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COMMON FIXED POINT THEOREMS FOR SIX MAPPINGS IN ORDERED *G*-METRIC SPACES

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Abstract: In this paper, we establish some common fixed point theorems for six mappings in the framework of ordered G-metric space satisfying some generalized contractive conditions which improve and generalize the results of Abbas et.al. [3] for three mappings in a complete G-metric space . Examples are presented to support our results.

Keywords: Common fixed point, partially ordered set, dominating maps, weakly annihilator maps, G-metric space.
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1. Introduction and preliminaries

The notion of G-metric space was introduced by Mustafa and Sims [5], [6] as a generalization of metric spaces. Afterwards Mustafa and Sims [7] proved fixed point theorems for mappings satisfying different contractive conditions in this space. The study of unique common fixed points of mappings satisfying strict contractive conditions has been at the center of rigorous research activity. In [1] Abbas and Rhoades studied common fixed point results for non-commuting mappings without continuity in G-metric spaces. Moreover, existence of fixed points in ordered metric spaces has been initiated by Ran and Reurings [9] and further studied by Nieto and

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Lopez [8]. Recently, Abbas et.al. [3] extended and generalized the results in [7] and proved common fixed point theorems for three mappings in complete G-metric space. The purpose of this article is to study common fixed point theorems for six mappings in ordered G-metric spaces without using weakly compatible. Our result generalize various results of Abbas et.al. [3]. Here we present the necessary definitions and results in G-metric spaces which will be useful for the rest of the paper. However, for details we refer to [5], [6].

Definition 1.1. [6] Let X be a nonempty set, and let $G : X^3 \to [0,\infty)$, be a function satisfying:

(G_1) $G(x,y,z) = 0$ if $x = y = z$,

(G_2) $0 < G(x,x,y)$, for all $x,y \in X$, with $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z), \forall x,y,z \in X$, with $z \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) \ldots$, (symmetry in all three variables),

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2.[6] Let (X,G) be a G-metric space, a sequence (x_n) is said to be

(i) G-convergent if for every $\varepsilon > 0$, there exists an $x \in X$ and $k \in \mathbf{N}$ such that for all $m,n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

(ii) G-Cauchy if for every $\varepsilon > 0$, there exists an $k \in \mathbf{N}$ such that for all $m,n,p \geq k$, $G(x_m, x_n, x_p) < \varepsilon$, that is $G(x_m, x_n, x_p) \to 0$ as $m,n,p \to \infty$.

(iii) A space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent.

Definition 1.3.[6] A G-metric space X is symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Lemma 1.4.[6] Let (X,G) be a G-metric space. Then the following are equivalent:

(i) (x_n) $is\ convergent\ to\ x$,

(ii) $G(x_n, x_n, x) \to 0\ as\ n \to \infty$,

(iii) $G(x_n, x, x) \to 0\ as\ n \to \infty$,

$(iv) G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 1.5.[6] Let (X, G) be a G-metric space. Then the following are equivalent:

(i) The sequence (x_n) is G-Cauchy,

(ii) for every $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for $m, n \geq k$.

Lemma 1.6.[6] Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 1.7.[6] *every G-metric space* (X,G) *will define a metric space* (X, d_G) *by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \forall x, y \in X.$$

Proposition 1.8.[6] *Let* (X, G)*be a G-metric space. Then for any* x, y, z,*and* $a \in X$, *it follows that*

(i) *if* $G(x, y, z) = 0$ *then* $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(x, x, y)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$,

Definition 1.9. [4] Let X be a nonempty set. Then $(X, \preccurlyeq, G)$is called an ordered G-metric space if (X, G) is a G- metric space and $(X, \preccurlyeq)$ is a partial order set.

Definition 1.10. Let $(X, \preccurlyeq)$ be a partial ordered set. Then two points $x, y \in X$ are said to be comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$.

In [2] Abbas et al. introduced the following definitions:

Definition 1.11. [2] Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping f is called weak annihilator of g if $fgx \preccurlyeq x$ for all $x \in X$.

Definition 1.12. [2] Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping f on X is called dominating if $x \preccurlyeq fx$ for all $x \in X$.

For examples illustrating the above definitions are given in [2].

Definition 1.13. A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable.

2. Common fixed point theorems

In this section, we establish common fixed point theorems for six mappings defined on an ordered G-metric space. We begin with the following theorem which generalize (Theorem 2.1, [3]).

Theorem 2.1. *Let* $(X, \preccurlyeq, G)$*be an ordered* G*-metric space and let* f, g, h, S, T *and* R *be self-maps on* X *satisfying the following condition*

$$G(fx, gy, hz) \leq kM(x, y, z), \tag{2.1}$$

where $k \in [0, \frac{1}{2})$ *and*

$$M(x, y, z) = max\{G(Sx, Ty, Rz), G(fx, fx, Sx), G(gy, gy, Ty), G(hz, hz, Rz),$$
$$(gy, gy, Sx), G(Ty, hz, hz), G(Rz, fx, fx)\}$$

for all comparable elements $x, y, z \in X$. *Suppose that*

$(i) f\ (X) \subseteq T(X), g(X) \subseteq R(X), h(X) \subseteq S(X),$

(ii)*dominating maps* f, g, h *are weak annihilators of* T, R, S *respectively***,**

(iii) one of $S(X)$, $T(X)$ *or* $R(X)$ *is a* G*-complete subspace of* X.

If, for a non-decreasing sequence $\{x_n\}$ *with* $x_n \preccurlyeq y_n$ *for all* $n \geq 0$ *and* $y_n \to q$ *implies that* $x_n \preccurlyeq q$, *then* f, g, h, S, T *and* R *have a common fixed point. Moreover, the set of common fixed points of* f, g, h, S, T *and* R *is well ordered if and only if* f, g, h, S, T *and* R *have one and only one common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $f\ (X) \subseteq T(X)$, $g(X) \subseteq R(X)$, $h(X) \subseteq S(X)$, we can choose $x_1, x_2, x_3 \in X$ such that $y_0 = fx_0 = Tx_1, y_1 = gx_1 = Rx_2$, and $y_2 = hx_2 = Sx_3$. Continuing this process, we define the sequences x_n and y_n in X by $y_{3n} = f\ x_{3n} = Tx_{3n+1}, y_{3n+1} = gx_{3n+1} = Rx_{3n+2}, y_{3n+2} = hx_{3n+2} = Sx_{3n+3}$, $for\ n \geq 0$. By given assumptions, we get

$$x_{3n} \preccurlyeq fx_{3n} = Tx_{3n+1} \preccurlyeq fTx_{3n+1} \preccurlyeq x_{3n+1},$$
$$x_{3n+1} \preccurlyeq gx_{3n+1} = Rx_{3n+2} \preccurlyeq gRx_{3n+2} \preccurlyeq x_{3n+2},$$
$$x_{3n+2} \preccurlyeq hx_{3n+2} = Sx_{3n+3} \preccurlyeq hSx_{3n+3} \preccurlyeq x_{3n+3}.$$

So, for all $n \geq 0$ we have $x_n \preccurlyeq x_{n+1}$. Suppose that $G(y_n, y_{n+1}, y_{n+2}) > 0$ for all $n \geq 0$. If not, then for some $m \geq 0$, $y_m = y_{m+1} = y_{m+2}$ and the sequence $\{y_n\}$ becomes constant for $n \geq m$.

Indeed, let $m = 3k$ then $y_{3k} = y_{3k+1} = y_{3k+2}$ and from (2.1) we obtain

$$G(y_{3k+3}, y_{3k+1}, y_{3k+2}) = G(fx_{3k+3}, gx_{3k+1}, hx_{3k+2}) \leq kM(x_{3k+3}, x_{3k+1}, x_{3k+2})$$

where

$$\begin{aligned}
&M(x_{3k+3}, x_{3k+1}, x_{3k+2})\\
&= \max\{G(Sx_{3k+3}, Tx_{3k+1}, Rx_{3k+2}), G(fx_{3k+3}, fx_{3k+3}, Sx_{3k+3}),\\
&G(gx_{3k+1}, gx_{3k+1}, Tx_{3k+1}), G(hx_{3k+2}, hx_{3k+2}, Rx_{3k+2}), G(gx_{3k+1}, gx_{3k+1}, Sx_{3k+3}),\\
&G(Tx_{3k+1}, hx_{3k+2}, hx_{3k+2}), G(Rx_{3k+2}, fx_{3k+3}, fx_{3k+3})\}\\
&= \max\{G(y_{3k+2}, y_{3k}, y_{3k+1}), G(y_{3k+3}, y_{3k+3}, y_{3k+2}), G(y_{3k+1}, y_{3k+1}, y_{3k}),\\
&G(y_{3k+2}, y_{3k+2}, y_{3k+1}), G(y_{3k+1}, y_{3k+1}, y_{3k+2}), G(y_{3k}, y_{3k+2}, y_{3k+2}), G(y_{3k+1}, y_{3k+3}, y_{3k+3})\}\\
&\leq \max\{0, G(y_{3k+1}, y_{3k+2}, y_{3k+3}), 0,0,0,0, G(y_{3k+1}, y_{3k+2}, y_{3k+3})\}\\
&= G(y_{3k+1}, y_{3k+2}, y_{3k+3}).
\end{aligned}$$

Hence

$$G(y_{3k+1}, y_{3k+2}, y_{3k+3}) \leq kG(y_{3k+1}, y_{3k+2}, y_{3k+3}).$$

Therefore $G(y_{3k+1}, y_{3k+2}, y_{3k+3}) = 0$, that is $y_{3k+1} = y_{3k+2} = y_{3k+3}$. Similarly, if $m = 3k+1$ one obtain that $y_{3k+2} = y_{3k+3} = y_{3k+4}$ and if $m = 3k+2$ we have $y_{3k+3} = y_{3k+4} = y_{3k+5}$. Thus $\{y_n\}$ becomes a constant sequence and y_{3n} is the common fixed point of f, g, h, S, T and R. Now, suppose that $G(y_n, y_{n+1}, y_{n+2}) > 0$ for all $n \geq 0$. Since $x_n \preccurlyeq x_{n+1}$ for all $n \geq 0$, then by (2.1) we have

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) = G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \leq kM(x_{3n}, x_{3n+1}, x_{3n+2})$$

for $n = 0,1,2,\cdots$, where

$$\begin{aligned}
&M(x_{3n}, x_{3n+1}, x_{3n+2})\\
&= max\{G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}), G(fx_{3n}, fx_{3n}, Sx_{3n}), G(gx_{3n+1}, gx_{3n+1}, Tx_{3n+1}),\\
&G(hx_{3n+2}, hx_{3n+2}, Rx_{3n+2}), G(gx_{3n+1}, gx_{3n+1}, Sx_{3n}),\\
&G(Tx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Rx_{3n+2}, fx_{3n}, fx_{3n})\}\\
&= max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n}, y_{3n-1}), G(y_{3n+1}, y_{3n+1}, y_{3n}),\\
&G(y_{3n+2}, y_{3n+2}, y_{3n+1}), G(y_{3n+1}, y_{3n+1}, y_{3n-1}), G(y_{3n}, y_{3n+2}, y_{3n+2}), G(y_{3n+1}, y_{3n}, y_{3n})\}\\
&\leq max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-1}, y_{3n}, y_{3n+1}),\\
&G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n-1}, y_{3n}, y_{3n+1})\}\\
&= max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2})\}.
\end{aligned}$$

If $max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2})\} = G(y_{3n}, y_{3n+1}, y_{3n+2})$ then we get

$$G(y_{3n},y_{3n+1},y_{3n+2}) \leq kG(y_{3n},y_{3n+1},y_{3n+2}),$$

which implies that $G(y_{3n},y_{3n+1},y_{3n+2}) = 0,$ a contradiction. Hence

$$max\{G(y_{3n-1},y_{3n},y_{3n+1}),G(y_{3n},y_{3n+1},y_{3n+2})\} = G(y_{3n-1},y_{3n},y_{3n+1})$$

and

$$G(y_{3n},y_{3n+1},y_{3n+2}) \leq kG(y_{3n-1},y_{3n},y_{3n+1}).$$

Similarly by replacing $x = x_{3n+3}$, $y = x_{3n+1}$, $z = x_{3n+2}$, in (2.1) we obtain

$$G(y_{3n+1},y_{3n+2},y_{3n+3}) \leq kG(y_{3n},y_{3n+1},y_{3n+2}).$$

Also, replacing $x = x_{3n+3}$, $y = x_{3n+4}$, $z = x_{3n+2}$, in (2.1) we have

$$G(y_{3n+2},y_{3n+3},y_{3n+4}) \leq kG(y_{3n+1},y_{3n+2},y_{3n+3}).$$

Therefore for all n we obtain

$$\begin{aligned} G(y_n,y_{n+1},y_{n+2}) &\leq kG(y_{n-1},y_n,y_{n+1}) \\ &\leq \cdots \leq k^nG(y_0,y_1,y_2). \end{aligned}$$

Now, for all l,m,n with $l > m > n$,

$$\begin{aligned} G(y_n,y_m,y_l) &\leq G(y_n,y_{n+1},y_{n+1}) + G(y_{n+1},y_{n+2},y_{n+2}) \\ &\quad + \cdots + G(y_{l-1},y_{l-1},y_l) \\ &\leq G(y_n,y_{n+1},y_{n+2}) + G(y_{n+1},y_{n+2},y_{n+3}) \\ &\quad + \cdots + G(y_{l-2},y_{l-1},y_l) \\ &\leq (k^n + k^{n+1} + \cdots + k^{l-2})G(y_0,y_1,y_2) \\ &\leq \frac{k^n}{1-k}G(y_0,y_1,y_2). \end{aligned}$$

Also, if $l = m > n$ and $l > m = n$ we obtain

$$G(y_n,y_m,y_l) \leq \tfrac{k^n}{1-k}G(y_0,y_1,y_2).$$

Hence $G(y_n,y_m,y_l) \to 0$ as $n,m,l \to \infty$. Therefore $\{y_n\}$ is a G-Cauchy sequence. Suppose that $S(X)$ is a G-complete subspace of X, then there exists a point $q \in S(X)$ such that $\lim_{n\to\infty} y_{3n+2} = \lim_{n\to\infty} Sx_{3n+3} = q$. Also, we can find a point $p \in X$ such that $Sp = q$. Since $\{y_n\}$ is a G-Cauchy sequence then $\lim_{n\to\infty} y_{3n} = \lim_{n\to\infty} y_{3n+1} = q$. We claim that $fp = q$. Since

$$x_{3n+2} \preccurlyeq hx_{3n+2} = y_{3n+2} \text{ and } \lim_{n\to\infty} y_{3n+2} = q \text{ then } x_{3n+2} \preccurlyeq q,$$

and since dominating map h is weak annihilators of S we have

$$x_{3n+2} \preccurlyeq q = Sp \preccurlyeq hSp \preccurlyeq p, \tag{2.2}$$

we conclude that $x_{3n+1} \preccurlyeq x_{3n+2} \preccurlyeq p$, hence from (2.1) we get

$$G(fp, y_{3n+1}, y_{3n+2}) = G(fp, gx_{3n+1}, hx_{3n+2}) \leq kM(p, x_{3n+1}, x_{3n+2})$$

where

$$\begin{aligned}
&M(p, x_{3n+1}, x_{3n+2})\\
&= max\{G(Sp, Tx_{3n+1}, Rx_{3n+2}), G(fp, fp, Sp), G(gx_{3n+1}, gx_{3n+1}, Tx_{3n+1}),\\
&G(hx_{3n+2}, hx_{3n+2}, Rx_{3n+2}), G(gx_{3n+1}, gx_{3n+1}, Sp),\\
&G(Tx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Rx_{3n+2}, fp, fp)\}\\
&= max\{G(q, y_{3n}, y_{3n+1}), G(fp, fp, q), G(y_{3n+1}, y_{3n+1}, y_{3n}),\\
&G(y_{3n+2}, y_{3n+2}, y_{3n+1}), G(y_{3n+1}, y_{3n+1}, q), G(y_{3n}, y_{3n+2}, y_{3n+2}), G(y_{3n+1}, fp, fp)\}.
\end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} M(p, x_{3n+1}, x_{3n+2}) = max\{0, G(fp, fp, q), 0,0,0,0, G(q, fp, fp)\} = G(fp, fp, q).$$

Hence

$$G(fp, q, q) \leq kG(fp, fp, q) \leq 2kG(fp, q, q).$$

Then $G(fp, q, q) \leq 0$. Hence $fp = q = Sp$. Since f is dominating map, $p \preccurlyeq fp = q$, and from (2.2) we have $p = q$. Therefore $fq = q = Sq$. Since $fq = q$ and $f(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = q$. We claim that $gu = q$. Since $x_{3n+2} \preccurlyeq q$, and since dominating map f is weak annihilators of T we obtain

$$x_{3n+2} \preccurlyeq q = Tu \preccurlyeq fTu \preccurlyeq u, \quad \text{implies} \quad x_{3n+2} \preccurlyeq q \preccurlyeq u, \tag{2.3}$$

so using (2.1) we get

$$G(q, gu, y_{3n+2}) = G(fq, gu, hx_{3n+2}) \leq kM(q, u, x_{3n+2})$$

where

$$\begin{aligned}
&M(q, u, x_{3n+2})\\
&= max\{G(Sq, Tu, Rx_{3n+2}), G(fq, fq, Sq), G(gu, gu, Tu), G(hx_{3n+2}, hx_{3n+2}, Rx_{3n+2}),\\
&G(gu, gu, Sq), G(Tu, hx_{3n+2}, hx_{3n+2}), G(Rx_{3n+2}, fq, fq)\}\\
&= max\{G(q, q, y_{3n+1}), 0, G(gu, gu, q), G(y_{3n+2}, y_{3n+2}, y_{3n+1}), G(gu, gu, q),\\
&G(q, y_{3n+2}, y_{3n+2}), G(y_{3n+1}, q, q)\}.
\end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} M(q, u, x_{3n+2}) = \max\{0,0, G(gu, gu, q), 0, G(gu, gu, q), 0,0\} = G(gu, gu, q).$$

Hence

$$G(q,gu,q) \leq kG(gu,gu,q) \leq 2kG(q,gu,q).$$

We get $G(q,gu,q) \leq 0$. Thus $gu = q = Tu$. Also, Since g is dominating map, $u \preccurlyeq gu = q$, and from (2.3) we have $u = q$. Therefore $gq = q = Tq$. Further, since $gq = q$ and $g(X) \subseteq R(X)$, there exists $v \in X$ such that $Rv = q$. We claim that $hv = q$. Since dominating map g is weak annihilators of R ones gets

$$q = Rv \preccurlyeq gRv \preccurlyeq v \text{ implies } q \preccurlyeq v, \tag{2.4}$$

by (2.1) we obtain

$$G(q,q,hv) = G(fq,gq,hv) \leq kM(q,q,v)$$

where

$$\begin{aligned} M(q,q,v) &= max\{G(Sq,Tq,Rv), G(fq,fq,Sq), G(gq,gq,Tq), G(hv,hv,Rv), \\ &G(gq,gq,Sq), G(Tq,hv,hv), G(Rv,fq,fq)\} \\ &= max\{0,0,0,G(hv,hv,q),0,G(q,hv,hv),0\} = G(q,hv,hv). \end{aligned}$$

Hence

$$G(q,q,hv) = G(fq,gq,hv) \leq kG(q,hv,hv) \leq 2kG(q,q,hv),$$

which gives that $G(q,q,hv) = 0$, and $hv = q = Rv$. Since h is dominating map, $v \preccurlyeq hv = q$, and from (2.4) we have $v = q$. Therefore $hq = q = Rq$. We conclude that q is a common fixed point of f,g,h,S,T and R.

Now, suppose that the set of common fixed points of f,g,h,S,T and R is well ordered. We show that a common fixed points of f,g,h,S,T and R is unique. Let w is another common fixed point of f,g,h,S,T and R. Thus from (2.1) it follows that

$$G(q,q,w) = G(fq,gq,hw) \leq kM(q,q,w)$$

where

$$\begin{aligned} M(q,q,w) &= max\{G(Sq,Tq,Rw), G(fq,fq,Sq), G(gq,gq,Tq), G(hw,hw,Rw), \\ &G(gq,gq,Sq), G(Tq,hw,hw), G(Rw,fq,fq)\} \\ &= max\{G(q,q,w),0,0,0,0,G(q,w,w),G(w,q,q)\} \\ &\leq max\{G(q,q,w),0,2G(q,q,w)\} = 2G(q,q,w). \end{aligned}$$

Hence

$$G(q,q,w) \leq 2kG(q,q,w),$$

so we have $G(q,q,w) = 0$ and $q = w$. Therefore, q is a unique common fixed point of f,g,h,S,T and R. Conversely, if f,g,h,S,T and R have one and only one common fixed point then it is singleton set, so it is well ordered. The proof is similar when $T(X)$ or $R(X)$ is a G-complete subspace of X.

If we put $S=T=R=I$ (where I is the identity mapping) we have the following Corollary.

Corollary 2.2 *Let $(X,\preccurlyeq,G)$ be a complete ordered G-metric space and let f,g and h be self-maps on X satisfying the following condition*

$$G(fx,gy,hz)\leq kM(x,y,z),$$

where $k\in[0,\frac{1}{2})$ and

$$M(x,y,z)=\max\{G(x,y,z),G(fx,fx,x),G(gy,gy,y),G(hz,hz,z),\\ (gy,gy,x),G(y,hz,hz),G(z,fx,fx)\}$$

for all comparable elements $x,y,z\in X$. Suppose that f,g and h are dominating maps. If, for a non-decreasing sequence $\{x_n\}$ with $x_n\to q$ implies that $x_n\preccurlyeq q$ for all n. Then f,g and h have a common fixed point. Moreover, the set of common fixed points of f,g and h is well ordered if and only if f,g and h have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. We define the sequence x_n by

$$fx_{3n}=x_{3n+1},gx_{3n+1}=x_{3n+2},hx_{3n+2}=x_{3n+3}\quad\text{for } n\geq 0.$$

By given assumptions, we get

$$x_{3n}\preccurlyeq fx_{3n}=x_{3n+1}\preccurlyeq gx_{3n+1}=x_{3n+2}\preccurlyeq hx_{3n+2}=x_{3n+3}.$$

So, for all $n\geq 0$ we have $x_n\preccurlyeq x_{n+1}$. Return the same proof of Theorem 2.1 in [3] we conclude that $\{x_n\}$ is a G-Cauchy sequence and $x_n\to q$ as $n\to\infty$. Since $x_n\preccurlyeq x_{n+1}$ for all $n\geq 0$ and $x_n\to q$ as $n\to\infty$ then $x_n\preccurlyeq q$ for all $n\geq 0$. Hence from the proof of Theorem 2.1 in [3] we conclude that q is a common fixed of f,g and h. Also, similarly as the proof of Theorem 2.1 we have the set of common fixed points of f,g and h is well ordered if and only if f,g and h have one and only one common fixed point.

Example 2.3 *Let $X=[0,\infty)$ with the G-metric defined by*

$G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\}$, and suppose that $\leq$ be the usual ordering on X. We define a new ordering $\preccurlyeq$ on X as follows

$$x \preccurlyeq y \Leftrightarrow y \leq x, \qquad \forall x, y \in X.$$

It is clearly that $(X, \preccurlyeq, G)$ is an ordered G-metric space. Let $f, g, h, S, T, R: X \to X$ be defined by

$$fx = \ln(1+x), \quad gx = \ln(1+\tfrac{x}{4}), \quad hx = \ln(1+\tfrac{x}{2}),$$

$$Tx = e^x - 1, \quad Rx = e^{2x} - 1, \quad and \quad Sx = e^{4x} - 1.$$

It is obvious that $f(X) = T(X) = g(X) = R(X) = h(X) = S(X) = X$. For each $x \in X$, we have

$$1 + x \leq e^x, \quad 1 + \tfrac{x}{4} \leq e^x, \quad 1 + \tfrac{x}{2} \leq e^x.$$

Hence

$$fx = \ln(1+x) \leq x, \quad gx = \ln(1+\tfrac{x}{4}) \leq x, \quad hx = \ln(1+\tfrac{x}{2}) \leq x.$$

Then $x \preccurlyeq fx, x \preccurlyeq gx$, and $x \preccurlyeq hx$. Therefore f, g and h are dominating mappings. Also, for each $x \in X$ we obtain

$$fT(x) = f(e^x - 1) = \ln e^x = x \geq x,$$

$$gR(x) = g(e^{2x} - 1) = \ln(\tfrac{3+e^{2x}}{4}) = \ln(e^x \tfrac{3e^{-x}+e^x}{4}) = x + \ln(\tfrac{3e^{-x}+e^x}{4}) \geq x,$$

$$hS(x) = h(e^{4x} - 1) = \ln(\tfrac{1+e^{4x}}{2}) = \ln(e^x \tfrac{e^{-x}+e^{3x}}{2}) = x + \ln(\tfrac{e^{-x}+e^{3x}}{2}) \geq x.$$

We conclude that $fT(x) \preccurlyeq x, gR(x) \preccurlyeq x$ and $hS(x) \preccurlyeq x$. Thus f, g, h are weak annihilators of T, R, S respectively. Moreover, for all $x, y, z \in X$ one obtain the following:

$$G(fx, gy, hz) = \max\{|fx - gy|, |gy - hz|, |hz - fx|\}$$

$$= \max\{|\ln(1+x) - \ln(1+\tfrac{y}{4})|, |\ln(1+\tfrac{y}{4}) - \ln(1+\tfrac{z}{2})|,$$

$$|\ln(1+\tfrac{z}{2}) - \ln(1+x)|\}$$

$$\leq \max\{|x - \tfrac{y}{4}|, |\tfrac{y}{4} - \tfrac{z}{2}|, |\tfrac{z}{2} - x|\}$$

$$= \tfrac{1}{4}\max\{|4x - y|, |y - 2z|, |2z - 4x|\}$$

$$\leq \tfrac{1}{4}\max\{|e^{4x} - e^y|, |e^y - e^{2z}|, |e^{2z} - e^{4x}|\}$$

$$= \tfrac{1}{4}\max\{|Sx - Ty|, |Ty - Rz|, |Rz - Sx|\}$$

$$= \tfrac{1}{4} G(Sx, Ty, Rz)$$

$$\leq \frac{1}{4} M(x,y,z),$$

where

$$M(x,y,z) = \max\{G(Sx,Ty,Rz), G(fx,fx,Sx), G(gy,gy,Ty), G(hz,hz,Rz), (gy,gy,Sx), G(Ty,hz,hz), G(Rz,fx,fx)\}.$$

The hypotheses of Theorem 2.1 are holds with contractive factor equal to $\frac{1}{4}$. Also, 0 is a unique common fixed point of f,g,h,S,T and R.

Theorem 2.4 *Let $(X,\preccurlyeq,G)$ be an ordered G-metric space and let f,g,h,S,T and R be self-maps on X satisfying the following condition*

$$G(fx,gy,hz) \leq kM(x,y,z), \tag{2.5}$$

where $k \in [0,\frac{1}{3})$ and

$$M(x,y,z) = \max\{G(Ty,fx,fx) + G(Sx,gy,gy), G(Rz,gy,gy) + G(Ty,hz,hz), G(Rz,fx,fx) + G(Sx,hz,hz)\}$$

for all comparable elements $x,y,z \in X$. Suppose that

(i) $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$, $h(X) \subseteq S(X)$,

(ii)*dominating maps f,g,h are weak annihilators of T,R,S respectively,*

(iii) *one of $S(X)$, $T(X)$ or $R(X)$ is a G-complete subspace of X.*

If for a non-decreasing sequence $\{x_n\}$ with $x_n \preccurlyeq y_n$ for all n and $y_n \to q$ implies that $x_n \preccurlyeq q$, then f,g,h,S,T and R have a common fixed point. Moreover, the set of common fixed points of f,g,h,S,T and R is well ordered if and only if f,g,h,S,T and R have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq T(X), g(X) \subseteq R(X), h(X) \subseteq S(X)$, we can choose $x_1,x_2,x_3 \in X$ such that $y_0 = fx_0 = Tx_1$, $y_1 = gx_1 = Rx_2$, and $y_2 = hx_2 = Sx_3$. Continuing this process, we define the sequences x_n and y_n in X by $y_{3n} = fx_{3n} = Tx_{3n+1}$, $y_{3n+1} = gx_{3n+1} = Rx_{3n+2}$, $y_{3n+2} = hx_{3n+2} = Sx_{3n+3}$ $for\ n \geq 0$.

By given assumptions, we get

$$x_{3n} \preccurlyeq fx_{3n} = Tx_{3n+1} \preccurlyeq f\,Tx_{3n+1} \preccurlyeq x_{3n+1},$$

$$x_{3n+1} \preccurlyeq gx_{3n+1} = Rx_{3n+2} \preccurlyeq gRx_{3n+2} \preccurlyeq x_{3n+2},$$

$$x_{3n+2} \preccurlyeq hx_{3n+2} = Sx_{3n+3} \preccurlyeq hSx_{3n+3} \preccurlyeq x_{3n+3}.$$

Hence, for all $n \geq 0$ we have $x_n \preccurlyeq x_{n+1}$. Suppose that $G(y_n, y_{n+1}, y_{n+2}) > 0$ for all $n \geq 0$. If not, then for some $m \geq 0$, $y_m = y_{m+1} = y_{m+2}$ the sequence $\{y_n\}$ is constant for $n \geq m$. Indeed, let $m = 3k$ then $y_{3k} = y_{3k+1} = y_{3k+2}$ and from (2.5) we obtain

$$G(y_{3k+3}, y_{3k+1}, y_{3k+2}) = G(fx_{3k+3}, gx_{3k+1}, hx_{3k+2}) \leq kM(x_{3k+3}, x_{3k+1}, x_{3k+2})$$

where

$$\begin{aligned}
&M(x_{3k+3}, x_{3k+1}, x_{3k+2}) \\
&= max\{G(Tx_{3k+1}, fx_{3k+3}, fx_{3k+3}) + G(Sx_{3k+3}, gx_{3k+1}, gx_{3k+1}), \\
&G(Rx_{3k+2}, gx_{3k+1}, gx_{3k+1}) + G(Tx_{3k+1}, hx_{3k+2}, hx_{3k+2}), \\
&G(Rx_{3k+2}, fx_{3k+3}, fx_{3k+3}) + G(Sx_{3k+3}, hx_{3k+2}, hx_{3k+2})\} \\
&= max\{G(y_{3k}, y_{3k+3}, y_{3k+3}) + G(y_{3k+2}, y_{3k+1}, y_{3k+1}), \\
&G(y_{3k+1}, y_{3k+1}, y_{3k+1}) + G(y_{3k}, y_{3k+2}, y_{3k+2}), \\
&G(y_{3k+1}, y_{3k+3}, y_{3k+3}) + G(y_{3k+2}, y_{3k+2}, y_{3k+2})\} \\
&= max\{G(y_{3k+1}, y_{3k+3}, y_{3k+3}), 0\} \\
&\leq max\{G(y_{3k+1}, y_{3k+2}, y_{3k+3}), 0\} = G(y_{3k+1}, y_{3k+2}, y_{3k+3}).
\end{aligned}$$

Hence

$$G(y_{3k+1}, y_{3k+2}, y_{3k+3}) \leq kG(y_{3k+1}, y_{3k+2}, y_{3k+3}).$$

Therefore $G(y_{3k+1}, y_{3k+2}, y_{3k+3}) = 0$, that is $y_{3k+1} = y_{3k+2} = y_{3k+3}$. Similarly, if $m = 3k + 1$ one obtain that $y_{3k+2} = y_{3k+3} = y_{3k+4}$ and if $m = 3k + 2$ we have $y_{3k+3} = y_{3k+4} = y_{3k+5}$. Thus, $\{y_n\}$ becomes a constant sequence and y_{3n} is the common fixed point of f, g, h, S, T and R. Now, suppose that $G(y_n, y_{n+1}, y_{n+2}) > 0$ for all $n \geq 0$. Since $x_n \preccurlyeq x_{n+1}$ for all $n \geq 0$, from (2.5) we have

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) = G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \leq kM(x_{3n}, x_{3n+1}, x_{3n+2})$$

for $n = 0,1,2, \cdots$, where

$$\begin{aligned}
&M(x_{3n}, x_{3n+1}, x_{3n+2})\\
&= max\{G(Tx_{3n+1}, fx_{3n}, fx_{3n}) + G(Sx_{3n}, gx_{3n+1}, gx_{3n+1}),\\
&G(Rx_{3n+2}, gx_{3n+1}, gx_{3n+1}) + G(Tx_{3n+1}, hx_{3n+2}, hx_{3n+2}),\\
&G(Rx_{3n+2}, fx_{3n}, fx_{3n}) + G(Sx_{3n}, hx_{3n+2}, hx_{3n+2})\}\\
&= max\{G(y_{3n}, y_{3n}, y_{3n}) + G(y_{3n-1}, y_{3n+1}, y_{3n+1}),\\
&G(y_{3n+1}, y_{3n+1}, y_{3n+1}) + G(y_{3n}, y_{3n+2}, y_{3n+2}),\\
&G(y_{3n+1}, y_{3n}, y_{3n}) + G(y_{3n-1}, y_{3n+2}, y_{3n+2})\}\\
&\leq max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}),\\
&G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n}, y_{3n}) + G(y_{3n}, y_{3n+2}, y_{3n+2})\}\\
&\leq max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}),\\
&G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2})\}\\
&= max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}),\\
&2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2})\}\\
&= 2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2}).
\end{aligned}$$

Then

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq k(2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2})).$$

Hence

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq \frac{2k}{1-k} G(y_{3n-1}, y_{3n}, y_{3n+1}).$$

Put $\lambda = \frac{2k}{1-k}$, clear $0 \leq \lambda < 1$. Therefore

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq \lambda G(y_{3n-1}, y_{3n}, y_{3n+1}).$$

Similarly we obtain

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq \lambda G(y_{3n}, y_{3n+1}, y_{3n+2}).$$

Also, we have

$$G(y_{3n+2}, y_{3n+3}, y_{3n+4}) \leq \lambda G(y_{3n+1}, y_{3n+2}, y_{3n+3}).$$

Therefore, for all n,

$$\begin{aligned}
G(y_n, y_{n+1}, y_{n+2}) &\leq \lambda G(y_{n-1}, y_n, y_{n+1})\\
&\leq \cdots \leq \lambda^n G(y_0, y_1, y_2).
\end{aligned}$$

Following similar arguments to those given in Theorem 2.1 , $G(y_n, y_m, y_l) \to 0$ as $n, m, l \to \infty$. Therefore $\{y_n\}$ is a G-Cauchy sequence. Suppose that $S(X)$ is a G-complete subspace of X, then there exists a point $q \in S(X)$ such that $\lim_{n\to\infty} y_{3n+2} = \lim_{n\to\infty} Sx_{3n+3} = q$. Also, we can find a point $p \in X$ such that $Sp = q$. We claim that $fp = q$. Since

$$x_{3n+2} \preccurlyeq hx_{3n+2} = y_{3n+2} \text{ and } \lim_{n\to\infty} y_{3n+2} = q \text{ then } x_{3n+2} \preccurlyeq q,$$

and since dominating map h is weak annihilators of S we have

$$x_{3n+2} \preccurlyeq q = Sp \preccurlyeq hSp \preccurlyeq p, \tag{2.6}$$

we conclude that $x_{3n+1} \preccurlyeq x_{3n+2} \preccurlyeq p$, thus by (2.5) we obtain

$$G(fp, y_{3n+1}, y_{3n+2}) = G(fp, gx_{3n+1}, hx_{3n+2}) \leq kM(p, x_{3n+1}, x_{3n+2})$$

where

$$\begin{aligned} &M(p, x_{3n+1}, x_{3n+2}) \\ &= max\{G(Tx_{3n+1}, fp, fp) + G(Sp, gx_{3n+1}, gx_{3n+1}), \\ &G(Rx_{3n+2}, gx_{3n+1}, gx_{3n+1}) + G(Tx_{3n+1}, hx_{3n+2}, hx_{3n+2}), \\ &G(Rx_{3n+2}, fp, fp) + G(Sp, hx_{3n+2}, hx_{3n+2})\} \\ &= max\{G(y_{3n}, fp, fp) + G(q, y_{3n+1}, y_{3n+1}), \\ &G(y_{3n+1}, y_{3n+1}, y_{3n+1}) + G(y_{3n}, y_{3n+2}, y_{3n+2}), \\ &G(y_{3n+1}, fp, fp) + G(q, y_{3n+2}, y_{3n+2})\}. \end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} \lim_{n\to\infty} M(p, x_{3n+1}, x_{3n+2}) &= max\{G(q, fp, fp), 0, G(q, fp, fp)\} \\ &= G(fp, fp, q). \end{aligned}$$

Hence

$$G(fp, q, q) \leq kG(fp, fp, q) \leq 2kG(fp, q, q).$$

That is $G(fp, q, q) = 0$. Hence $fp = q = Sp$. Since f is dominating map, $p \preccurlyeq fp = q$, and from (2.6) we have $p = q$. Therefore $fq = q = Sq$. Since $fq = q$ and $f(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = q$. We claim that $gu = q$. Since $x_{3n+2} \preccurlyeq q$, and since dominating map f is weak annihilators of T we obtain

$$x_{3n+2} \preccurlyeq q = Tu \preccurlyeq fTu \preccurlyeq u, \text{ implies } x_{3n+2} \preccurlyeq q \preccurlyeq u. \tag{2.7}$$

Using (2.5) we have

$$G(q, gu, y_{3n+2}) = G(fq, gu, hx_{3n+2}) \leq kM(q, u, x_{3n+2})$$

where

$$\begin{aligned} M(q, u, x_{3n+2}) &= \max\{G(Tu, fq, fq) + G(Sq, gu, gu), \\ &G(Rx_{3n+2}, gu, gu) + G(Tu, hx_{3n+2}, hx_{3n+2}), \\ &G(Rx_{3n+2}, fq, fq) + G(Sq, hx_{3n+2}, hx_{3n+2})\} \\ &= \max\{G(q, gu, gu), G(y_{3n+1}, gu, gu) + G(q, y_{3n+2}, y_{3n+2}), \\ &G(Rx_{3n+2}, q, q) + G(q, y_{3n+2}, y_{3n+2}\}. \end{aligned}$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} M(q,u,x_{3n+2}) \quad = max\{G(q,gu,gu), G(q,gu,gu),0\} = G(gu,gu,q).$$

Hence

$$G(q,gu,q) \le kG(gu,gu,q) \le 2kG(q,gu,q).$$

Thus $gu = q = Tu$. Also, Since g is dominating map, $u \preccurlyeq gu = q$, and from (2.7) we have $u = q$. Therefore $gq = q = Tq$. Further, since $gq = q$ and $g(X) \subseteq R(X)$, there exists $v \in X$ such that $Rv = q$. We claim that $hv = q$. Since dominating map g is weak annihilators of R ones gets

$$q = Rv \preccurlyeq gRv \preccurlyeq v, \text{implies } q \preccurlyeq v. \tag{2.8}$$

From (2.5) we have

$$G(q,q,hv) = G(fq,gq,hv) \le kM(q,q,v)$$

where

$$\begin{aligned} M(q,q,v) \quad &= \max\{G(Tq,fq,fq) + G(Sq,gq,gq), G(Rv,gq,gq) + G(Tq,hv,hv), \\ &G(Rv,fq,fq) + G(Sq,hv,hv)\} \\ &= \max\{0, G(q,hv,hv), G(q,hv,hv)\} = G(q,hv,hv). \end{aligned}$$

Hence

$$G(q,q,hv) = G(fq,gq,hv) \le kG(q,hv,hv) \le 2kG(q,q,hv),$$

which gives that $G(q,q,hv) = 0$, and $hv = q = Rv$. Since h is dominating map, $v \preccurlyeq hv = q$, and from (2.8) we have $v = q$. Therefore $hq = q = Rq$. We conclude that q is a common fixed point of f,g,h,S,T and R.

Now, suppose that the set of common fixed points of f,g,h,S,T and R is well ordered. We show that a common fixed points of f,g,h,S,T and R is unique. Let w is another common fixed point of f,g,h,S,T and R. Thus from (2.5) one obtain

$$G(q,q,w) = G(fq,gq,hw) \le kM(q,q,w)$$

where

$$\begin{aligned} M(q,q,w) \quad &= max\{G(Tq,fq,fq) + G(Sq,gq,gq), G(Rw,gq,gq) + G(Tq,hw,hw), \\ &G(Rw,fq,fq) + G(Sq,hw,hw\} \\ &= max\{0, G(w,q,q) + G(q,w,w), G(w,q,q) + G(q,w,w\} \\ &= G(w,q,q) + G(q,w,w). \end{aligned}$$

Hence

$$G(q,q,w) \le k(G(w,q,q) + G(q,w,w)) \le 3kG(q,q,w).$$

Thus we have $G(q,q,w)=0$ and $q=w$. Therefore, q is a unique common fixed point of f,g,h,S,T and R. Conversely, if f,g,h,S,T and R have one and only one common fixed point then it is singleton set, so it is well ordered. The proof is similar when $T(X)$ or $R(X)$ is a G-complete subspace of X.

If we put $S=T=R=I$ (where I is the identity mapping) we have the following corollary.

Corollary 2.5 *Let* $(X,\preccurlyeq,G)$ *be a complete ordered* G*-metric space and let* f,g *and* h *be self-maps on* X *satisfying the following condition*

$$G(fx,gy,hz)\leq kM(x,y,z),$$

where $k\in[0,\frac{1}{3})$ *and*

$$M(x,y,z)=\max\{G(y,fx,fx)+G(x,gy,gy),G(z,gy,gy)+G(y,hz,hz),\\ G(z,fx,fx)+G(x,hz,hz)\}$$

for all comparable elements $x,y,z\in X$. *Suppose that* f,g *and* h *are dominating maps. If, for a non-decreasing sequence* $\{x_n\}$ *with* $x_n\to q$ *implies that* $x_n\preccurlyeq q$ *for all* n. *Then* f,g *and* h *have a common fixed point. Moreover, the set of common fixed points of* f,g *and* h *is well ordered if and only if* f,g *and* h *have one and only one common fixed point.*

Proof. Let x_0 be an arbitrary point in X. We define the sequence x_n by

$$fx_{3n}=x_{3n+1},\qquad gx_{3n+1}=x_{3n+2},\qquad hx_{3n+2}=x_{3n+3}\ \text{ for } n\geq 0.$$

By given assumptions, we get

$$x_{3n}\preccurlyeq fx_{3n}=x_{3n+1}\preccurlyeq gx_{3n+1}=x_{3n+2}\preccurlyeq hx_{3n+2}=x_{3n+3}\ \text{ for } n\geq 0.$$

So, for all $n\geq 0$ we have $x_n\preccurlyeq x_{n+1}$. Return the same proof of Theorem 2.4 in [3] we conclude that $\{x_n\}$ is a G-Cauchy sequence and $x_n\to q$ as $n\to\infty$. Since $x_n\preccurlyeq x_{n+1}$ for all $n\geq 0$ and $x_n\to q$ as $n\to\infty$ then $x_n\preccurlyeq q$ for all $n\geq 0$. Hence from the proof of Theorem 2.4 in [3] we conclude that q is a common fixed of f,g and h. Also, similarly as the proof of Theorem 2.4 we have the set of common fixed points of f,g and h is well ordered if and only if f,g and h have one and only one common fixed point.

Example 2.6 *Let* $X=[0,\infty)$ with the G-metric defined by

$$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\},$$

and suppose that $\leq$ be the usual ordering on X. We define an ordering $\preccurlyeq$ on X as follows

$$x \preccurlyeq y \Leftrightarrow y \leq x, \quad \forall x, y \in X.$$

It is clearly that $(X, \preccurlyeq, G)$ is an ordered G-metric space. Let $f, g, h, S, T, R: X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{12} & \text{if x} \in [0,1) \\ \frac{x}{8} & \text{if x} \in [1,\infty) \end{cases}, \quad gx = \begin{cases} \frac{x}{4} & \text{if x} \in [0,1) \\ \frac{x}{6} & \text{if x} \in [1,\infty) \end{cases}, \quad hx = \begin{cases} \frac{x}{2} & \text{if x} \in [0,1) \\ x & \text{if x} \in [1,\infty) \end{cases},$$

$$Sx = \begin{cases} 4x & \text{if x} \in [0,1) \\ 6x & \text{if x} \in [1,\infty) \end{cases}, \quad Tx = \begin{cases} 12x & \text{if x} \in [0,1) \\ 8x & \text{if x} \in [1,\infty) \end{cases}, \quad Rx = \begin{cases} 24x & \text{if x} \in [0,1) \\ 48x & \text{if x} \in [1,\infty) \end{cases}.$$

We see that f, g, h, S, T and R are discontinuous maps. It is obvious that $f(X) = T(X) = g(X) = R(X) = h(X) = S(X) = X$. For each $x \in X$, we have

$$fx \leq x, \quad gx \leq x, \quad hx \leq x.$$

Then $x \preccurlyeq fx, x \preccurlyeq gx$, and $x \preccurlyeq hx$. Therefore f, g and h are dominating mappings. Also, for each $x \in X$ we obtain

$$fT(x) = x \geq x, \quad gR(x) = \begin{cases} 6x \geq x & \text{ifx} \in [0,1) \\ 8x \geq x & \text{ifx} \in [1,\infty) \end{cases},$$

$$hS(x) = \begin{cases} 2x \geq x & \text{ifx} \in [0,1) \\ 6x \geq x & \text{ifx} \in [1,\infty) \end{cases}.$$

We conclude that $fT(x) \preccurlyeq x, gR(x) \preccurlyeq x$ and $hS(x) \preccurlyeq x$. Thus f, g, h are weak annihilators of T, R, S respectively. Now, for all $x, y, z \in X$ we check the following cases:

(1) If $x, y, z \in [0,1)$ we have

$$\begin{aligned} G(fx, gy, hz) &= max\{|\frac{x}{12} - \frac{y}{4}|, |\frac{y}{4} - \frac{z}{2}|, |\frac{z}{2} - \frac{x}{12}|\} \\ &= \frac{1}{48} max\{|4x - 12y|, |12y - 24z|, |24z - 4x|\} \\ &\leq \frac{1}{48} max\{|4x - \frac{y}{4}| + |12y - \frac{x}{12}| + |\frac{x}{12} - \frac{y}{4}|, \\ & |12y - \frac{z}{2}| + |24z - \frac{y}{4}| + |\frac{y}{4} - \frac{z}{2}|, |24z - \frac{x}{12}| + |4x - \frac{z}{2}| + |\frac{z}{2} - \frac{x}{12}|\} \\ &= \frac{1}{48} max\{G(Sx, gy, gy) + G(Ty, fx, fx) + |\frac{x}{12} - \frac{y}{4}|, \\ & G(Ty, hz, hz) + G(Rz, gy, gy) + |\frac{y}{4} - \frac{z}{2}|, \end{aligned}$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+|\frac{z}{2}-\frac{x}{12}|\}$$

$$\leq \frac{1}{48} max\{G(Sx,gy,gy)+G(Ty,fx,fx)+G(fx,gy,hz),$$

$$G(Ty,hz,hz)+G(Rz,gy,gy)+G(fx,gy,hz),$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+G(fx,gy,hz)\}$$

$$=\frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Hence, $G(fx,gy,hz)\leq \frac{1}{47}M(x,y,z)$, where

$$M(x,y,z)=max\{G(Sx,gy,gy)+G(Ty,fx,fx),$$

$$G(Ty,hz,hz)+G(Rz,gy,gy), G(Rz,fx,fx)+G(Sx,hz,hz)\}$$

(2) If $x,y,z\in[1,\infty)$ we have

$$G(fx,gy,hz)=max\{|\frac{x}{8}-\frac{y}{6}|,|\frac{y}{6}-z|,|z-\frac{x}{8}|\}$$

$$=\frac{1}{48} max\{|6x-8y|,|8y-48z|,|48z-6x|\}$$

$$\leq \frac{1}{48} max\{G(Sx,gy,gy)+G(Ty,fx,fx)+|\frac{x}{8}-\frac{y}{6}|,$$

$$G(Ty,hz,hz)+G(Rz,gy,gy)+|\frac{y}{6}-z|,$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+|z-\frac{x}{8}|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Hence, $G(fx,gy,hz)\leq \frac{1}{47}M(x,y,z)$.

(3) If $x,y\in[0,1)$ and $z\in[1,\infty)$, one gets

$$G(fx,gy,hz)=max\{|\frac{x}{12}-\frac{y}{4}|,|\frac{y}{4}-z|,|z-\frac{x}{12}|\}$$

$$=\frac{1}{48} max\{|4x-12y|,|12y-48z|,|48z-4x|\}$$

$$\leq \frac{1}{48} max\{G(Sx,gy,gy)+G(Ty,fx,fx)+|\frac{x}{12}-\frac{y}{4}|,$$

$$G(Ty,hz,hz)+G(Rz,gy,gy)+|\frac{y}{4}-z|,$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+|z-\frac{x}{12}|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Therefore, $G(fx,gy,hz)\leq \frac{1}{47}M(x,y,z)$.

(4) If $x,z \in [0,1)$ and $y \in [1,\infty)$, then

$$G(fx,gy,hz) = max\{|\frac{x}{12}-\frac{y}{6}|,|\frac{y}{6}-\frac{z}{2}|,|\frac{z}{2}-\frac{x}{12}|\}$$

$$= \frac{1}{48}max\{|4x-8y|,|8y-24z|,|24z-4x|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Thus, $G(fx,gy,hz) \leq \frac{1}{47}M(x,y,z)$.

(5) If $y,z \in [0,1)$ and $x \in [1,\infty)$, we obtain

$$G(fx,gy,hz) = max\{|\frac{x}{8}-\frac{y}{4}|,|\frac{y}{4}-\frac{z}{2}|,|\frac{z}{2}-\frac{x}{8}|\}$$

$$= \frac{1}{48}max\{|6x-12y|,|12y-24z|,|24z-6x|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Hence, $G(fx,gy,hz) \leq \frac{1}{47}M(x,y,z)$.

(6) If $x \in [0,1)$ and $y,z \in [1,\infty)$, then

$$G(fx,gy,hz) = max\{|\frac{x}{12}-\frac{y}{6}|,|\frac{y}{6}-z|,|z-\frac{x}{12}|\}$$

$$= \frac{1}{48}max\{|4x-8y|,|8y-48z|,|48z-4x|\}$$

$$\leq \frac{1}{48}max\{G(Sx,gy,gy)+G(Ty,fx,fx)+|\frac{x}{12}-\frac{y}{6}|,$$

$$G(Ty,hz,hz)+G(Rz,gy,gy)+|\frac{y}{6}-z|,$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+|z-\frac{x}{12}|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Therefore, $G(fx,gy,hz) \leq \frac{1}{47}M(x,y,z)$.

(7) If $y \in [0,1)$ and $x,z \in [1,\infty)$, one obtains

$$G(fx,gy,hz) = max\{|\frac{x}{8}-\frac{y}{4}|,|\frac{y}{4}-z|,|z-\frac{x}{8}|\}$$

$$= \frac{1}{48}max\{|6x-12y|,|12y-48z|,|48z-6x|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

Thus, $G(fx,gy,hz) \leq \frac{1}{47}M(x,y,z)$.

*(8)*If $z \in [0,1)$ and $x, y \in [1,\infty)$, we get

$$G(fx,gy,hz) = max\{|\frac{x}{8}-\frac{y}{6}|,|\frac{y}{6}-\frac{z}{2}|,|\frac{z}{2}-\frac{x}{8}|\}$$

$$= \frac{1}{48}max\{|6x-8y|,|8y-24z|,|24z-6x|\}$$

$$\leq \frac{1}{48}max\{G(Sx,gy,gy)+G(Ty,fx,fx)+|\frac{x}{8}-\frac{y}{6}|,$$

$$G(Ty,hz,hz)+G(Rz,gy,gy)+|\frac{y}{6}-\frac{z}{2}|,$$

$$G(Rz,fx,fx)+G(Sx,hz,hz)+|\frac{z}{2}-\frac{x}{8}|\}$$

$$\leq \frac{1}{48}(M(x,y,z)+G(fx,gy,hz)).$$

So, $G(fx,gy,hz) \leq \frac{1}{47}M(x,y,z)$.

The hypotheses of Theorem 2.4 are holds with constant $k = \frac{1}{47}$. Also, 0 is a unique common fixed point of f, g, h, S, T and R.

REFERENCES

[1] M. Abbas, B. E. Rhoades, Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Appl. Math. and Computation, 215 (2009) 262–269.

[2] M. Abbas, T. Nazir, and S. Radenovic′, Common fixed points of four maps in partially ordered metric spaces, Appl. Math. Lett. 24 (2011) 1520–1526.

[3] M. Abbas, T. Nazir, and R. Saadati, Common fixed point results for three maps in generalized metric space, Advances in Difference Equations 2011, 49 (2011).

[4] M. Abbas , Y.J. Cho and T. Nazir, Common Fixed Points of Ciric-type Contractive Mappings in Two Ordered Generalized Metric Spaces, Fixed Point Theory and Applications, 139 (2012).

[5] Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, Intern. Conf. Fixed Point. Theory and Applications, Yokohama (2004) 189–198.

[6] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Analysis, 7 (2006) 289–297.

[7] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, (2009) Article ID 917175, 2009, 10 pages.

[8] J. J. Nieto and R. Rodrguez-Lopez, Contractive mapping theorems in partially ordered sets and

applications to ordinary differential equations, Order. 22 (2005), 223–239.

[9] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc Am Math Soc., 132 (2004), 1435–1443.