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CONVERGENCE THEOREMS OF MODIFIED TWO-STEP ITERATION PROCESS FOR TWO ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we establish a weak convergence theorem and some strong convergence theorems of modified two-step iteration process for two asymptotically quasi-nonexpansive mappings to converge to common fixed points in the setting of real Banach spaces. The results presented in this paper extend, improve and generalize some previous results from the existing literature.

Keywords: Asymptotically quasi-nonexpansive mapping, modified two-step iteration process, common fixed point, strong convergence, weak convergence, Banach space.

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1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, $y \in K$;

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(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$;

(iii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\|$$

for all x, $y \in K$ and $n \geq 1$;

(iv) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \quad \leq \quad k_n \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L \|x - y\|$$

for all x, $y \in K$ and $n \geq 1$.

From the above definitions, it is clear that each of a nonexpansive, a quasi-nonexpansive, an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping. However, the converse of each of above statements may be not true.

The class of asymptotically nonexpansive mappings which is an important generalization of that nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. Moreover, the set $F(T)$ of fixed points of T is closed and convex.

In 1991, Schu [11] introduced the following Mann-type iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n x_n, \end{aligned} \tag{1}$$

where $T \colon K \to K$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition

$\delta \leq \alpha_n \leq 1-\delta$ for all $n \geq 1$ for some $\delta > 0$. Hence conclude that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see, for example, [5, 6, 8, 9, 10, 11, 12, 18] and references therein).

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Later on, many authors, for example Khan and Takahashi [8], Shahzad and Udomene [13] and Takahashi and Tamura [17] have studied the two mappings case of iterative schemes for different types of mappings.

Ishikawa-type iteration process

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1, \end{aligned} \tag{3}$$

for two mappings has also been studied by many authors including [3], [8], [16].

Recently, Khan et al. [7] modified the iteration process (2) to the case of two mappings as follows:

$$
\begin{aligned}
x_1 &= x \in K, \\
x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\
y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \; n \geq 1,
\end{aligned} \tag{4}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Remarks. (i) Note that (4) reduces to (2) when $S = T$. Similarly, the process (4) reduces to (1) when $T = I$.

(ii) The process (2) does not reduce to (1) but (4) does. Thus (4) not only covers the results proved by (2) but also by (1) which are not covered by (2).

(iii) The process (4) is independent of (3) neither of them reduces to the other.

In this paper, we prove some strong convergence theorems for two asymptotically quasi-nonexpansive mappings using iteration process (4) in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize several known results given in the existing literature.

2. Preliminaries

For the sake of convenience, we restate the following concepts.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \; \|y\| = 1, \; \varepsilon = \|x-y\| \right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Two mappings $S, T\colon K \to K$, where K is a subset of a normed space E, are said to satisfy the condition (A') [4] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,F))$

or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\big\{\|x - p\| : p \in F = F(S) \cap F(T)\big\}$.

A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

A Banach space E has the Kadec-Klee property [14] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Now, we state the following useful lemmas to prove our main results:

Lemma 2.1. (See [11]) Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2. (See [15]) Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of nonnegative numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. If one of the following conditions is satisfied:

1. $\alpha_{n+1} \leq \alpha_n + \beta_n$, $n \geq 1$,
2. $\alpha_{n+1} \leq (1+\beta_n)\alpha_n$, $n \geq 1$,

then the limit $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.3. (See [19]) Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 2.4. (See [14]) Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and p, $q \in W_w(x_n)$ (where $W_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty}$ $\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.

Lemma 2.5. (See [14]) Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each Lipschitzian mapping $T\colon K \to K$ with the Lipschitz constant L,

$$\|tTx + (1-t)Ty - T(tx + (1-t)y)\| \leq L\phi^{-1}\Big(\|x - y\| - \frac{1}{L}\|Tx - Ty\| \Big)$$

for all x, $y \in K$ and all $t \in [0,1]$.

3. Strong Convergence Theorems

In this section, we prove some strong convergence theorems of the iteration scheme (4) for two asymptotically quasi-nonexpansive mappings in the framework of real Banach spaces. In the sequel, we need the following lemma in order to prove our main theorems.

Lemma 3.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by (4). Then $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$.*

Proof. Let $q \in F$. Then from (4), we have

$$\begin{aligned}
\|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n \|T^n x_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n l_n \|x_n - q\| \\
&\leq (1-\beta_n) l_n \|x_n - q\| + \beta_n l_n \|x_n - q\| \\
&\leq l_n \|x_n - q\|, \qquad (5)
\end{aligned}$$

which implies that

$$\|y_n - q\| \leq l_n \|x_n - q\| \leq k_n l_n \|x_n - q\|. \qquad (6)$$

Again using (4) and (6), we obtain

$$\begin{aligned}
\|x_{n+1}-q\| &= \|(1-\alpha_n)T^n x_n+\alpha_n S^n y_n-q\| \\
&\le (1-\alpha_n)\|T^n x_n-q\|+\alpha_n\|S^n y_n-q\| \\
&\le (1-\alpha_n)l_n\|x_n-q\|+\alpha_n k_n\|y_n-q\| \\
&\le (1-\alpha_n)k_n l_n\|x_n-q\|+\alpha_n k_n l_n\|y_n-q\| \\
&\le (1-\alpha_n)k_n l_n\|x_n-q\|+\alpha_n k_n l_n[k_n l_n\|x_n-q\|] \\
&\le (1-\alpha_n)k_n^2 l_n^2\|x_n-q\|+\alpha_n k_n^2 l_n^2\|x_n-q\| \\
&\le k_n^2 l_n^2\|x_n-q\| \\
&= [1+(k_n^2 l_n^2-1)]\|x_n-q\|. \qquad (7)
\end{aligned}$$

By putting $\theta_n=(k_n^2 l_n^2-1)$ the last inequality can be written as follows

$$\|x_{n+1}-q\| \le (1+\theta_n)\|x_n-q\|. \qquad (8)$$

By hypothesis of the theorem, we find

$$\begin{aligned}
\sum_{n=1}^{\infty}\theta_n &= \sum_{n=1}^{\infty}(k_n^2 l_n^2-1) \\
&= \sum_{n=1}^{\infty}(k_n l_n+1)(k_n l_n-1) \\
&\le (N_1N_2+1)\sum_{n=1}^{\infty}(k_n l_n-1)<\infty.
\end{aligned}$$

Denote $\alpha_n=\|x_n-q\|$ in (8) we have

$$\alpha_{n+1}\le(1+\theta_n)\alpha_n,$$

and since $\sum_{n=1}^{\infty}\theta_n<\infty$, by Lemma 2.2 we know that the limit $\lim_{n\to\infty}\alpha_n$ exists. This means the limit

$$\lim_{n\to\infty}\|x_n-q\|=r$$

exists, where $r\ge 0$ is some constant. This completes the proof.

Theorem 3.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by (4). Then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.*

Proof. The necessity of the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ is obvious. Let us prove the sufficiency part of the theorem. Since $S, T\colon K \to K$ are uniformly L-Lipschitzian mappings, so S and T are continuous mappings. Therefore the sets $F(S)$ and $F(T)$ are closed. Hence $F = F(S) \cap F(T)$ is a nonempty closed set.

For any given $q \in F$, we have

$$\|x_{n+1} - q\| \;\leq\; (1 + \theta_n)\,\|x_n - q\|\,. \tag{9}$$

where $\theta_n = (k_n^2 l_n^2 - 1)$ with $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have

$$d(x_{n+1}, F) \;\leq\; (1 + \theta_n) d(x_n, F). \tag{10}$$

From (10) and Lemma 2.2, we obtain that $\lim_{n\to\infty} d(x_n, F)$ exists.
By condition $\liminf_{n\to\infty} d(x_n, F) = 0$, we get

$$\lim_{n\to\infty} d(x_n, F) \;=\; \liminf_{n\to\infty} d(x_n, F) = 0. \tag{11}$$

Now, we will prove that the sequence $\{x_n\}$ converges to a common fixed point of the mappings S and T. In fact, due to $1 + x \leq exp(x)$ for all $x > 0$, and from (11), we obtain

$$\|x_{n+1} - q\| \;\leq\; \exp(\theta_n)\,\|x_n - q\|\,. \tag{12}$$

Hence for any positive integers m, n and from (12) with $\sum_{n=1}^{\infty}\theta_n < \infty$, we have

$$\begin{aligned}
\|x_{n+m}-q\| &\leq \exp(\theta_{n+m-1})\,\|x_{n+m-1}-q\| \\
&\leq \exp(\theta_{n+m-1})[\exp(\theta_{n+m-2})\,\|x_{n+m-2}-q\| \\
&\leq \exp(\theta_{n+m-1}+\theta_{n+m-2})\,\|x_{n+m-2}-q\| \\
&\leq \dots \\
&\leq \exp\Big(\sum_{k=n}^{n+m-1}\theta_k\Big)\,\|x_n-q\| \\
&\leq \exp\Big(\sum_{k=1}^{\infty}\theta_k\Big)\,\|x_n-q\|, \qquad (13)
\end{aligned}$$

which implies that

$$\|x_{n+m}-q\| \leq Q\,\|x_n-q\| \qquad (14)$$

for all $q \in F$, where $Q = \exp\big(\sum_{k=1}^{\infty}\theta_k\big) < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$, then for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_{n_0}, F) < \frac{\varepsilon}{Q}. \qquad (15)$$

Therefore there exists a $q_1 \in F$ such that

$$\|x_{n_0}-q_1\| < \frac{\varepsilon}{Q}. \qquad (16)$$

Consequently, for all $n \geq n_0$ from (14), we have

$$\begin{aligned}
\|x_n-q_1\| &\leq Q\,\|x_{n_0}-q_1\| \\
&< Q.\frac{\varepsilon}{Q} = \varepsilon, \qquad (17)
\end{aligned}$$

which implies the strong convergence limit of the sequence $\{x_n\}$ to a common fixed point q_1 of the mappings S and T. This completes the proof.

Theorem 3.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $F =$*

$F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Then $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$. Assume that $\lim_{n\to\infty} \|x_n - q\| = r$. If $r = 0$, the conclusion is obvious. Now suppose $r > 0$. We claim $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $x_n - q$, $y_n - q \in B_R(0)$ for all $n \geq 1$. Using (4) and Lemma 2.3, we have

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1-\beta_n)T^n x_n + \beta_n x_n - q\|^2 \\
&\leq (1-\beta_n)\|T^n x_n - q\|^2 + \beta_n \|x_n - q\|^2 \\
&\quad - W_2(\beta_n) g(\|T^n x_n - x_n\|) \\
&\leq (1-\beta_n)\|T^n x_n - q\|^2 + \beta_n \|x_n - q\|^2 \\
&\leq (1-\beta_n) l_n^2 \|x_n - q\|^2 + \beta_n \|x_n - q\|^2 \\
&\leq (1-\beta_n) l_n^2 \|x_n - q\|^2 + \beta_n l_n^2 \|x_n - q\|^2 \\
&\leq l_n^2 \|x_n - q\|^2. \qquad (18)
\end{aligned}$$

Again using (4), (18) and Lemma 2.3, we have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\|^2 \\
&\le (1-\alpha_n)\|T^n x_n - q\|^2 + \alpha_n \|S^n y_n - q\|^2 \\
&\quad -W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \\
&\le (1-\alpha_n) l_n^2 \|x_n - q\|^2 + \alpha_n k_n^2 \|y_n - q\|^2 \\
&\quad -W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \\
&\le (1-\alpha_n) l_n^2 \|x_n - q\|^2 + \alpha_n k_n^2 [l_n^2 \|x_n - q\|^2] \\
&\quad -W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \\
&\le (1-\alpha_n) k_n^2 l_n^2 \|x_n - q\|^2 + \alpha_n k_n^2 [l_n^2 \|x_n - q\|^2] \\
&\quad -W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \\
&\le k_n^2 l_n^2 \|x_n - q\|^2 - W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \\
&\le \|x_n - q\|^2 + \theta_n R^2 - W_2(\alpha_n) g(\|T^n x_n - S^n y_n\|) \qquad (19)
\end{aligned}
$$

where $\theta_n = (k_n^2 l_n^2 - 1)$.

Observe that $W_2(\alpha_n) \ge \delta^2$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. Now (19) implies that

$$
\delta^2 \sum_{n=1}^{\infty} g(\|T^n x_n - S^n y_n\|) < \|x_1 - q\|^2 + R^2 \sum_{n=1}^{\infty} \theta_n < \infty. \qquad (20)
$$

Therefore, we have $\lim_{n\to\infty} g(\|T^n x_n - S^n y_n\|) = 0$. Since g is strictly increasing and continuous at 0, it follows that

$$
\lim_{n\to\infty} \|T^n x_n - S^n y_n\| = 0. \qquad (21)
$$

Now taking limsup on both the sides of (6), we obtain

$$
\limsup_{n\to\infty} \|y_n - q\| \le r. \qquad (22)
$$

Since T is asymptotically quasi-nonexpansive, we can get that

$$
\|T^n x_n - q\| \le l_n \|x_n - q\|. \qquad (23)
$$

for all $n \geq 1$. Taking limsup on both the sides of (23), we obtain

$$\limsup_{n\to\infty} \|T^n x_n - q\| \leq r. \tag{24}$$

Now

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\| \\ &= \|(T^n x_n - q) + \alpha_n (S^n y_n - T^n x_n)\| \\ &\leq \|T^n x_n - q\| + \alpha_n \|S^n y_n - T^n x_n\| \end{aligned}$$

yields that

$$r \leq \liminf_{n\to\infty} \|T^n x_n - q\|. \tag{25}$$

So that (24) gives $\lim_{n\to\infty} \|T^n x_n - q\| = r$.

On the other hand, since S is asymptotically quasi-nonexpansive, we have

$$\begin{aligned} \|T^n x_n - q\| &\leq \|T^n x_n - S^n y_n\| + \|S^n y_n - q\| \\ &\leq \|T^n x_n - S^n y_n\| + k_n \|y_n - q\|, \end{aligned}$$

so we have

$$r \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{26}$$

By using (22) and (26), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = r. \tag{27}$$

Thus $r = \lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \|(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)\|$ gives by Lemma 2.1 that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{28}$$

Now

$$\|y_n - x_n\| = \beta_n \|T^n x_n - x_n\|.$$

Hence by (28), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{29}$$

Also note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - x_n\| \\ &\leq \|T^n x_n - x_n\| + \alpha_n \|T^n x_n - S^n y_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{30}$$

so that

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{31}$$

Furthermore, from

$$\begin{aligned} \|x_{n+1} - S^n y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\quad + \|T^n x_n - S^n y_n\| \end{aligned}$$

using (21), (28) and (30), we find that

$$\lim_{n\to\infty} \|x_{n+1} - S^n y_n\| = 0. \tag{32}$$

Then

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| \\ &\quad + \left\|T^{n+1}x_n - Tx_{n+1}\right\| \\ &\leq \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|x_{n+1} - x_n\| \\ &\quad + L\|T^n x_n - x_{n+1}\| \\ &= \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|x_{n+1} - x_n\| \\ &\quad + L\alpha_n\|T^n x_n - S^n y_n\| \end{aligned}$$

yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{33}$$

Now

$$\begin{aligned}
\|x_n - S^n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n y_n\| \\
&\quad + \|S^n y_n - S^n x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n y_n\| \\
&\quad + L\|y_n - x_n\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + \left\|S^{n+1}x_{n+1} - Sx_{n+1}\right\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\|S^n x_{n+1} - x_{n+1}\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\Big(\|S^n x_{n+1} - S^n y_n\| \\
&\quad + \|S^n y_n - x_{n+1}\|\Big) \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L^2\|x_{n+1} - y_n\| \\
&\quad + L\|S^n y_n - x_{n+1}\|
\end{aligned}$$

implies

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{34}$$

This completes the proof.

Theorem 3.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$.*

From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). If at least one of the mappings S and T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. Without loss of generality, we may assume that T is semi-compact. This with (33) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\} \to x^* \in K$. Since S and T are continuous, then from (33) and (34), we find

$$\|x^* - Tx^*\| \;=\; \lim_{n_k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0, \tag{35}$$

and

$$\|x^* - Sx^*\| \;=\; \lim_{n_k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0. \tag{36}$$

This shows that $x^* \in F = F(S) \cap F(T)$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n_k\to\infty} \|x_{n_k} - x^*\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$. This completes the proof.

Applying Theorem 3.1, we obtain strong convergence of the process (4) under the condition (A') as follows:

Theorem 3.4. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Let S and T satisfy the condition (A'), then the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. We proved in Theorem 3.2 that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \;\; \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{37}$$

From the condition (A') and (37), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Now all the conditions of Theorem 3.1 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a common fixed point of S and T. This completes the proof.

4. Weak Convergence Theorem

In this section, we prove a weak convergence theorem of the iteration process (4) in the framework of real uniformly convex Banach spaces.

Lemma 4.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n l_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Then $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all p, $q \in F$ and $t \in [0,1]$.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p-q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 4.1 for $t \in (0,1)$. For all $x \in K$, we define the mapping $G_n \colon K \to K$ by

$$G_n x = (1-\alpha_n)T^n x + \alpha_n S^n((1-\beta_n)x + \beta_n T^n x).$$

Then

$$\|G_n x - G_n y\| \leq \lambda_n \|x-y\|, \tag{38}$$

for all $x, y \in K$, where $\lambda_n = (1+\theta_n)$ and $\theta_n = (k_n^2 l_n^2 - 1)$ with $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\lambda_n \to 1$ as $n \to \infty$. Setting

$$S_{n,m} = G_{n+m-1}G_{n+m-2}\dots G_n, \quad m \geq 1 \tag{39}$$

and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)S_{n,m}q)\|. \tag{40}$$

From (38) and (39), we have

$$\begin{aligned} \|S_{n,m}x - S_{n,m}y\| &\leq \lambda_n\lambda_{n+1}\dots\lambda_{n+m-1}\|x-y\| \\ &\leq \Big(\prod_{i=n}^{n+m-1}\lambda_i\Big)\|x-y\| \\ &= H_n\|x-y\| \qquad (41) \end{aligned}$$

for all $x, y \in K$, where $H_n = \prod_{i=n}^{n+m-1}\lambda_i$ and $S_{n,m}x_n = x_{n+m}$, $S_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m} + H_n a_n(t). \qquad (42) \end{aligned}$$

It follows from (40), (41) and Lemma 2.5 that

$$b_{n,m} \leq H_n\phi^{-1}(\|x_n - p\| - H_n^{-1}\|x_{n+m} - p\|).$$

By Lemma 3.1 and $\lim_{n\to\infty} H_n = 1$, we have $\lim_{n,\,m\to\infty} b_{n,\,m} = 0$ and so

$$\limsup_{m\to\infty} a_m(t) \le \lim_{n,\,m\to\infty} b_{n,\,m} + \liminf_{n\to\infty} H_n a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$. This completes the proof.

Theorem 4.1. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $F = F(S) \cap F(T) \neq \emptyset$. Suppose $N_1 = \lim_n k_n \ge 1$ and $N_2 = \lim_n l_n \ge 1$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Theorem 3.2, we have

$$\lim_{n\to\infty} \left\|x_{n_j} - Sx_{n_j}\right\| = 0,\ \lim_{n\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0.$$

Since the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p,\, q \in w_w(x_n)$. By Lemma 4.1, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 2.4. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof.

Remark 4.1. Theorems of this paper can also be proved with error terms.

Remark 4.2. Our results extend, improve and generalize many known results given in the existing literature.

Example 1. Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x\, cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |x||cos\, x| \leq |x| = \|x - z\|\,,$$

and hence T is asymptotically quasi-nonexpansive mapping with constant sequence $\{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$\|Tx - Ty\| = \left\|\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi\right\| = \pi,$$

whereas

$$\|x - y\| = \left\|\frac{\pi}{2} - \pi\right\| = \frac{\pi}{2}.$$

Example 2. Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2}\, cos\, \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\, cos\, \frac{1}{x}$. Thus $2 = cos\, \frac{1}{x}$. This is not hold. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |\frac{x}{2}||cos\, \frac{1}{x}| \leq \frac{|x|}{2} < |x| = \|x - z\|\,,$$

and hence T is asymptotically quasi-nonexpansive mapping with constant sequence $\{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$\|Tx - Ty\| = \left\|\frac{1}{3\pi}\, cos\, \frac{3\pi}{2} - \frac{1}{2\pi}\, cos\, \pi\right\| = \frac{1}{2\pi},$$

whereas

$$\|x-y\| = \left\|\frac{2}{3\pi} - \frac{1}{\pi}\right\| = \frac{1}{3\pi}.$$

5. Conclusion

The class of asymptotically quasi-nonexpansive mapping is more general than the class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings. Hence the results presented in this paper are good improvement and generalization of the corresponding previous results from the existing literature (see, e.g., [6, 8, 10, 11, 13] and many others).

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