda(T^n x, T^{n-1}x, T^{n-1}x)] \\
&= \quad 3\xi[\omega_\lambda(T^{n-1}x, T^n x, T^n x) + \omega_\lambda(T^n x, T^{n-1}x, T^{n-1}x)] \\
&\vdots \\
&\leq \quad (3\xi)^n[\omega_\lambda(x, Tx, Tx) + \omega_\lambda(Tx, x, x)].
\end{aligned}
$$

Therefore,

$$\omega_\lambda(T^n x, T^{n+1}x, T^{n+1}x) + \omega_\lambda(T^{n+1}x, T^n x, T^n x) \leq (3\xi)^n[\omega_\lambda(x, Tx, Tx) + \omega_\lambda(Tx, x, x)].$$

Then, we have

$$\lim_{n\to\infty}[\omega_\lambda(T^n x, T^{n+1}x, T^{n+1}x) + \omega_\lambda(T^{n+1}x, T^n x, T^n x)] = 0, \ \forall x \in A \cup B \cup C.$$

Using Lemma 3.8, we find that $F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset$ for all $\varepsilon > 0$. $\square$

Theorem 4.9. *Let (X, G) be a modular G-metric space. Suppose that the cyclic mapping $T : A \cup B \cup C \to A \cup B \cup C$ is a modular $G-MNC$ operator. Then for every $\varepsilon > 0$,*

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq 2\varepsilon\frac{1+\eta}{1-\eta},$$

where $\eta = max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α, β, γ as in Definition 4.7

Proof. In the proof of Theorem 4.8, we have already shown that if T satisfies at least one of the conditions $i)$, $ii)$, $iii)$ from Definition 4.7, then

$$
\begin{aligned}
[\omega_\lambda(Tx, Ty, Ty) + \omega_\lambda(Ty, Tx, Tx)] \quad &\leq \quad 2\eta[\omega_\lambda(x, Tx, Tx) + \omega_\lambda(Tx, x, x)] \\
&+ \quad \eta[\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x)]
\end{aligned}
$$

and

$$
\begin{aligned}
[\omega_\lambda(Tx, Ty, Ty) + \omega_\lambda(Ty, Tx, Tx)] \quad &\leq \quad 2\eta[\omega_\lambda(y, Ty, Ty) + \omega_\lambda(Ty, y, y)] \\
&+ \quad \eta[\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x)]
\end{aligned}
$$

hold.

Let $\varepsilon > 0$. We will only verify that condition ii) in Lemma 3.9 is satisfied, as i) holds, see the Proof of Theorem 4.8. Let $\theta > 0$ and $x,y \in F^{\varepsilon}_{\omega_\lambda}(T)$ and assume that

$$\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x) - \omega_\lambda(Tx,Ty,Ty) - \omega_\lambda(Ty,Tx,Tx) \leq \theta \Rightarrow$$

$$\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x) \leq \omega_\lambda(Tx,Ty,Ty) + \omega_\lambda(Ty,Tx,Tx) + \theta \Rightarrow$$

$$\begin{aligned}\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x) &\leq 2\eta[\omega_\lambda(x,Tx,Tx) + \omega_\lambda(Tx,x,x)] \\ &+ \eta[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x)] + \theta \Rightarrow\end{aligned}$$

$$(1-\eta)[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x)] \leq 2\eta\varepsilon + \theta$$

$$[\omega_\lambda(x,y,y) + \omega_\lambda(y,x,x)] \leq \frac{2\eta\varepsilon + \theta}{1-\eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\eta\varepsilon+\theta}{1-\eta} > 0$ such that

$$\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq \theta \Rightarrow \omega_\lambda(x,y,y)+\omega_\lambda(y,x,x) \leq \phi(\theta).$$

Now by Lemma 3.9, it follows that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq 2\varepsilon\frac{1+\eta}{1-\eta}, \forall \varepsilon > 0.$$

□

Remark 4.10. Example 4.4 shows that T satisfies all the conditions of Theorems 4.8 and 4.9 and thus for every $\varepsilon > 0, F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset$ and $\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq 2\varepsilon\frac{1+\eta}{1-\eta}$ respectively.

Definition 4.11. Let A,B and C be non-empty subsets of a moudlar $G-$metric space X. The cyclic mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a modular $G-$MN-semi contraction if there exists $\alpha \in]0, \frac{1}{2}[$ such that

$$[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq \alpha[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)]+[\omega_\lambda(x,Tx,Tx)+\omega_\lambda(Tx,x,x)].$$

Theorem 4.12. *Let A, B and C be non-empty subsets of a modular G-metric space X. Suppose that the cyclic mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a modular $G-$MN-semi contraction. Then for every $\varepsilon > 0$, $F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset$.*

Proof. Let $\varepsilon > 0$ and $x \in A\cup B\cup C$.

$$\begin{aligned}
[\omega_\lambda(T^n x, T^{n+1}x, T^{n+1}x) + \omega_\lambda(T^{n+1}x, T^n x, T^n x)] &= [\omega_\lambda(T(T^{n-1})x, T(T^n)x, T(T^n)x) \\
&+ \omega_\lambda(T(T^n)x, T(T^{n-1})x, T(T^{n-1})x)] \\
&\leq 2\alpha[\omega_\lambda(T^{n-1}x, T^n x, T^n x) \\
&+ \omega_\lambda(T^n x, T^{n-1}x, T^{n-1}x)] \\
&\leq \\
&\vdots \\
&\leq (2\alpha)^n[\omega_\lambda(x, Tx, Tx) + \omega_\lambda(Tx, x, x)].
\end{aligned}$$

But $\alpha \in]0, \frac{1}{2}[$. Therfore

$$\lim_{n\to\infty}[\omega_\lambda(T^n x, T^{n+1}x, T^{n+1}x) + \omega_\lambda(T^{n+1}x, T^n x, T^n x)] = 0,\ \forall x \in A\cup B\cup C.$$

Now by Lemma 3.8, it follows that $F^{\varepsilon}_{\omega_\lambda}(T) \neq \emptyset, \forall \varepsilon > 0$. □

Theorem 4.13. *Let (X, G) be a modular G-metric space. Suppose that the cyclic mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a modular $G-$MNC-semi contraction. Then for every $\varepsilon > 0$,*

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \varepsilon\frac{2+\alpha}{1-\alpha}.$$

Proof. Let $\varepsilon > 0$. We will only verify that condition ii) in Lemma 3.9 is satisfied. Let $\theta > 0$ and $x, y \in F^{\varepsilon}_{\omega_\lambda}(T)$ and assume that

$$\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x) - \omega_\lambda(Tx, Ty, Ty) - \omega_\lambda(Ty, Tx, Tx) \leq \theta \Rightarrow$$

$$\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x) \leq \omega_\lambda(Tx, Ty, Ty) + \omega_\lambda(Ty, Tx, Tx) + \theta \Rightarrow$$

$$\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x) \leq \alpha[\omega_\lambda(x, y, y) + \omega_\lambda(y, x, x) + \omega_\lambda(x, Tx, Tx) + \omega_\lambda(Tx, x, x)] + \theta \Rightarrow$$

$$(1-\alpha)[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)] \leq 2\alpha\varepsilon+\theta$$

$$[\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)] \leq \frac{2\alpha\varepsilon+\theta}{1-\alpha}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha\varepsilon+\theta}{1-\alpha} > 0$ such that

$$\omega_\lambda(x,y,y)+\omega_\lambda(y,x,x)-[\omega_\lambda(Tx,Ty,Ty)+\omega_\lambda(Ty,Tx,Tx)] \leq \theta \Rightarrow \omega_\lambda(x,y,y)+\omega_\lambda(y,x,x) \leq \phi(\theta).$$

Now by Lemma 3.9, it follows that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \phi(2\varepsilon), \forall\varepsilon > 0,$$

which means exactly that

$$\delta(F^{\varepsilon}_{\omega_\lambda}(T)) \leq \varepsilon\frac{2+\alpha}{1-\alpha}, \forall\varepsilon > 0.$$

□

Remark 4.14. Example 4.4 shows that T satisfies all the conditions of Theorems 4.12 and 3.9 and thus for every $\varepsilon > 0, F^{\varepsilon}_{G}(T) \neq \emptyset$ and $\delta(F^{\varepsilon}_{G}(T)) \leq \varepsilon\frac{2+\alpha}{1-\alpha}$ respectively.

5. Conclusions

Nowadays, fixed points and approximate fixed points play an important role in different areas of mathematics and its applications, particularly in mathematics, physics, differential equation, game theory, and dynamic programming.

In this work, we introduced the new classes of operators and contraction maps and gave results about approximate fixed points and diameter approximate fixed point on moudlar G-metric spaces. Also, by using two general lemmas regarding approximate fixed Point of cyclic maps on moudlar G-metric spaces we proved several approximate fixed point theorems and diameter approximate fixed point for a new class of operators and contraction mapping on moudlar G-metric spaces. We accompanied our theoretical results by some applied examples.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] H. Aydi, M. Postolache, W. Shatanawi, Coupled fixed point results for $(\psi,\varphi)-$weakly contractive mappings in ordered G-metric spaces, Comput. Math. Appl. 63 (2012), 298-309.

[2] B. Azadifar, M. Maramaei, and Gh. Sadeghi, On the modular G-metric spaces and fixed point theorems, J. Nonlinear Sci. Appl. 6 (2013), 293-304.

[3] V. Berinde, On the Approximation of Fixed Points of Weak Contractive Mappings, Carpathian J. Math. 19 (2003), 7-22.

[4] M. Berinde, Approximate fixed point theorems. Studia univ. "babes- bolyai", Mathmatica, Volume LI, Number 1, March 2006.

[5] S. Chandok, Z. Mustafa, M. Postolache, Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces, U. Politeh. Buch. Ser. A, 75 (2013), 13-26.

[6] S.K. Chatterjea, Fixed-point Theorems, C.R. Acad. Bulgare Sci. 25 (1972), 727-730.

[7] V. Chistyakov, Modular metric spaces generated by F-modulars, Folia Math. 14 (2008), 3-25.

[8] V.V. Chistyakov, Modular metric spaces I Basic concepts, Nonlinear Anal. 72 (2010), 1-14.

[9] P. Harjulehto, P. Hasto, M. Koskenoja, S. Varonen, The dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, Potent. Anal. 25 (2006), 205–222.

[10] J. Heinonen, T. Kilpelinen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, Oxford, UK, 1993.

[11] R. Kannan, Some Results on Fixed Points, Bull. Calcutta Math. Soc. 10 (1968), 71-76.

[12] M.A. Khamsi, W.K. Kozlowski, S. Reich, Fixed point theory in modular function spaces, Nonlinear Anal. 14 (1990), 935–953.

[13] M.A. Miandaragh, M. Postolache, Sh. Rezapour, Some approximate fixed point results for generalized α-contractive mappings, U. Politeh. Buch. Ser. A, 75 (2013), 3-10.

[14] M.A. Miandaragh, M. Postolache, Sh. Rezapour, Approximate fixed points of generalized convex contractions, Fixed Point Theory Appl. 2013 (2013), 255.

[15] S.A.M. Mohsenalhosseini, H. Mazaheri, M.A. Dehghan, Approximate best proximity pairs in metric space, Abstr. Appl. Anal. 2011 (2011), 596971.

[16] S.A.M. Mohsenalhosseini, H. Mazaheri, Fixed Point for Completely Norm Space and Map T_α, Math. Moravica. 16 (2012), 25-35.

[17] S.A.M. Mohsenalhosseini, T. Ahmadi, Approximate fixed point in $G-$metric spaces for various types of operators, J. Math. Comput. Sci. 6 (2016), 767-782.

[18] S.A.M. Mohsenialhosseini, Approximate fixed points of operators on G-metric spaces, U.P.B. Sci. Bull., Ser. A, 79 (2017), 85-96.

[19] H. Nakano, On the stability of functional equations, Aequat. Math. 77 (2009), 33-88.

[20] Z. Mustafa, B. Sims, A new approach to a generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[21] Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, in: Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia, Spain, July 2003, pp. 189-198.

[22] T. Zamfirescu, Fixed point theorems in metric spaces, Arch. Math. (Basel), 23 (1972), 292-298.

[23] W. Orlicz, Collected Papers, Vols. I, II, PWN, Warszawa, 1988.

[24] W. Shatanawi, M. Postolache, Some fixed point results for a G-weak contraction in G-metric spaces, Abstr. Appl. Anal. 2012 (2012), 815870.

[25] Y.N. Raffoul, Stability in functional difference equations using fixed point theory, Commun. Korean Math. Soc. 29 (2014), 195–204.

[26] I.A. Rus, Generalized contractions and applications, Cluj University Press, Cluj-Napoca, 2001.