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A COMMON FIXED POINT THEOREM UNDER φ-CONTRACTIVE CONDITIONS

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Abstract. In this paper, common fixed point theorems for weakly compatible mappings under generalized φ-contractive condition without the concept of boundedness of orbit are obtained.

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1. Introduction and Preliminaries

Let (X,d) be a metric space. Two mappings $S,T : X \to X$ are said to satisfy quasi-contractive condition whenever there exists $h \in (0,1)$ such that

$$d(Tx,Ty) \leq h\max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)\} \tag{1.1}$$

for all $x,y \in X$. Das and Naik [5] proved common fixed point theorem for commuting mappings using the contractive condition (1.1). Two mappings $S,T : X \to X$ are said to satisfy generalized φ-contractive condition if

$$d(Tx,Ty) \leq \varphi(\max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)\}) \tag{1.2}$$

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for all $x, y \in X$ and $\varphi : R_+ \to R_+$ is continuous. Using this φ-contractive condition (1.2), Verinde [1,2] proved common fixed point theorems for weakly commuting mappings and compatible mappings. The contractive condition (1.1) is a special case of (1.2) when $\varphi(t) = ht$, where $0 \le h < 1$.

Definition 1.1. Let $\varphi : R_+ \to R_+$ be such that

(a) φ is nondecreasing upper semi continuous

(b) $\varphi(t) < t$ for $t > 0$.

If φ in (1.2) is defined in definition 1.1, then φ contractive condition due to Browder [3]

$$d(Tx,Ty) \le \varphi(\max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Sx,Ty) + d(Sy,Tx)]\}), \tag{1.3}$$

which implies (1.2) as $\max\{a,b,c,\frac{1}{2}(e+f)\} \le \max\{a,b,c,e,f\}$ for any real numbers a,b,c,e, and f. If $S = I$, the identity map, then (1.1) is reduced to

$$d(Tx,Ty) \le h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.4}$$

for $x, y \in X$, which is due to Ciric [4]. In proving theorems, Ciric [4], Das and Naik [5], Phaneendra [6], Verinde [2] etc. used the concept of orbit. The orbit of T is the set $O_T(x) = \{x, Tx, T^2x, ...\}$ and orbit of S and T is the set $\{y_1, y_2, ...\}$, where $Sx_n = Tx_{n+1} = y_n$. It was shown in [7] that the condition (1.4) does assure that the orbit of T is bounded. Also it is known from lemma 2.2 [5] that the condition (1.1) does assure that the orbit of S and T is bounded. Using (1.2), Verinde [2] proved the following theorem.

Theorem 1.2. *Let (X,d) be a complete metric space and $S,T : X \to X$ be two compatible mappings with bounded orbits. Suppose that T is continuous and satisfy the conditions*

$$d(Sx,Sy) \le \varphi(M(x,y)), \quad \forall x,y \in X, \tag{1.5}$$

where

$$M(x,y) = \max\{d(Tx,Ty), d(Tx,Sx), d(Ty,Sy), d(Tx,Sy), d(Ty,Sx)\}$$

with $\varphi : R_+ \to R_+$ a continuous function. If $S(X) \subset T(X)$, then T and S have a unique common fixed point.

It is an open question whether or not two mappings S and T satisfying (1.2) with $\varphi : R_+ \to R_+$ an arbitrary function have bounded orbits. Therefore, it is of interest to prove existence of common fixed point for two mappings with an arbitrary function $\varphi : R_+ \to R_+$. For this end, we need the following.

Definition 1.3. Let $\varphi : R_+ \to R_+$ be such that

(a) φ is nondecreasing upper semi continuous

(b) $\varphi(2t) < t$ for $t > 0$.

For $t > 0$, we conclude that $\varphi(2t) < t$, which implies that $\varphi(t) < t$ but not conversely. Let $\varphi : R_+ \to R_+$ be defined by $\varphi(t) = \frac{2}{3}t$. Then $\varphi(t) < t$ is true. In view of $\varphi(2t) = \frac{2}{3}2t = \frac{4}{3}t > t$, we find, $\varphi(t) < t \nRightarrow \varphi(2t) < t$.

In this work, we prove common fixed point theorems for two weakly compatible mappings using generalized φ-contractive condition (1.2) with φ as defined in Definition 1.3 and dropping the condition of boundedness of orbit. Also we extend our result to four weakly compatible mappings.

2. Main results

Theorem 2.1. *Let X be a complete metric space. Let $S,T : X \to X$ be two weakly compatible mappings such that $\overline{T(X)} \subset S(X)$ and satisfying*

$$d(Tx,Ty) \leq \varphi(\max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)\}), \quad \forall x,y \in X, \tag{2.1}$$

where φ as defined in definition (1.3). Then the mappings S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}, n = 0,1,2,....$ Let $d_n = d(Tx_n,Tx_{n+1}), n = 0,1,2,....$ Then, we find that

$$\begin{aligned} d_n &\leq \varphi(\max\{d(Sx_n,Sx_{n+1}),d(Tx_n,Sx_n),d(Tx_{n-1},Sx_{n-1}),d(Tx_n,Sx_{n-1}),d(Tx_{n-1},Sx_n)\}) \\ &\leq \varphi(\max\{d(Tx_n,Tx_{n-2}),d(Tx_n,Tx_{n-1}),d(Tx_{n-1},Tx_{n-2}),d(Tx_n,Tx_{n-2}),d(Tx_{n-1},Tx_{n-1})\}) \\ &\leq \varphi(d_n + d_{n+1}). \end{aligned}$$

Suppose $d_n > d_{n-1}$, then $d_n \leq \varphi(2d_n) < d_n$, which leads to a contradiction. Hence $d_n \leq d_{n-1}, n = 0,1,2,...$ Therefore $\{d_n\}$ is a decreasing sequence of positive number which is bounded below by zero. Therefore, we find that $\lim_{n\to\infty} d_n$ exists. Let $\lim_{n\to\infty} d_n = L$. Suppose $L > 0$. From $d_n \leq \varphi(2d_{n-1})$, we have $L \leq \varphi(2L) < L$, which is a contradiction. Hence $L = 0$. Thus, $\lim_{n\to\infty} d_n = 0$ i.e. $\lim_{n\to\infty} d(Tx_n,Tx_{n-1}) = 0$.

Now, we are in a position to show that $\{Tx_n\}$ and $\{Sx_n\}$ are Cauchy sequences in X. If $\{Tx_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ of positive integers with $m_i > n_i > i$ and

$$d(Tx_{m_i},Tx_{n_i}) \geq \varepsilon \tag{2.2}$$

for $i = 1,2,3,....$ Suppose m_i is the smallest integer exceeding n_i which satisfies (2.2), that is,

$$d(Tx_{m_i-1},Tx_{n_i}) < \varepsilon. \tag{2.3}$$

Notice that

$$\varepsilon \leq d(Tx_{m_i},Tx_{n_i}) \leq d(Tx_{m_i},Tx_{m_i-1}) + d(Tx_{m_i-1},Tx_{n_i}) < \varepsilon + d(Tx_{m_i},Tx_{m_i-1}).$$

Since $\lim_{n\to\infty} d(Tx_{n_i}, Tx_{n_i-1}) = 0$, we, therefore, find that $\lim_{n\to\infty} d(Tx_{n_i}, Tx_{m_i}) = \varepsilon$. Notice that

$$\begin{aligned}\lim_{n\to\infty} d\big(Tx_{n_i}, Tx_{m_i}) \leq & \varphi(\max\{d(Tx_{m_i-1}, Tx_{n_i-1}),\\ & d(Tx_{m_i-1}, Tx_{m_i}), d(Tx_{n_i-1}, Tx_{n_i})\\ & d(Tx_{m_{i-1}}, Tx_{n_i}), d(Tx_{n_i-1}, Tx_{m_i})\}\big).\end{aligned}$$

Since $d_n \leq d_{n-1}$ and $m_i > n_i$, we have $d(Tx_{m_i-1}, Tx_{m_i}) \leq d(Tx_{n_i-1}, Tx_{n_i})$.

Therefore, $d(Tx_{n_i}, Tx_{m_i}) \leq \varphi(\varepsilon + d(Tx_{n_i}, Tx_{n_i-1})$. Notice that φ is upper semi continuous and $\varphi(2t) < t$. Taking limit as $n_i \to \infty$, we have $\varepsilon \leq \varphi(\varepsilon) < \varepsilon$, a contradiction. Therefore $\{Tx_n\}$ is a Cauchy sequence in X. Similarly $\{Sx_n\}$ is also a Cauchy sequence in X. Then there exists a point $u \in X$ such that

$$\lim_{n\to\infty} Tx_n = u = \lim_{n\to\infty} Sx_n.$$

In view of $\overline{T(X)} \subset S(X)$, we find that $z \in X$, where $u = Sz$. It follows that

$$\begin{aligned}d(Tz, u) &= \lim_{n\to\infty} d(Tz, Tx_n)\\ &\leq \lim_{n\to\infty} [\varphi(\max\{d(Sz, Sx_n), d(Tz, Sz), d(Tx_n, Sx_n), d(Tz, Sx_n), d(Tx_n, Sz)\})]\\ &\leq \varphi(d(Tz, u)).\end{aligned}$$

Suppose $d(Tz, u) > 0$. We find $d(Tz, u) \leq \varphi(d(Tz, u)) < d(Tz, u)$, which is a contradiction. Hence $Tz = u = Sz$. Since S and T are weakly compatible, therefore $STz = TSz$ i.e. $Su = Tu = p$ (say). Again the weak compatibility of S and T implies

$$Tp = TSu = STu = Sp.$$

Suppose $Tp \neq p$. It follows that

$$\begin{aligned}d(Tp, p) &= d(Tp, Tu)\\ &\leq \varphi(max\{d(Sp, Su), d(Tp, Sp), d(Tu, Su), d(Tp, Su), d(Tu, Sp)\}).\end{aligned}$$

That is,

$$d(Tp, p) \leq \varphi(d(Tp, p)) < d(Tp, p).$$

This is a contradiction. Hence $Tp = p = Sp$. Let q be another fixed point of S and T. Suppose $p \neq q$. Then

$$d(p, q) = d(Tp, Tq) \leq \varphi(d(p, q)) < d(p, q),$$

which is a contradiction. Hence $p = q$. This completes the proof.

Next, we give an example to support our result.

Example 2.2. Let $X = [0, 1]$ and d a usual metric on X. Consider $S, T : X \to X$ defined by $Tx = \frac{x}{9}, x \in [0, 1]$ and $Sx = \frac{x}{3}$ for $0 \leq x \leq \frac{1}{2}$, $Sx = \frac{1}{3}$ for $\frac{1}{2} < x \leq 1$, where S and T are weakly compatible. Let $\varphi(t) = \frac{t}{3}$. Then all the conditions in theorem 2.1 holds. It is obvious that 0 is the unique common fixed point of S and T.

Now we extend theorem 2.1 for two mappings to four mappings as follows.

Theorem 2.3. *Let X be a complete metric space. Let $A,B,S,T : X \to X$ be four mappings such that (A,S) and (B,T) are weakly compatible such that $\overline{A(X)} \subset T(X)$, $\overline{B(X)} \subset S(X)$ and*

$$d(Ax,By) \le \varphi(\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(By,Sx),d(Ax,Ty)\}) \tag{2.4}$$

for all $x,y \in X$, where φ is as defined in definition (1.3). Then A,B,S,T have a unique common fixed point.

Proof. Let $x_0 \in X$. Let us consider the case that the sequences $\{x_n\}$ and $\{y_n\}$ in X defined by $y_{2n} = Sx_{2n} = Bx_{2n-1}$, $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ which is possible by (i). Let $d_{2n} = d(y_{2n}, y_{2n+1})$ and $d_{2n-1} = d(y_{2n-1}, y_{2n})$. Following the proof in Theorem 2.1, one can immediately obtain the result. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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